ELSEVIER

17 August 1998

Physics Letters A 245 (1998) 301-306

PHYSICS LETTERS A

# Quantum interference in parallel Josephson junction arrays: a perturbative analysis for finite inductances

R. De Luca [a], T. Di Matteo [b], A. Tuohimaa [b], J. Paasi [b]

[a] *Dipartimento di Fisica and INFM, Università degli Studi di Salerno, I-84081 Baronissi (Salerno), Italy*
[b] *Laboratory of Electricity and Magnetism, Tampere University of Technology, FIN-33101 Tampere, Finland*

Received 20 March 1998; accepted for publication 13 May 1998
Communicated by J. Flouquet

**Abstract**

We study the magnetic field dependence of the maximum Josephson current in a homogeneous parallel array of Josephson junctions in the limit of very small values of the characteristic inductance parameter $\beta_L$. We show that the usual interference patterns obtained for $\beta_L = 0$ and for vanishingly small junction to loop area ratios are enriched by new features when $\beta_L$ is finite, but still small enough to allow a perturbative analysis of the problem, and when the single junction interference pattern is taken into account. © 1998 Elsevier Science B.V.

*PACS:* 74.50.+r; 85.25.Dq; 74.25.Ha
*Keywords:* Josephson effects; SQUIDs; Magnetic properties

## 1. Introduction

The magnetic field dependence of the maximum Josephson current $I_c$ of a parallel array of $N+1$ junctions is of interest for its practical applications in the field of electronic devices based on flux-flow [1]. Moreover, it can also be regarded for its purely scientific interest, given the analogy existing between this subject and the problem of diffraction gratings in optics.

The simplest parallel connection of Josephson junctions is the d.c. SQUID, which has been extensively studied in the past [2,3]. Even in this simple case, though, an exact solution of the problem can only be given in the case of negligibly small inductance of the superconducting loop containing the two junctions. In the present work, therefore, we start from the circuital model shown in Fig. 1 containing $N + 1$ junctions and $N$ loops. We show that, for negligible values of the generalized SQUID parameter $\beta_L$, this general approach gives the usual results, already known from the literature [4,5]. In this case, indeed, the $I_c$ versus $H$ curves, where $H$ is the externally applied magnetic field, show an interference pattern with unitary periodicity in the quantity $\mu_0 H S_0$, $S_0$ being the loop area. In addition, by taking into account the single junction maximum Josephson current field dependence, these interference patterns are shown to be modulated by a Fraunhofer-like pattern with pseudo-periodicity $\mu_0 H S_J$, $S_J$ being the effective junction area. Finally, by assuming small enough values of $\beta_L$, a perturbative analytic approach is developed to study this limiting case and predictions are made in the cases $N = 1$ and 2.


*PII* S0375-9601(98)00399-5

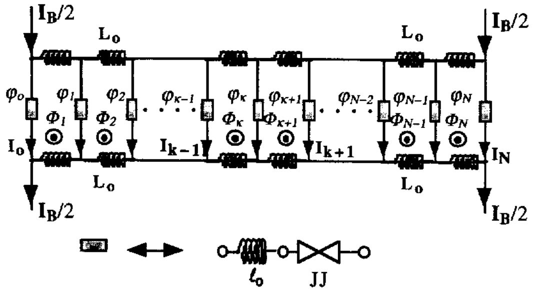

Fig. 1. Parallel Josephson junction array. The current bias is applied at the top vertices of the array. The Josephson junctions are contained in the rectangular boxes together with the inductance parameter of the vertical branches, as shown in the inset.

## 2. The model and the equations

Consider the homogeneous parallel Josephson junction array shown in Fig. 1. In this system, each rectangular box in the vertical branches corresponds to a Josephson junction (JJ) and an inductor of inductance $l_0$, as shown in the inset of Fig. 1. The effective inductance of the horizontal branches is taken to be $L_0$. Furthermore, the number of junctions is $N+1$ and the number of loops $N$. Each loop of the network has a surface area equal to $S_0$. An external field, $\boldsymbol{H}$, is applied perpendicular to the loop surface and the flux linked to each loop is $\Phi_m$, $m = 1, 2, \ldots, N$. A bias current, $I_B$, is applied at the top vertices of the array and driven at the bottom ones, as shown in Fig. 1. Finally, a current $I_k$ goes through the $k$th junction, to which a gauge invariant superconducting phase difference $\varphi_k$, with $k$ ranging from 0 to $N$, is associated.

By imposing fluxoid quantization to each loop enclosed between the 0th and the $k$th vertical branch, one can write the $N$ relations between the normalized flux variables $\Psi_k = \Phi_k/\Phi_0$, where $\Phi_0$ is the elementary flux quantum, and the superconducting phases $\varphi_k$ as follows,

$$\varphi_k = \varphi_0 - 2\pi \sum_{m=1}^{k} \Psi_m + 2\pi n_k, \tag{1}$$

where $n_k$ is an integer and $k = 1, \ldots, N$.

The flux variables can be linked to the branch currents and to the applied field $\boldsymbol{H}$ according to

$$\Psi_k = \beta_L \left( a(i_{k-1} - i_k) - i_B + 2\sum_{m=0}^{k-1} i_m \right) + \Psi_{\text{ex}}, \tag{2}$$

where $\Psi_{\text{ex}} = \mu_0 H S_0/\Phi_0$, $a = l_0/L_0$ and $\beta_L = L_0 I_{J0}/\Phi_0$ with $I_{J0}$ being the zero-field maximum Josephson current of the junctions. All the currents are normalized with respect to $I_{J0}$, so that $i_k = I_k/I_{J0}$ and $i_B = I_B/I_{J0}$. Let us introduce the non-linear Josephson operator $O_{J_k}$, defined as

$$O_{J_k}(\cdot) = \frac{\Phi_0}{2\pi R}\frac{\mathrm{d}}{\mathrm{d}t}(\cdot) + I_{J0} f_k \sin(\cdot), \tag{3}$$

where the resistive parameter, $R$, is taken to be the same for all JJs and $f_k$ is a function accounting for the Fraunhofer-like dependence of the maximum Josephson current from the local field. It can be shown that, for linear field distributions inside the junctions, $f_k$ may be written as follows,

$$f_k = \frac{\sin(\pi \Psi_{J_k})}{\pi \Psi_{J_k}}, \tag{4}$$

where

$$\Psi_{J_k} = \frac{S_J}{S_0} \frac{\Psi_k + \Psi_{k+1}}{2} \tag{5}$$

with $S_J$ being the effective junction area. Neglecting the capacitance of the junctions, the equations of the motion for the $N+1$ phase variables can be written as

$$O_{J_k}(\varphi_k) = I_k. \tag{6}$$

In what follows we shall maximize the bias current $I_B$ with respect to the phase variables $\varphi_k$. Notice that, from Kirchoff's law, the normalized bias current can be written, under stationary conditions, as follows,

$$i_B = \sum_{k=0}^{N} i_k = \sum_{k=0}^{N} f_k \sin(\varphi_k). \tag{7}$$

Eqs. (1)-(7) thus completely define the problem.

## 3. A perturbative approach

In the present section we shall derive the dependence of the normalized maximum Josephson current $i_c$ from the external magnetic flux $\Psi_{\text{ex}}$ in the limit of small $\beta_L$ values.

Let us then start by noticing that all the phase variables can be expressed in terms of the single phase $\varphi_0$ in a recursive way by means of Eqs. (1)-(6). This procedure is more evident in the case of null inductance, for which $\Psi_k = \Psi_{ex}$. Indeed, in this case, Eq. (1) can be rewritten as follows,

$$\varphi_k = \varphi_0 - 2\pi k\Psi_{ex} + 2\pi n_k. \tag{8}$$

In this way, Eq. (7), can be written in the following form,

$$i_B = \sum_{k=0}^{N} i_k = \sum_{k=0}^{N} f_k \sin(\varphi_0 - 2\pi k\Psi_{ex}). \tag{9}$$

For perfectly identical junctions and for a slowly varying field distribution in the array, we can set $f_k = f_0$ for $k = 0, 1, 2, \ldots, N$, so that, the sum in Eq. (9) can be carried out exactly, giving

$$i_B = f_0 \sin(\varphi_0 - N\pi\Psi_{ex}) \frac{\sin[(N+1)\pi\Psi_{ex}]}{\sin(\pi\Psi_{ex})}. \tag{10}$$

It is now easy to maximize $i_B$, so that

$$i_c = \left| f_0 \frac{\sin[(N+1)\pi\Psi_{ex}]}{\sin(\pi\Psi_{ex})} \right|. \tag{11}$$

A typical interference pattern is represented, for $N = 9$, in Fig. 2, where we notice that the envelope of the maxima is given by the presence of the single junction Fraunhofer-like pattern $|f_0|$.

In the case of finite inductances, we could still apply a recursive approach to define the phase variables in terms of the single phase $\varphi_0$. In this case, however, we would get into a nested type of operation, and an exact solution of the problem would not be possible anymore. Nevertheless, we can show that, if the inductance parameter $\beta_L$ is small enough, we can still try to get to a semi-analytic result by the following procedure. First we express the phase variables with index $k$ greater than 0 in terms of the variable $\varphi_0$ through Eq. (1). By making use of Eq. (2) one writes

$$\varphi_k = \varphi_0 - 2\pi k\Psi_{ex} + 2\pi\beta_L\left(k i_B - a(i_0 - i_k) - 2\sum_{m=0}^{k-1}(k-m)i_m\right). \tag{12}$$

In what follows we shall take the inductances of the vertical branches to be negligible ($a = 0$). In the above expression, the terms in large parentheses depend both

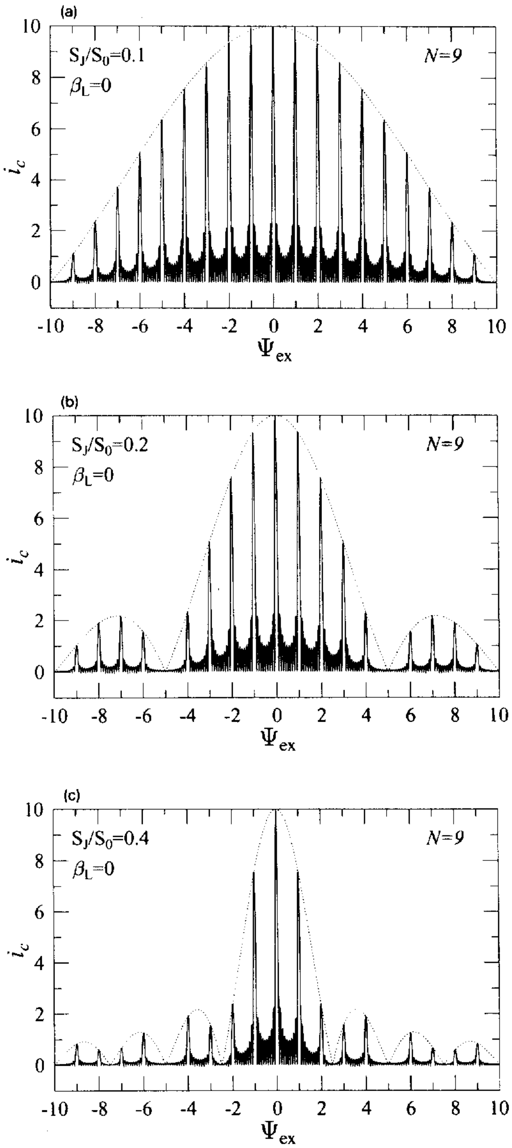


Fig. 2. Maximum Josephson current of the parallel array of Fig. 1 as a function of the normalized applied flux $\Psi_{ex}$, for $N = 9$, $\beta_L = 0$ and for the following values of the $S_J/S_0$ ratio: (a) 0.1; (b) 0.2; (c) 0.4. The dotted line represents the single junction Fraunhofer-like envelope.

on $i_B$ and on the currents $i_m$. It is therefore evident that $i_B$ can now only be implicitly defined by Eq. (7). Moreover, we still take all the junctions to be identical and assume a slowly varying flux distribution in the array, so that we may set $f_k = f_0$, as done before, and write

$$i_B=\sum_{k=0}^{N} i_k=\sum_{k=0}^{N} f_0 \sin\Big[\varphi_0-2\pi k\Psi_{ex} + 2\pi\beta_L\Big(k i_B-2\sum_{m=0}^{k-1}(k-m)i_m\Big)\Big]. \quad (13)$$

By trigonometric identities we can recast Eq. (13) in the following form,

$$i_B=f_0\sum_{k=0}^{N}\Big\{\sin(\varphi_0-2\pi k\Psi_{ex}) \times \cos\Big[2\pi\beta_L\Big(k i_B-2\sum_{m=0}^{k-1}(k-m)i_m\Big)\Big] + \cos(\varphi_0-2\pi k\Psi_{ex}) \times \sin\Big[2\pi\beta_L\Big(k i_B-2\sum_{m=0}^{k-1}(k-m)i_m\Big)\Big]\Big\}. \quad (14)$$

By expanding the sine and cosine in terms of the parameter $\beta_L$ up to the second order, we may write, after having gathered the coefficients of order 0, 1 and 2 in $i_B$,

$$A i_B^2 + B i_B + C = 0, \quad (15)$$

where

$$A = 2\pi^2 f_0 \beta_L^2 \sum_{k=0}^{N} k^2 \sin(\varphi_0-2\pi k\Psi_{ex}), \quad (16)$$

$$B = 1-2\pi f_0\beta_L\Big(\sum_{k=0}^{N} k\cos(\varphi_0-2\pi k\Psi_{ex}) + 4\pi f_0\beta_L\sum_{k=0}^{N}\sum_{m=0}^{k-1} k(k-m)\sin(\varphi_0-2\pi k\Psi_{ex}) \times \sin(\varphi_0-2\pi m\Psi_{ex}) - 4\pi f_0\beta_L\sum_{k=0}^{N}\sum_{m=0}^{k-1} m(k-m) \times \cos(\varphi_0-2\pi m\Psi_{ex})\cos(\varphi_0-2\pi k\Psi_{ex})\Big) \quad (17)$$

and

$$\frac{C}{f_0} = 8\pi^2 f_0^2\beta_L^2\sum_{k=0}^{N}\Big(\sum_{m=0}^{k-1}\sum_{m'=0}^{k-1}(k-m) \times (k-m')\sin(\varphi_0-2\pi k\Psi_{ex}) \times \sin(\varphi_0-2\pi m\Psi_{ex})\sin(\varphi_0-2\pi m'\Psi_{ex}) - 2\sum_{m=0}^{k-1}\sum_{m'=0}^{m-1}(k-m)(m-m')\cos(\varphi_0-2\pi k\Psi_{ex}) \times \sin(\varphi_0-2\pi m'\Psi_{ex})\cos(\varphi_0-2\pi m\Psi_{ex})\Big) + 4\pi f_0\beta_L\sum_{k=0}^{N}\sum_{m=0}^{k-1}(k-m)\sin(\varphi_0-2\pi m\Psi_{ex}) \times \cos(\varphi_0-2\pi k\Psi_{ex}) - \sum_{k=0}^{N}\sin(\varphi_0-2\pi k\Psi_{ex}). \quad (18)$$

From the above expressions for $A$, $B$, and $C$ we notice that, for vanishing $f_0$ values, $i_B$ vanishes. For non-zero values of $f_0$, on the other hand, we may solve Eq. (15) and look for the maximum of $i_B$ at a fixed value of the externally applied flux $\Psi_{ex}$ by letting $\varphi_0$ vary in the interval $[0, 2\pi]$.

## 4. Results and discussion

Having derived the equation for the normalized $i_B$ current (Eq. (15)) by a perturbative expansion up to second-order terms in the parameter $\beta_L$, we can now numerically study the magnetic field dependence of the maximum Josephson current of the array shown in Fig. 1. Naturally, this semi-analytic approach is valid only if the parameter $\beta_L$ is such as to allow the perturbative expansion. We shall take a rough estimate of the range of validity of our analysis by writing $\beta_L < 1/2\pi N^2$. We therefore notice that, while for small $N$ values this method can be used up to significant values of $\beta_L$, this is not anymore true for large $N$'s. We shall now discuss the main features of the numerical results obtained via this approach.

The maximum normalized Josephson current $i_c$ is derived as a function of the normalized external magnetic flux $\Psi_{ex}$ with the aid of Eqs. (15)-(18) by a rather simple numerical algorithm. This algorithm, as already specified in the previous section, is constructed

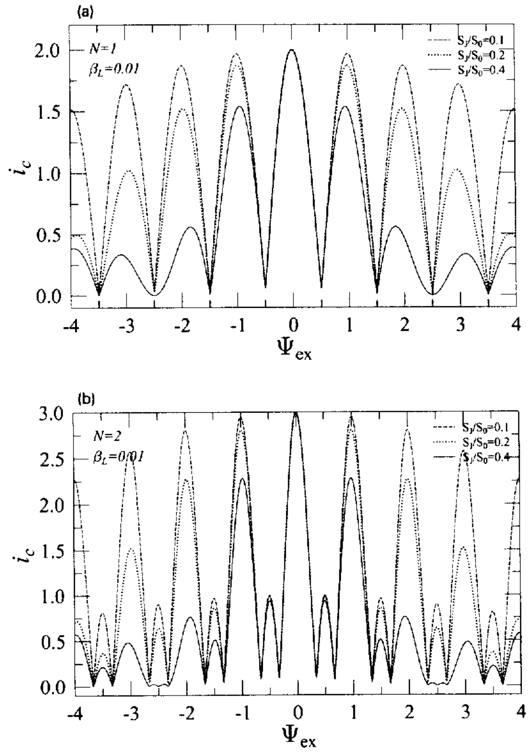


Fig. 3. Maximum Josephson current of the parallel array of Fig. 1 as a function of the normalized applied flux $\Psi_{ex}$, for $S_J/S_0 = 0.1, 0.2, 0.4$, $\beta_L = 0.01$ and for the following values for $N$: (a) $N = 1$; (b) $N = 2$.

in such a way as to allow the variable $\varphi_0$ to vary in the interval $[0, 2\pi]$ at a fixed step; before a new increment of this variable is made, the $i_c$ value is recorded and compared with the one obtained in the previous step. Only the greatest value of $i_c$ is retained at the end of the whole process for a fixed value of the normalized flux. In this way, we have derived the $i_c$ versus $\Psi_{ex}$ curves for different values of the SQUID parameter $\beta_L$ and of the surface ratio $S_J/S_0$ and for a different number of loops $N$.

First, let us consider the $\beta_L = 0$ case for $N = 9$. Resulting $i_c$ versus $\Psi_{ex}$ curves are shown, for values of the surface ratio $S_J/S_0 = 0.1$, $S_J/S_0 = 0.2$ and $S_J/S_0 = 0.4$, in Figs. 2a, 2b and 2c, respectively. The single junction Fraunhofer-like pattern determines the envelope under which the quantum interference of the junctions causes an interference pattern with a unitary pseudo-periodicity in $\Psi_{ex}$. The height of the maximum at $\Psi_{ex} = 0$, as is well known, is equal to the number of junctions in the array and the interference minima are

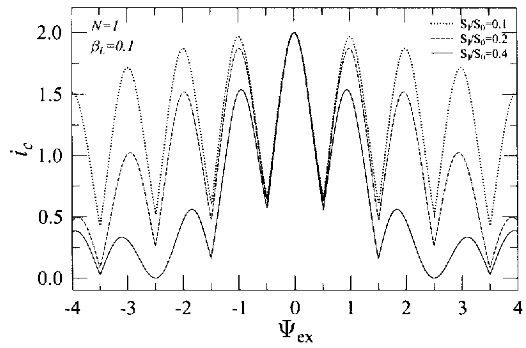


Fig. 4. Maximum Josephson current of the parallel array of Fig. 1 as a function of the normalized applied flux $\Psi_{ex}$, for $S_J/S_0 = 0.1, 0.2, 0.4$, $\beta_L = 0.1$ and for $N = 1$.

always zero. Moreover, a change of the surface ratio $S_J/S_0$ results in a change of the pseudo-periodicity of $|f_0|$ as a function of $\Psi_{ex}$ as it is seen from Figs. 2a–2c where the function $|f_0|$ is shown by means of a dotted line.

At finite $\beta_L$ values, on the other hand, higher lying minima are seen to appear in the $i_c$ curves. In Figs. 3a and 3b, for example, the maximum Josephson current versus the applied flux $\Psi_{ex}$ is reported for $N = 1$ and $N = 2$, respectively, when $\beta_L = 0.01$. The surface ratio values $S_J/S_0 = 0.1$, $S_J/S_0 = 0.2$ and $S_J/S_0 = 0.4$ were chosen, and the curves are shown in the $\Psi_{ex}$ range of $[-4, 4]$. Due to the single-junction envelopes, the $i_c$ maxima of the interference pattern are depressed as the $S_J/S_0$ ratio grows as in the $\beta_L = 0$ case, while the $i_c$ minima become larger than the null value attained for $\beta_L = 0$. The depression and the splitting of the secondary peak at $\Psi_{ex} = 2.5$ in Fig. 3b for $S_J/S_0 = 0.4$ is a consequence of a zero in the Fraunhofer-like envelope. Indeed, this external envelope has zeroes for $\Psi_{ex} = nS_0/S_J$ where $n$ is an integer. For $n = 1$ and $S_J/S_0 = 0.4$, the half-integer secondary peak of the interference pattern is therefore suppressed, and, as a consequence, is split into two parts.

For increasing $\beta_L$ values the rise of the minima becomes more and more evident for decreasing values of the junction loop area ratio, as it can be seen in Fig. 4, where we report the magnetic field dependence of the maximum Josephson current when $N = 1$, $\beta_L = 0.1$, and $S_J/S_0 = 0.1, 0.2, 0.4$. The minima are now significantly different from zero.

In Fig. 5 we compare the $i_c$ curves for different values of $\beta_L = 0, 0.05, 0.1$, when $N = 1$ and $S_J/S_0 = 0.1$.

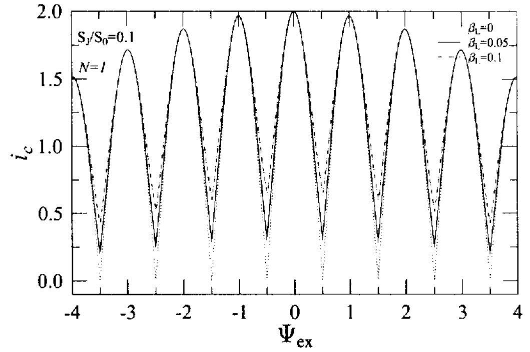


Fig. 5. Maximum Josephson current of the parallel array of Fig. 1 as a function of the normalized applied flux $\Psi_{ex}$, for $N = 1$, $\beta_L = 0, 0.05, 0.1$ and $S_J/S_0 = 0.1$.

In this figure the minima are higher in the $\beta_L = 0.1$ case than in the other two cases. Here it is worthwhile to remark that the rise of the minima is a consequence of the broadening of the major peaks due to the finite self-flux of the loop currents, which tend to break phase coherence.

We would like to point out that the numerical solution as given by solving the self-consistent problem (Eq. (13)) is in agreement with the numerical results obtained with the perturbative analysis carried out for the above cases. Of course, numerical solutions obtained by self-consistent algorithms require a much more computer time. The study of the magnetic response of these types of systems for values of the parameter $\beta_L$ which do not fulfill the requirement put forth by our analysis can be carried out by means of the dynamical equations for the phase variables (Eq. (3)). In this way, one can treat cases involving non-uniform initial field distributions in the array and have a better defined correspondence between numerical results and experiments. This approach will be pursued in future works.

Finally, despite the fact that our analysis can be applied in a rather small range of variation of the parameter $\beta_L$, the results obtained by this semi-analytic approach qualitatively agree with experimental data of Ref. [6].

## 5. Conclusions

In order to study the magnetic field dependence of the maximum Josephson current in a homogeneous parallel array of $N + 1$ Josephson junctions, we developed a perturbative analytic approach up to second order terms in the parameter $\beta_L$. For finite $\beta_L$ values, but still small enough to allow a perturbative analysis to the problem, we found a rise of the $i_c$ minima in the $I_c(H)$ pattern and a broadening of the major peaks as $\beta_L$ increases.


## Acknowledgement

We thank Professor S. Pace for many helpful discussions and comments.


## References


[1] K.K. Likharev, V.K. Semenov, O.V. Snigerev, B.N. Todorov, IEEE Trans. on Magn. MAG-15 (1979) 420.

[2] A. Barone, G. Paternó, Physics and Application of the Josephson Effect (Wiley, New York, 1982).

[3] C.D. Tesche, J. Clarke, J. Low Temp. Phys. 29 (1977) 310.

[4] J.H. Miller, Jr., G.H. Gunaratne, J. Huang, T.D. Golding, Appl. Phys. Lett. 59 (1991) 3330.

[5] W.T. Tsang, T. Van Duzer, J. Appl. Phys. 46 (1975) 4573.

[6] R. Gerdemann, L. Alff, A. Beck, O.M. Froehlich, B. Mayer, R. Gross, IEEE Trans. on Appl. Supercond. 5 (1995) 3292.