# Polar Actions on Symmetric Spaces

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## Polar representations

- $H$ compact connected Lie group acting on
  $V$ real vector space with $H$-invariant inner product
- $\pi : H \to O(V)$ representation
- $v \in V$, $\Sigma_v \subset V$ cross-section of action at $v$
- $\Sigma_v$ minimal $\iff \dim H \cdot v$ maximal

*Definition.* $\pi : H \to O(V)$ **polar** if all orbits intersect a minimal cross-section orthogonally

*Examples.*

- standard representation $\pi : SO_2 \to O(\mathbb{R}^2)$ is polar
- $M = G/K$ Riemannian symmetric space, $o \in M$ with
  $K \cdot o = o$, isotropy representation $\pi : K \to O(T_oM)$ is polar

**Dadok 1985**: Polar representations on $\mathbb{R}^n$ are orbit equivalent to isotropy representations of Riemannian symmetric spaces

## Polar actions

$M$ connected Riemannian manifold, $H \subset I(M)$ connected subgroup

*Definition.* The action of $H$ on $M$ is **polar** if there exists a connected closed submanifold $\Sigma$ of $M$ such that

- $\forall p \in M : \Sigma \cap H \cdot p \neq \emptyset$
- $\forall p \in \Sigma : T_p\Sigma \subset \nu_p(H \cdot p)$

Such a submanifold $\Sigma$ is called a **section** of the action.

*Fact.* Sections are *totally geodesic* submanifolds

*Definition.* A polar action is **hyperpolar** if it admits a flat section.

*Problem.* Classification of polar actions on Riemannian symmetric spaces

$S^n$ and $\mathbb{R}H^n$: apply Dadok's result

## Compact symmetric spaces

**Podestà, Thorbergsson 1999**: Classification of *polar* actions on projective spaces

**Kollross 2002**: Classification of *hyperpolar* actions on irreducible Riemannian symmetric spaces of compact type and rank $\geq 2$

Every *polar* action on an irreducible Riemannian symmetric spaces of compact type and rank $\geq 2$ is hyperpolar

- **Podestà-Thorbergsson 2002**: $SO_{n+2}/SO_nSO_2$, $n \geq 3$
- **Biliotti-Gori 2005**: $SU_{n+k}/S(U_nU_k)$, $n \geq k \geq 2$
- **Biliotti 2006**: Hermitian symmetric spaces
- **Kollross 2007**: Simple isometry group
- **Kollross 2009**: $G_2$, $F_4$, $E_6$, $E_7$, $E_8$
- **Lytchak 2011**: Cohomogeneity is $\geq 3$
- **Kollross-Lytchak 2011**: Cohomogeneity is 2

## Compact vs noncompact

Some observations:

- *Cohomogeneity one actions*: Every Riemannian symmetric space of noncompact type admits cohomogeneity one actions (not true for compact type)
- *Polar and hyperpolar actions*: Every Riemannian symmetric space of noncompact type admits polar actions which are not hyperpolar (not true for compact type and higher rank)
- Concept of *duality* between symmetric spaces of compact type and of noncompact type is useful only for special situations, e.g. actions by algebraic reductive subgroups (**Kollross 2011**)
- In the compact case one can restrict to actions of compact groups (well understood!), whereas in the noncompact case one needs to consider noncompact groups (not well understood!)

## Current state of affairs

| | regular foliation | singular foliation |
|---|---|---|
| cohom 1 | explicit classification | general construction |
| hyperpolar | explicit classification | ? |
| polar | ? | ? |
| | $\mathbb{C}H^n$: classification | $\mathbb{C}H^2$: classification |

Joint work with

- José Carlos Díaz-Ramos (Santiago de Compostela)
- Hiroshi Tamaru (Hiroshima)

## Polar foliations of complex hyperbolic spaces

- $\mathbb{C}H^n = SU_{n,1}/S(U_n U_1) = G/K$
- $\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ restricted root space decomposition
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ Iwasawa decomposition, $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$
- $\mathbb{C}H^n = AN$ solvable Lie group with left-invariant metric
- $V = \{0\}$ or $V = \mathfrak{a}$; $\mathfrak{w} \subset \mathfrak{g}_\alpha \cong \mathbb{C}^{n-1}$ real subspace
- $\mathfrak{s}_{V,\mathfrak{w}} = (\mathfrak{a} \ominus V) \oplus (\mathfrak{n} \ominus \mathfrak{w})$ subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$
- $S_{V,\mathfrak{w}}$ corresponding subgroup of $AN$

**Berndt-DiazRamos 2012**:

- The orbits of $S_{V,\mathfrak{w}}$ form a homogeneous polar foliation of $\mathbb{C}H^n$
- Every homogeneous polar foliation of $\mathbb{C}H^n$ is holomorphically congruent to one of these foliations

**Proof** relies on following result (**Gorodski 2004** for compact case):

Let $M = G/K$ be a Riemannian symmetric space of noncompact type and $H$ be a connected closed subgroup of $G$ whose orbits form a regular foliation $\mathcal{F}$ of $M$. Consider the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and define

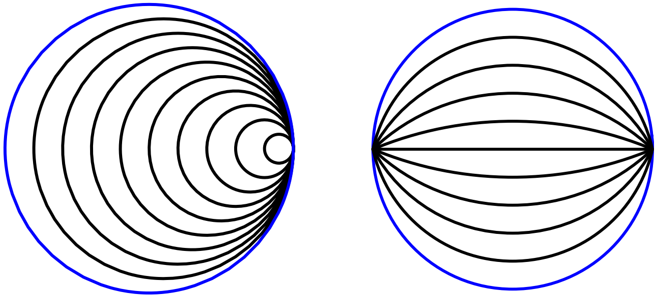
$$\mathfrak{h}_{\mathfrak{p}}^{\perp} = \{ \xi \in \mathfrak{p} : \langle \xi, Y \rangle = 0 \text{ for all } Y \in \mathfrak{h} \}.$$

Then the action of $H$ on $M$ is polar if and only if

- $\mathfrak{h}_{\mathfrak{p}}^{\perp}$ is a Lie triple system in $\mathfrak{p}$, and
- $\mathfrak{h}$ is orthogonal to the subalgebra $[\mathfrak{h}_{\mathfrak{p}}^{\perp}, \mathfrak{h}_{\mathfrak{p}}^{\perp}] \oplus \mathfrak{h}_{\mathfrak{p}}^{\perp}$ of $\mathfrak{g}$.

In this case, let $H_{\mathfrak{p}}^{\perp}$ be the connected subgroup of $G$ with Lie algebra $[\mathfrak{h}_{\mathfrak{p}}^{\perp}, \mathfrak{h}_{\mathfrak{p}}^{\perp}] \oplus \mathfrak{h}_{\mathfrak{p}}^{\perp}$. Then the orbit $\Sigma = H_{\mathfrak{p}}^{\perp} \cdot o$ is a section of the $H$-action on $M$.

## The case of codimension one



- horosphere foliation
- foliation with exactly one minimal leaf $S$ = ruled real hypersurface associated to a horocycle in a totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^n$

## Polar actions on $\mathbb{C}H^2$

- $N$ horosphere in $\mathbb{C}H^2$; $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$; $N$ is a 3-dim Heisenberg group
- $S$ ruled real hypersurface in $\mathbb{C}H^2$ generated by a horocycle in $\mathbb{R}H^2 \subset \mathbb{C}H^2$; $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{g}_\alpha^{\mathbb{R}} \oplus \mathfrak{g}_{2\alpha}$
- $N \cap S$ is a Euclidean plane $\mathbb{E}^2$ embedded in $N$ as a minimal surface and in $\mathbb{C}H^2$ as a real surface with nonzero constant mean curvature; $\mathfrak{n} \cap \mathfrak{s} = \mathfrak{g}_\alpha^{\mathbb{R}} \oplus \mathfrak{g}_{2\alpha}$

**Berndt-DiazRamos 2012**: Every polar action on $\mathbb{C}H^2$ is orbit equivalent to the action of the invariance group of one of the following geometric objects in $\mathbb{C}H^2$:

- Cohom 1: $\{o\}$, $\mathbb{C}H^1$, $\mathbb{R}H^2$, $N$, $S$
- Cohom 2: $\{o\} \subset \mathbb{C}H^1$ (full flag), $\mathbb{R}H^1$, horocycle in $\mathbb{C}H^1$, $\mathbb{E}^2$

## Outline of proof

- Possible cohomogeneity is 1 or 2
- Cohomogeneity 1: known by earlier work
- Assume cohomogeneity 2
- 0-dimensional orbit: group is compact and action has a fixed point, only possibility is $S(U_1 U_1 U_1)$
- 1-dimensional orbit, no fixed point: Lie-theoretical arguments, technical
- regular foliation: known by earlier work

## The general setting

- $M = G/K$ connected irreducible Riemannian symmetric space of noncompact type
  $G$ noncompact semisimple real Lie group
  $K$ maximal compact subgroup of $G$
  $o \in M$ with $K \cdot o = o$
- $H$ connected closed subgroup of $G$ acting on $M$ polarly

# Parabolic subalgebras (I)

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition
- $\mathfrak{a}$ maximal abelian subspace of $\mathfrak{p}$
- restricted root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left( \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha \right)$$

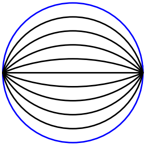
- $\Lambda$ set of simple roots for $\Sigma$
- $\Phi$ subset of $\Lambda$, $\Sigma_\Phi = \Sigma \cap \mathrm{span}\{\Phi\}$
- $\mathfrak{l}_\Phi = \mathfrak{g}_0 \oplus \left( \bigoplus_{\alpha \in \Sigma_\Phi} \mathfrak{g}_\alpha \right)$ , $\mathfrak{n}_\Phi = \bigoplus_{\alpha \in \Sigma^+ \setminus \Sigma^+_\Phi} \mathfrak{g}_\alpha$
  $\mathfrak{l}_\Phi$ reductive subalgebra, $\mathfrak{n}_\Phi$ nilpotent subalgebra
- $\mathfrak{q}_\Phi = \mathfrak{l}_\Phi \oplus \mathfrak{n}_\Phi$ **parabolic subalgebra** (**Chevalley decomposition**)
- Every parabolic subalgebra of $\mathfrak{g}$ is conjugate to $\mathfrak{q}_\Phi$ for some subset $\Phi \subset \Lambda$

## Parabolic subalgebras (II)

- $\mathfrak{l}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi$ with $\mathfrak{a}_\Phi$ split component of $\mathfrak{l}_\Phi$
  $\mathfrak{m}_\Phi$ reductive subalgebra, $\mathfrak{a}_\Phi$ abelian subalgebra
- $\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ (**Langlands decomposition**)
- $M_\Phi \cdot o = B_\Phi$ semisimple symmetric space with rank equal to $|\Phi|$, totally geodesic in $M$, **boundary component** of $M$ with respect to maximal Satake compactification
- $A_\Phi \cdot o = \mathbb{E}^{r-|\Phi|}$ Euclidean space, totally geodesic in $M$
- $L_\Phi \cdot o = F_\Phi = B_\Phi \times \mathbb{E}^{r-|\Phi|}$ totally geodesic in $M$
- $M = B_\Phi \times \mathbb{E}^{r-|\Phi|} \times N_\Phi$ (**horospherical decomposition**)
- The action of $N_\Phi$ on $M$ is polar
- The action of $N_\Phi$ on $M$ is hyperpolar $\iff \Phi = \emptyset$

## Examples of hyperpolar foliations

- $V$ linear subspace of $\mathbb{E}^m$
  $\Longrightarrow \mathcal{F}_V^m = \{p + V \mid p \in \mathbb{E}^m\}$ homogeneous hyperpolar foliation of $\mathbb{E}^m$
- $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, $M = G/K = \mathbb{F}H^n$
  $\mathfrak{s} = \mathfrak{a} \oplus (\mathfrak{g}_\alpha \ominus \ell) \oplus \mathfrak{g}_{2\alpha}$, $\ell$ line in $\mathfrak{g}_\alpha$
  $\Longrightarrow \mathcal{F}_{\mathbb{F}}^n$ homogeneous codimension one foliation of $\mathbb{F}H^n$ with unique minimal leaf



- $\mathcal{F}_{\mathbb{F}_1}^{n_1} \times \cdots \times \mathcal{F}_{\mathbb{F}_k}^{n_k} \times \mathcal{F}_V^m$ homogeneous hyperpolar foliation of $\mathbb{F}_1 H^{n_1} \times \cdot \times \mathbb{F}_k H^{n_k} \times \mathbb{E}^m$

## Examples of hyperpolar foliations (II)

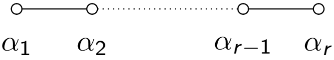
- $M = G/K$ symmetric space of noncompact type
- $\Phi$ orthogonal set of simple roots, $k = |\Phi|$
- $\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ Langlands decomposition of parabolic subalgebra $\mathfrak{q}_\Phi$ of $\mathfrak{g}$
- $F_\Phi \cong \underbrace{\mathbb{F}_1 H^{n_1} \times \cdots \times \mathbb{F}_k H^{n_k}}_{M_\Phi \cdot o} \times \underbrace{\mathbb{E}^{r-k}}_{A_\Phi \cdot o}$
- $\mathcal{F}^{n_1}_{\mathbb{F}_1} \times \cdots \times \mathcal{F}^{n_k}_{\mathbb{F}_k} \times \mathcal{F}^{r-k}_V$ homogeneous hyperpolar foliation of $F_\Phi$
- $\mathcal{F}_{\Phi,V} = \mathcal{F}^{n_1}_{\mathbb{F}_1} \times \cdots \times \mathcal{F}^{n_k}_{\mathbb{F}_k} \times \mathcal{F}^{r-k}_V \times N_\Phi$ homogeneous hyperpolar foliation of $M = F_\Phi \times N_\Phi$
- $\mathcal{F}_{\emptyset,\{0\}}$ horocycle foliation of $M$

## Classification of homogeneous hyperpolar foliations

**Berndt-DiazRamos-Tamaru 2010**: *Let $M$ be a symmetric space of noncompact type. Every homogeneous hyperpolar foliation on $M$ is isometrically congruent to $\mathcal{F}_{\Phi,V}$ for some orthogonal set $\Phi$ of simple roots and some linear subspace $V \subset \mathbb{E}^{r-|\Phi|}$.*

## The symmetric space $SL_{r+1}(\mathbb{R})/SO_{r+1}$

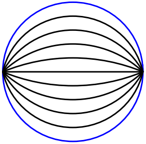
- Dynkin diagram




- $\Phi \subset \Lambda = \{\alpha_1, \ldots, \alpha_r\}$ orthogonal, $k = |\Phi|$
- horospherical decomposition:
  $SL_{r+1}(\mathbb{R})/SO_{r+1} \cong \underbrace{\mathbb{R}H^2 \times \ldots \times \mathbb{R}H^2}_{k \text{ factors}} \times \mathbb{E}^{r-k} \times N_\Phi$
- $N_\Phi$ corresponds to the set of all upper block diagonal matrices with certain $2 \times 2$ and $1 \times 1$ diagonal blocks, diagonal entries are 1

## The symmetric space $SL_{r+1}(\mathbb{R})/SO_{r+1}$

- horospherical decomposition: $SL_{r+1}(\mathbb{R})/SO_{r+1} \cong \underbrace{\mathbb{R}H^2 \times \ldots \times \mathbb{R}H^2}_{k \text{ factors}} \times \mathbb{E}^{r-k} \times N_\Phi$
- On each $\mathbb{R}H^2$ select the foliation



- On $\mathbb{E}^{r-k}$ select a foliation by parallel affine subspaces
- On $N_\Phi$ select the foliation with one leaf $N_\Phi$
- The product foliation is hyperpolar, and every homogeneous hyperpolar foliation of $SL_{r+1}(\mathbb{R})/SO_{r+1}$ arises in this way