CORRELATION FUNCTIONS IN INTEGRABLE QUANTUM FIELD THEORY

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ABSTRACT OF THE DISSERTATION

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The general aim of this work is to understand and develop methods for studying correlation functions in integrable quantum field theory, and to verify in particular cases expected general principles. Two models are considered: the $SU(2)$-Thirring model and the Ising model on a two-dimensional space of constant negative curvature. The former is a model describing the low-energy behavior of electrons in a Mott insulator. Two-point correlation functions of Fermi fields are studied at all distances using a form-factor expansion and conformal perturbation theory. The validity of both methods is verified by observing an agreement at intermediate distance scales. The latter model is a paradigm for understanding the effects of space curvature on critical behaviors. The magnetization and the two-point correlation functions of spin fields are evaluated exactly, generalizing methods valid on flat space. From these results, the thermodynamical properties of the model are analyzed. Technically, the results provide solutions to the connection problem of particular Painlevé VI equations.

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Dedication

À mes parents.

Table of Contents

General introduction

One of the most widely used tools of modern theoretical physics is relativistic quantum field theory. Its power comes from the principle of universality: a given model of quantum field theory describes the large distance physics of many systems with very different microscopic descriptions. Its popularity is also due to its wide applicability: quantum field theory can be used to describe fundamental particles, electronic excitations in materials or thermal fluctuations in statistical systems with many degrees of freedom. The different concepts appearing naturally from each application give new insight into quantum field theory.

However, it is often hard to obtain numerically accurate results from a model of quantum field theory, and sometimes even to deduce the physically relevant low-energy degrees of freedom. Moreover, as mathematical constructions, most models of relativistic quantum field theory are in fact not well defined. Notable exceptions for these two difficulties are models of two-dimensional quantum field theory with conformal invariance, and models of free relativistic particles. Often only from these exceptions, from perturbative calculations and from physically sound arguments, the properties of more complicated models are analyzed. In fact, the general structure of quantum field theory is essentially inferred from this: the space of states, properties of local fields or the analytical structure of correlation functions.

But a special class of models, two-dimensional integrable models, offers examples in which it is possible to evaluate exactly some quantities, for instance the scattering matrix, matrix elements of local fields, and in some cases the correlation functions. The main characteristic of models in this class is that they admit an infinite number of conservation laws preventing particle production in scattering processes. Many integrable models have direct applications in condensed matter physics and in statistical mechanics. For instance, two such models which will be studied in this work are the $SU(2)$-Thirring model, which is a model for the low-energy properties of one-dimensional Mott insulators, and the Ising model, a paradigm for a class of critical behaviors of statistical systems. Some more recent results of integrable quantum field theory are also of interest in string theory. But a deeper reason to study integrable models is to verify expected properties of quantum field theory, to uncover new structures and to develop an intuition for more complicated and more realistic (generally non-integrable) interacting models.

i Correlation functions

Roughly speaking, a model of quantum field theory is described by a set of states forming a Hilbert space; a set of operator valued distributions acting on the Hilbert space; and a Hamiltonian, a particular operator generating time translations, whose spectrum is bounded from below by the vacuum state. In general, one is given a microscopic description of a model: a Hamiltonian, for instance, expressed in terms of local, microscopic degrees of freedom. The problem is to find useful information from this dynamics: the set of particles and their masses (or the fundamental excitations, eigenstates of the Hamiltonian), the scattering amplitudes, and the correlation functions of local fields.

The objects in a model of quantum field theory that encode most directly all physical information are the correlation functions of local fields. In principle, all quantities of interest in a model can be obtained, in more or less simple ways, from the knowledge of correlation functions. The main problem in the study of a model of quantum field theory can then be seen as the reconstruction of its correlation functions.

Of course, for some purposes it is not necessary to know about the correlation functions of local fields. Many useful physical quantities can be calculated directly from the knowledge of the fundamental excitations. For instance, the free energy is related to the set of eigenvalues of the Hamiltonian, and the thermodynamics of the model can be deduced from it.

However, physical quantities that are related to the *dynamics* of the physical constituents of the model or to the *response* of the model to external perturbations require more direct knowledge about the local fields and their correlation functions. For instance, from the fermion two-point function in the $SU(2)$-Thirring model, one finds the spectral density, which tells about the probability for an electron to create a fundamental excitation, or the probability for an excited state to emit an electron. The spectral density can be measured experimentally, and in fact, of interest to the present work, is an important part of characterizing a material as a one-dimensional metal or insulator. In the statistical mechanics interpretation, correlation functions are related to linear responses to applied local external fields (and to susceptibilities if the field is extended on the whole sample). The measurement of linear responses is one of the main ways of studying a material.

Hence the study of correlation functions is of great interest. Since integrability gives stringent constraints, it is worth developing its consequences on correlation functions.

ii On-shell structure of massive integrable models

In integrable models of two-dimensional quantum field theory, the full "on-shell" description is often available (albeit sometimes conjecturally): the spectrum of particles and the scattering matrix (for reviews, see [100, 35]). In general, such an on-shell description should completely define a model (integrable or not), and in principle, all local fields can be identified and their correlation functions calculated. However, it is very nontrivial to obtain correlation functions of local fields from the on-shell description. In integrable models, there is a way of obtaining matrix elements of local fields in the basis of asymptotic states, or "form factors" [123, 72, 118]. From these matrix elements, one can obtain the full long-distance expansion of correlation functions. There is then a programme of reconstructing correlation functions from their long-distance expansion given by the form factors. It turns out that this programme often leads to a numerically accurate description of correlation functions up to quite short distances [133, 144, 1, 8]. But it is in general not possible to re-sum this long distance expansion in order to have close expressions, and it is not clear, from general principles, how efficiently an approximation by a finite sum describes correlation functions at short distance scales. In the rest of this section, I will explain the main results concerning the on-shell structure of massive integrable models.

Asymptotic states and scattering matrix

In massive quantum field theory on a space of infinite volume, there is a convenient basis for the Hilbert space: the set of *asymptotic states* (see, for instance, [127]). An intuitive picture of asymptotic states can be obtained as follows. Assume that at any finite time from the time of the "experimentation," during which particles interact, the particles involved are separated by finite distances. Hence in the infinite past or in the infinite future, they are whether infinitely separated, or some may stay at finite distances forever, forming bound states. Bound states can in turn be seen as particles of their own, so that in the infinite past or in the infinite future, one can consider that the particles are all infinitely separated; since interactions are local, they propagate freely. By definition, an asymptotic state is such that in the infinite past (*in*-states), or in the infinite future (*out*-states), one finds a state with a well-defined set of particles infinitely separated, propagating freely at fixed rapidities[1]. This is an eigenstate of the Hamiltonian, since it has a well-defined energy. Since at finite time all particles should come close to each other, in the infinite past they are ordered in space from left to right by decreasing values of their rapidity, and in the infinite future they are in opposite order; this is what characterizes *in*- and *out*-states. Note that if one finds a state with a definite number of particles at definite rapidities in the infinite past, then in the infinite future one generically finds a superposition of such states.

Of course, if particles have definite momenta, they cannot have definite positions; this intuitive picture must be made more precise. This can be done by constructing space-ordered wave packets around definite momenta, and by taking the limit of infinite wave-packet extent and infinite separation between the central positions of the wave packets. For every particle of mass M, there is a (not necessarily unique) local fields $\Psi(x,t)$ that creates this particle. This field has the quantum numbers of the particle and the Fourier transform of its two-point correlation function $\langle\Psi(x,t)\Psi(0,0)\rangle \equiv \langle\text{vac}|\mathcal{T}\left(\Psi(x,t)\Psi(0,0)\right)|\text{vac}\rangle$ has a pole, as function of the squared two-momentum p^2, at the position of the square of the mass of the particle M^2, and no other poles at lower values of p^2 (in super-renormalizable models, it can be uniquely determined by requiring in addition that it has the lowest scaling dimension). This means that correlation functions involving this field satisfy the Klein-Gordon equation (or the Dirac equation, or the appropriate equation for particles transforming under higher spin representations of the Lorentz group) asymptotically at large space-like distance:

$$(M^2 - \Box)\langle\Psi(x,t)\cdots\rangle = O\left(e^{-M'\sqrt{x^2-t^2}}\right) \quad \text{as} \quad x^2 - t^2 \to \infty$$

where M' is the lowest mass, greater than M, of a state created by this field, and

[1]Because this is the case of interest in this dissertation, I consider models in one space dimension.

$\cdots$ represents fields at other fixed positions. Consider the operators

$$A(\theta)^{(in,out)} = \lim_{L\to\infty} \lim_{t\to\mp\infty} \int dx \, (f_\theta(x,t)\partial_t\Psi(x,t) - \partial_t f_\theta(x,t)\Psi(x,t)) \qquad \text{(ii.1)}$$

with

$$f_\theta(x,t) = \exp\left[iM\cosh(\theta)\, t - iM\sinh(\theta)\, x - \frac{(x-\tanh(\theta)t)^2}{L^2} \right] \ ,$$

as well as their hermitian conjugate $A^\dagger(\theta)^{(in,out)}$. The operators $A(\theta)^{(in)}$, $A^\dagger(\theta)^{(in)}$ satisfy canonical (anti-)commutation relations, for instance $[A(\theta)^{(in)}, A^\dagger(\theta')^{(in)}] = 4\pi\delta(\theta-\theta')$, and similarly for the operators $A(\theta)^{(out)}$, $A^\dagger(\theta)^{(out)}$. They are also eigenoperators of the Hamiltonian, $[H, A^\dagger(\theta)^{(in,out)}] = M\cosh(\theta)A^\dagger(\theta)^{(in,out)}$. Similar operators can be defined for all particles of the theory, and operators corresponding to different particle types commute with each other. The Hilbert space is the Fock space over the algebra of all such *in*-operators, which is isomorphic to the Fock space over the algebra of all *out*-operators.

Hence, there are two bases of eigenstates of the Hamiltonian $\mathbf{H}$ parametrized by a set of rapidities θ_i, with eigenvalues given by the sum of the associated energies:

$$\mathbf{H}|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle_{in,out} = \sum_{k=1}^n M_{a_k}\cosh(\theta_k)\,|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle_{in,out}$$

where A_a represents a particle, and the index a labels the various types of particles, characterized by their masses M_a and by possible other quantum numbers. The overlaps between *in*-states and *out*-states form the scattering matrix, or S-matrix.

In fact, the physical scattering matrix also depends on the impact parameters of the scattering. In our construction of the asymptotic states, we have specified the impact parameters of the scattering by making all particles "collide at one point" under an extrapolation of their free trajectories. We could have defined the operators $A(\theta)^{(in,out)}$ by shifting slightly the central positions of the wave packets in order to have different impact parameters; some particles could "collide first" in an extrapolation of the free trajectories. Any choice leads to a different but legitimate basis of the Hilbert space, and the overlap between the associated *in*-states and *out*-states form the scattering matrix for physical scattering with different impact parameters.

In addition to the energy operator, the momentum operator is also diagonal in the basis of asymptotic states. In integrable quantum field theory, energy and momentum conservation laws are part of an infinite family of local conservation laws in involution. These conservation laws can be described as "deformation of free laws" [83], in the sense that when applied to states where the particles are very far apart from each other at the infinite past or future, they are conservation laws of the free propagation of these particles. For free propagation, there is a set of conservation laws such that the action on a state can be taken as the sum of terms formed by taking, for each particle, any integral power of its momentum (the same for all particles) times any integral power of its energy. It is conservation laws in this set that are deformed in interacting integrable models to form an infinite commuting family. It is convenient to organize conservation laws in this set by representing them by eigenoperators of the boost generator. A conservation law of spin s acts as follows on asymptotic *in*- and *out*-states:

$$Q_s|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle_{in,out} = \sum_{k=1}^{n} M_{a_k}^{(s)} \exp(\theta_k\, s)|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle_{in,out}\ . \tag{ii.2}$$

In fact, this form of the eigenvalues is a consequence of relativistic covariance and of the locality of the conserved charges. The infinite set of spins s for which there are local conserved charges and the one-particle eigenvalues $M_a^{(s)}$ are important characteristics of an integrable model. Along with the fact that $dQ_s/dt = 0$, the equations above imply that the number of particles, in a given scattering process, sharing the same set of one-particle eigenvalues $\{M_a^{(s)}\}$ remains unchanged after scattering, and that the final set of rapidities of these particles is the same as the initial one. That is, the scattering is purely elastic [5, 46, 107, 83].

It turns out that these properties of scattering processes along with the existence of an infinite number of local conservation laws constrain the scattering matrix enough to make it "factorizable" [83, 135, 139, 115]. That is, multi-particle S-matrix elements can be expressed in terms of two-particle S-matrix elements, and the scattering matrix does not depend on impact parameters. This is the one-dimensional counterpart of the Coleman-Mandula theorem [28], according to which the existence of at least one additional (Lie-algebraic) local conserved charge of spin higher than one in a model in more than one space dimension implies that the scattering matrix must be trivial.

The features of the factorization in one dimension can be made plausible by the following argument [115]. First, consider acting with an exponentiation of a conserved quantity of higher spin on an operator $\tilde{A}(\theta)^{(in)}$ defined as in (ii.1) *without* taking the limit $t \to -\infty$ (but at large negative time t). After taking the limit $t \to -\infty$ on the resulting operator, this results in a wave packet with a slightly shifted central position, the shifting *depending on the rapidity* θ. It is possible, by this procedure, to make a multi-particle state where each wave packet has a central position independently shifted. That is, the conserved charges of higher spins generate in this way changes in the impact parameters. Of course, the action of a conserved quantity on a state at time t is independent of the time, so that these shifted multi-particle states are just the same asymptotic states as defined above: asymptotic states corresponding to different impact parameters are proportional to each other. If the central positions of wave packets are shifted enough by this process, as time evolves there will be a well-defined first subprocess occurring, in which only two wave packets enter in collision while the others propagate freely. Since no particle is created or destroyed in two-particle scattering processes and (here only by energy and momentum conservation) the set of rapidities is preserved, there is still a well-defined rapidity configuration after the collision. The argument can be repeated until all wave packets have inverted their order. Since in one space dimension particles propagate on a line, the full scattering process will then be a successions of all necessary two-particle scattering sub-processes separated by free propagations. This immediately leads to the factorization property of the S-matrix[2].

If the *in*-state has particles with rapidities $\theta_1, \ldots, \theta_n$ with $\theta_1 > \cdots > \theta_n$, the two-particle scattering processes will involve couples θ_j, θ_k for all $j < k$. Taking the shifting of central positions of wave packets in different ways, the two-particle scattering processes will occur in different orders. Any order may occur, and for any order one should obtain the same multi-particle scattering. Hence the two-particle S-matrix must satisfy some relations, and it turns out that it is sufficient to ask for consistency of the three-particle scattering. The resulting cubic relations are the so-called Yang-Baxter equations (with implied summation over repeated indices):

$$S_{a_1,a_2}^{b_1,b_2}(\theta_1-\theta_2)S_{b_1,a_3}^{c_1,b_3}(\theta_1-\theta_3)S_{b_2,b_3}^{c_2,c_3}(\theta_2-\theta_3) = S_{a_2,a_3}^{b_2,b_3}(\theta_2-\theta_3)S_{a_1,b_3}^{b_1,c_3}(\theta_1-\theta_3)S_{b_1,b_2}^{c_1,c_2}(\theta_1-\theta_2)\,.$$

The two-particle S-matrix is defined by

$$|A_{a_1}(\theta_1)A_{a_2}(\theta_2)\rangle_{in} = S_{a_1,a_2}^{b_1,b_2}(\theta_1-\theta_2)|A_{b_1}(\theta_1)A_{b_2}(\theta_2)\rangle_{out}$$

for $\theta_1 > \theta_2$.

In fact, it is possible to show that the S-matrix is purely elastic and that it factorizes into two-particle S-matrices solely from the presence of *two* non-trivial conserved charges of spin higher than one[3] [105].

The factorization properties of the S-matrix can be implemented by identifying the *in* and *out* bases for the Hilbert space with two bases for the universal

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[2]Note that in higher dimensions, independence of the scattering matrix from the impact parameters implies that the scattering must be trivial – this is the essence of the Coleman-Mandula theorem.

[3]The argument involves using these conserved charges for shifting the central positions of wave packets, as above, as well as a "macro-causality" principle.

enveloping algebra of Zamolodchikov's algebra [139], generated by elements $A_a(\theta)$ with exchange relations

$$A_{a_1}(\theta_1)A_{a_2}(\theta_2) = S_{a_1,a_2}^{b_1,b_2}(\theta_1-\theta_2)A_{b_2}(\theta_2)A_{b_1}(\theta_1)\ .$$

One identifies the *in*-basis as the basis formed by products of elements $A_a(\theta)$ with rapidities in decreasing order from left to right, and the *out*-basis as the basis formed by similar products with rapidities in increasing order. Associativity of this algebra gives the Yang-Baxter equations[4].

In addition to the Yang-Baxter equations, the two-particle S-matrix must also satisfy some fundamental conditions (here we consider models invariant under charge conjugation) [43, 139]:

Real analyticity:	$(S_{a_1,a_2}^{b_1,b_2}(\theta))^* = S_{a_1,a_2}^{b_1,b_2}(-\theta)$;
Unitarity:	$S_{a_1,a_2}^{b_1,b_2}(\theta)S_{b_1,b_2}^{c_1,c_2}(-\theta) = \delta_{a_1}^{c_1}\,\delta_{a_2}^{c_2}$;
Crossing symmetry:	$S_{a_1,a_2}^{b_1,b_2}(i\pi-\theta) = C_{a_1,c}S_{d,a_2}^{c,b_2}(\theta)C^{d,b_1}$,
	C is the charge conjugation matrix, with $C_{a,b}C^{b,c} = \delta_a^c$;
Analytic structure:	the function $S_{a_1,a_2}^{b_1,b_2}(\theta)$ is analytic in the strip $0 < \Im m\,\beta < \pi$ except for poles at positions $\theta = i\ \arccos\left(\frac{M^2-M_1^2-M_2^2}{2M_1M_2}\right)$ corresponding to particles of mass M formed by bound states of particles of masses M_1 and M_2. The residues at these poles are related to the probability amplitude for producing the bound states.

Real analyticity, crossing symmetry and the analytic structure are consequences of general principles of quantum field theory, whereas unitarity as written above uses also the absence of particle production, since only the two-particle S-matrix

[4]There are actual operators $A_a(\theta)$ satisfying these algebra relations, and their hermitian conjugate $A_a^\dagger(\theta)$ satisfying conjugate relations, that act on the Hilbert space. With appropriate exchange relations involving products of $A_{a_1}(\theta_1)$ with $A_{a_2}^\dagger(\theta_2)$ [47], they are creation and annihilation operators for asymptotic states. The algebra generated by the operators $A_a(\theta)$ and their hermitian conjugates is called the Zamolodchikov-Faddeev algebra. The Hilbert space can be seen as the Fock space over this algebra.

is involved. Note that the unitarity equation above is in fact consequence of the usual unitarity along with parity-time reversal (PT) symmetry (which is present since we consider relativistic models (CPT-invariant) with charge conjugation (C) symmetry) and real analyticity.

The presence of bound states identified as asymptotic particles already in the spectrum of the theory leads, by arguments similar to those leading to factorized scattering, to a set of additional constraints on the analytic properties of the S-matrix, on the masses of the particles, on the infinite set of spins s of the conserved charges Q_s defined in (ii.2) and their eigenvalues (some are considered in the early works [114, 70, 71], a more extensive discussion can be found in [134], and the review [100] gives a nice treatment). These constraints are called "bootstrap equations."

The programme of reconstructing the two-particle S-matrix (hence all S-matrices by factorization) as well as the set of particles from the properties of the S-matrix above, from internal symmetries of a model and from bootstrap equations has now been carried out in several models (see for instance [101, 25, 136] and the reviews cited above). Note however that identifying a given S-matrix, solution to these conditions, with that of a theory with a known local dynamical description (for instance, with a known local Hamiltonian) is usually not a trivial task. One must often rely on conjectures and on approximative methods to make such an identification.

The spectrum of particles can sometimes be obtained from semiclassical calculations, which turn out to be exact in integrable models [32, 81, 93, 94], and the spectrum and S-matrix can also sometimes be explicitly calculated from exact Bethe ansatz solutions [9, 3, 78, 4] or from the related quantum inverse scattering method [117] (see also the book [79]).

Local fields and form factors

In principle, the knowledge of the full on-shell structure of a model should be enough to determine all local fields and their correlation functions. The relation between the on-shell structure and local fields, however, is usually very nontrivial. In integrable models, there is a way to explicitly exhibit this relation. Consider particular matrix elements, in the asymptotic state basis, of a local operator $\mathcal{O}(x)$ at the point $x = 0$ (say):

$$\langle \mathrm{vac}|\mathcal{O}(0)|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle_{in} \tag{ii.3}$$

These matrix elements are the form factors of the local operator $\mathcal{O}$. Using crossing symmetry and the scattering matrix, it is possible to express all matrix elements of $\mathcal{O}(0)$ between asymptotic *out*- and *in*-states in terms of the form factors, hence to completely describe the local operator once its form factors are known.

In many integrable models, the data of the factorized scattering theory is sufficient to reconstruct form factors. Consider a tensor-valued function

$$F_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n)$$

which equals the form factor (ii.3) when all θ_i's are real and ordered as $\theta_1 > \cdots > \theta_n$. This function can be analytically continued to complex values of θ_i's. The reconstruction of the form factors is based on constructing the function $F_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n)$ from its expected properties on the complex θ_i-planes, properties that are essentially consequences of factorized scattering.

In [72] (see also [137]), direct consequences of factorized scattering were considered, in addition to consequences of general principle of quantum field theory, in order to express a set of axioms on form factors and to find some exact form factors in the sine-Gordon model solely from the knowledge of the two-particle scattering matrix. However, the method did not allow to calculate form factors with more than three particles.

In classical integrable field theory, the Gelfand-Levitan-Marchenko (GLM) equations allow to express local classical fields in terms of the action-angle variables (the "scattering data" of the inverse scattering method), generalizing the inverse Fourier transform expression of free models. The quantum version of GLM equations was found in Ref. [31], giving the local fields in terms of creation and annihilation operators for asymptotic particles. Breather and soliton form factors were calculated in the sine-Gordon model [119, 120] using this method. Based on an analysis of the quantum GLM equations, it was shown in Ref. [76, 77], in different models, that an operator $\mathcal{O}(x)$ is local provided that its (analytically continued) form factors satisfy a set of axioms. Currently the most powerful programme, for general integrable models, is then to reconstruct form factors from properties that ensure that the associated field is local [118]; this does not involve explicitly quantum GLM equations.

These properties of form factors ensuring locality are expressed in the following axioms:

1. The function $F_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n)$ is analytic in the variables $\theta_i-\theta_j$ inside the strip $0<\Im m\,(\theta_i-\theta_j)<2\pi$ except for simple poles.

2. From relativistic invariance, the function F satisfies the relation

$$F_{a_1,\ldots,a_n}(\theta_1+\theta,\ldots,\theta_n+\theta)=e^{s\theta}F_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n)$$

   where s is the spin of the operator $\mathcal{O}$.

3. The function F satisfies the symmetry property (generalized Watson's theorem)

$$F_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n)= \\ S_{a_k,a_{k+1}}^{b_k,b_{k+1}}(\theta_k-\theta_{k+1})\,F_{a_1,\ldots,b_{k+1},b_k,\ldots,a_n}(\theta_1,\ldots,\theta_{k+1},\theta_k,\ldots,\theta_n)\;.$$

4. The function F satisfies the locality property

$$F_{a_1,\ldots,a_{n-1},a_n}(\theta_1,\ldots,\theta_{n-1},\theta_n+2\pi i)=e^{2\pi i\omega(\mathcal{O},\Psi)}F_{a_n,a_1,\ldots,a_{n-1}}(\theta_n,\theta_1,\ldots,\theta_{n-1})$$

where $\omega(\mathcal{O},\Psi)$ is the mutual locality index for the operator $\mathcal{O}$ and the elementary field Ψ associated to the particle labelled by a_n. The mutual locality index between two fields is the phase gained in correlation functions when one of the fields is brought around the other field counterclockwise back to its initial position.

5. As a function of θ_n, the function F has simple poles (kinematic poles) at the points $\theta_n = \theta_j + i\pi,\ j = 1,\ldots,n-1$, with the following residues:

$$
\begin{aligned}
&i\,F_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n) \sim \\
&\Big[\delta_{a_1}^{b_1}\cdots\delta_{a_{j-1}}^{b_{j-1}} S_{a_{j+1},a_j}^{b_{j+1},c_j}(\theta_{j+1}-\theta_j) S_{a_{j+2},c_j}^{b_{j+2},c_{j+1}}(\theta_{j+2}-\theta_j)\cdots S_{a_{n-1},c_{n-3}}^{b_{n-1},b_j}(\theta_{n-1}-\theta_j) - \\
&e^{2\pi i\omega(\mathcal{O},\Psi)}\delta_{a_{n-1}}^{b_{n-1}}\cdots\delta_{a_{j+1}}^{b_{j+1}} S_{a_j,a_{j-1}}^{c_j,b_{j-1}}(\theta_j-\theta_{j-1}) S_{c_j,a_{j-2}}^{c_{j-1},b_{j-2}}(\theta_j-\theta_{j-2})\cdots S_{c_3,a_1}^{b_j,b_1}(\theta_j-\theta_1)\Big] \\
&\times\, C_{a_n,b_j}\ \frac{F_{b_1,\ldots,\hat{b}_j,\ldots,b_{n-1}}(\theta_1,\ldots,\hat{\theta}_j,\ldots,\theta_{n-1})}{\theta_n-\theta_j-i\pi}
\end{aligned}
\qquad \text{(ii.4)}
$$

where hats mean omissions. There are other poles corresponding to bound states of mass M at the positions $\theta_n - \theta_j = i\ \arccos\left(\frac{M^2-M_n^2-M_j^2}{2M_nM_j}\right) \in i\,[0,\pi]$.

Axioms 4 and 5 are the main additions compared to the early attempt of Ref. [72]. The locality property, axiom 4, is the most nontrivial axiom coming from the analysis based on quantum GLM equations, and has not been shown from first principles of quantum field theory. The kinematic pole condition, axiom 5, is the axiom allowing the reconstruction of all form factors by "bootstrap" methods. Note that the form factors may have additional poles corresponding to bound states created by the operator considered. Bootstrap equations relate the residues at such poles with other form factors. It is believed that the space of local fields of an integrable model can be identified with the space of solutions to the form factor axioms along with bootstrap equations [26, 82, 89]

Reconstructing form factors from these axioms allow one to write long-distance expansions for correlation functions, by inserting between operators the resolution of the identity on the *in*-basis (say). For instance, the two-point function of fields at a space-like Minowsky interval $r=\sqrt{x^2-t^2}$ is given by

$$\langle\mathcal{O}(x,t)\mathcal{O}(0)\rangle = \sum_{n=0}^{\infty}\sum_{a_1,\ldots,a_n}\int\frac{d\theta_1\cdots d\theta_n}{n!\,(2\pi)^n}\,e^{-r\sum_j M_j\cosh(\theta_j)}\times$$
$$\times\ \langle\text{vac}|\mathcal{O}(0)|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle_{in}\ {}_{in}\langle A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)|\mathcal{O}(0)|\text{vac}\rangle\ .$$

All integrals are nonsingular and convergent. The series is expected to be convergent as well.

The axioms for form factors were solved in various works for different operators in different models (see, for instance, the review [100]). In particular, a scheme of solution was proposed in [89, 18] and used in [90, 92] in order to find integral expressions for form factors of exponential fields and of topologically charged fields in the sine-Gordon model (whose Euclidean action is given in (2.3.4)), as well as the related form factors of the Fermi fields in the $SU(2)$-Thirring model (2.2.8). In the first part of this dissertation, they are evaluated in order to describe the long-distance behaviors of the two-point function of the Fermi fields in the $SU(2)$-Thirring model.

In the second part of this dissertation, the concept of form factor is generalized in a natural way to the situation of a model on a maximally symmetric infinite-volume curved space. Form factors of interacting fields in a free fermion theory on such a space are evaluated using a property similar to the locality property in point 4 above, and using relations between analyticity properties and the spectrum of the theory in a fashion similar to that in the axioms above.

iii Conformal field theory and conformal perturbation theory

By analogy to the on-shell description, one can completely specify a model of quantum field theory by a "local" description. That is, one gives a set of local fields $\mathcal{O}_a(x)$ with their vacuum expectation values $\langle\mathcal{O}_a(x)\rangle$ and a (consistent) operator product algebra, where a product of two fields is written as an infinite sum of local fields of increasing dimension [128, 108]:

$$\mathcal{O}_a(x,t)\mathcal{O}_b(0) = \sum_c C_{a,b}^c(x,t)\mathcal{O}_c(0) \ .$$

The space-time dependent coefficients $C_{a,b}^c(x,t)$ are called the structure functions. Since the dimension of fields is bounded from below, the sum is singly-infinite. By dimensional analysis and the assumption of scale invariance at short distances, this provides a short-distance expansion for two-point correlation functions.

In some two-dimensional models with conformal invariance, the space of local fields and their operator product expansions can be specified completely. Conformally invariant models are most adequately studied using the local description, where conformal invariance dictates a wide and stringent structure. The consequences of conformal invariance on quantum field theory models in any number of dimensions were developed in [109]; in two dimensions, see for instance the seminal paper [10], the collection of articles [65] and the textbook [34].

Conformal symmetry in two dimensions is an infinite-dimensional symmetry corresponding to invariance under local scale transformations and generating the Virasoro algebra [124, 57] (the relation between conformal invariance and the Virasoro algebra was considered in [49] in the context of the Thirring model, and is explained most clearly and in full generality in [10]). It is always present whenever there is global scale, rotation and translation invariance. The space of states on closed lines is then a lowest-weight module for the Virasoro algebra, and for other symmetry algebras if other symmetries are present. The requirement of unitarity (when desired) [55, 56] and the requirement of modular invariance [23], essentially invariance under the choice of a time direction, fix the possible modules, so that the space of states can be determined. The space of states on closed lines is isomorphic to the space of local fields, hence local fields are completely determined.

Local fields split into left-moving and right-moving factors: their correlation functions are, in general, finite sums of products of holomorphic and anti-holomorphic functions, the "conformal blocks." Conformal blocks can be seen as correlation functions of left-moving and of right-moving "semi-local" fields, that is, fields for which correlation functions have nontrivial monodromy properties. In so-called minimal models [10], which correspond to particular choices of the central charge of the Virasoro algebra and of the lowest weights, conformal blocks satisfy linear differential equations coming from the reducibility of Verma modules for the Virasoro algebra. From these differential equations, conformal blocks can be fixed exactly; in particular, the operator product expansions are determined.

The application of conformal field theory to the study of two-dimensional statistical models (see, for instance, [21]) gives the exact critical behaviors of many universality classes of second order phase transitions, since the main characteristics of such transitions is the disappearance of a length scale. Conformal field theory also has deep connections with the mathematical study of infinite-dimensional Lie algebras and their representations. In this context, an axiomatic formulation that embodies the main characteristics of conformal field theory, paralleling the axiomatic formulation of quantum field theory, is fruitfully used: the theory of vertex operator algebras. I present an introduction to this theory in Appendix C, along with a short description of results I obtained with my collaborators in this context.

An application of conformal field theory that will play an important role in this work is in the study of the short distance behavior of correlation functions in massive models, or off-critical models. Since at short distances the mass scale does not play an important role, the short distance asymptotic behavior of correlation functions is described by conformal field theory. A massive model can then be viewed as a conformal model perturbed by a relevant operator. Then, the structure functions in models with a scale can be evaluated perturbatively in their dimensionful perturbation parameters from structure functions in conformally invariant models. By dimensional analysis, this description is a short distance expansion for the structure functions.

This method, called conformal perturbation theory, is a powerful alternative to the usual perturbation theory about a free model. In addition, it is very rewarding for the construction and classification of integrable models to describe them by perturbed conformal field theory [134]. The structure functions are in fact expected to be entire functions of the dimensionful parameters of the model, and all non-analyticity in these parameters is encoded into the one-point functions of local operators [116]. Note that it is in fact a very nontrivial task to evaluate one-point functions exactly in interacting models, and even in integrable models there are no known systematic ways of doing this[5]. In the rest of this section, I will describe the main ideas for the perturbative evaluation of structure functions as exposed in Ref. [144] (see also [60])

Conformal perturbation theory for structure functions

For illustration, consider a super-renormalizable theory defined as a conformal field theory with formal (Euclidean) action $\mathcal{A}_{CFT}$ perturbed by a relevant operator $\varphi(x)$:

$$\mathcal{A} = \mathcal{A}_{CFT} + g \int d^2x \, \varphi(x) \ . \tag{iii.1}$$

In perturbation theory, one expands, inside correlation functions evaluated in the conformal field theory, the exponential of minus the interacting part of the action in powers of the coupling g. The problem with perturbing about a conformal field theory is that the resulting integrals generically have infrared divergences. Hence, consider putting the theory on a finite circular surface of radius R with center at the position $x = 0$ in the plane (correlation functions in this theory will

[5]Some recent conjectures [91, 48] give exact expressions for vacuum expectation values in the sine-Gordon model, for instance.

be denoted $\langle\cdots\rangle_g^{(R)}$).

In the conformal field theory described by $\mathcal{A}_{CFT}$, choose a basis of fields $\tilde{\mathcal{O}}_a$ and form their dual $\tilde{\mathcal{O}}^a$ under the inner product obtained by putting one field at the position $x=0$ in the plane and another field at infinity:

$$\langle\tilde{\mathcal{O}}^a(\infty)\tilde{\mathcal{O}}_b(0)\rangle_{CFT} = \delta^a_b \ .$$

In the theory (iii.1), fields $\tilde{\mathcal{O}}_a$ will need, in general, ultraviolet regularization; denote the regularized fields by $\mathcal{O}_a$, with $\mathcal{O}_a = \tilde{\mathcal{O}}_a + O(g)$. Then, the matrix

$$\langle\tilde{\mathcal{O}}^a(\infty)\mathcal{O}_b(0)\rangle_g^{(R)}$$

has a well-defined perturbative expansion in g. In particular, we can define fields $\mathcal{O}^a_{(R)}$ by adding to $\tilde{\mathcal{O}}^a$ a perturbative series in g with finite field-dependent and R-dependent coefficients, in such a way that

$$\langle\mathcal{O}^a_{(R)}(\infty)\mathcal{O}_b(0)\rangle_g^{(R)} = \delta^a_b \ . \tag{iii.2}$$

Consider the operator product expansion of a field $\Phi(x)$ with itself, and suppose for simplicity that the field does not renormalize in the theory (iii.1). This operator product expansion can be expressed in terms of the renormalized fields $\mathcal{O}_a(x)$, and has the form

$$\Phi(x)\Phi(0) = \sum_a C^a_{\Phi\Phi}(x)\mathcal{O}_a(0) \ .$$

We are interested in a perturbative expansion for the structure functions $C^a_{\Phi\Phi}(x)$. Formally, the structure functions can be written

$$C^a_{\Phi\Phi}(x) = \langle\mathcal{O}^a_{(R)}(\infty)\Phi(x)\Phi(0)\rangle_g^{(R)} \ .$$

Since the field $\Phi(x)$ does not renormalize, this expression has a well-defined perturbative expansion in g. The assertion is that every coefficients in this expansion have a finite limit as $R\to\infty$, and in fact are *independent* of R. That is, the structure functions do not see the boundary, and are purely local objects.

This assertion can be illustrated to first order in g. First, note that we can write

$$\mathcal{O}^a_{(R)}(\infty) = \tilde{\mathcal{O}}^a(\infty) + g\sum_b \int_{UV}^{(R)} d^2x\,\langle\tilde{\mathcal{O}}^a(\infty)\phi(x)\tilde{\mathcal{O}}_b(0)\rangle_{CFT}\ \tilde{\mathcal{O}}^b(\infty) + O(g^2)\ ,$$

where the symbol $\int_{UV}^{(R)}$ means that we must take the integral in the region $|x| < R$ with an ultraviolet cutoff, say $|x| > \epsilon$, that we must subtract appropriate R-independent divergent terms in ϵ and that we must take the limit $\epsilon \to 0$. Then, to first order in g, we have

$$\begin{aligned} C^a_{\Phi\Phi}(x) \;=\;& \langle\tilde{\mathcal{O}}^a(\infty)\Phi(x)\Phi(0)\rangle_{CFT} - g\int^{(R)} d^2y\,\langle\tilde{\mathcal{O}}^a(\infty)\phi(y)\Phi(x)\Phi(0)\rangle_{CFT} \\ & + g\sum_b\int_{UV}^{(R)} d^2y\,\langle\tilde{\mathcal{O}}^a(\infty)\phi(y)\tilde{\mathcal{O}}_b(0)\rangle_{CFT}\,\langle\tilde{\mathcal{O}}^b(\infty)\Phi(x)\Phi(0)\rangle_{CFT}\ . \end{aligned}$$

On the first line, the symbol $\int^{(R)}$ means that the integral is performed in the region $|y| < R$. On the second line, using completeness of the set of operators $\tilde{\mathcal{O}}_b$ and performing the integral in the region $|x| < |y| < R$ (evaluating the correlation functions by operator product expansions convergent in the region $|x| < |y| < R$), we obtain exactly the negative of the result of the integral on the first line evaluated in the region $|x| < |y| < R$. The rest of the integrals is independent of R, since subtractions in $\int_{UV}^{(R)}$ are R-independent. Hence the full structure function to first order is indeed R-independent.

The integral on the second line cannot be evaluated in the region $|y| < |x|$ using completeness of the operators $\mathcal{O}_b$. Rather, we must evaluate each factor in each term of the sum over b independently using the operator product expansions of the operators involved and the appropriate subtractions. Then we must sum over b. Evaluating the integral on the first line in the region $|y| < |x|$, one generically finds a nonzero value for the sum of the integrals on both lines evaluated in the region $|y| < |x|$ at order g.

It is a simple matter to observe that the integral on the second line can be made to vanish as $R \to \infty$ by formally increasing the dimension of the operator

$\tilde{\mathcal{O}}^a$. Indeed, the factor $\int_{UV}^{(R)} d^2y\, \langle\tilde{\mathcal{O}}^a(\infty)\phi(y)\tilde{\mathcal{O}}_b(0)\rangle_{CFT}$ can only depend on R, so it must be a power of R times a pure number, and this power can be made to be negative for all b's by choosing the dimension of the operator $\tilde{\mathcal{O}}^a$ high enough. Hence, the full first-order correction to the structure function can be obtained solely from the integral on the first line in the limit $R \to \infty$, by analytically continuing in the dimension of the operator $\tilde{\mathcal{O}}^a$ from a convergent region to the "physical" region (this procedure being the meaning of the prime on the integral below):

$$C^a_{\Phi\Phi}(x) = \langle\tilde{\mathcal{O}}^a(\infty)\Phi(x)\Phi(0)\rangle_{CFT} - g\int' d^2y\, \langle\tilde{\mathcal{O}}^a(\infty)\phi(y)\Phi(x)\Phi(0)\rangle_{CFT}\ .$$

This is valid as long as the infrared divergences are power law divergences.

Note that the identity operator $\mathbf{1}$ is self-dual under (iii.2). From this, one can see that usual perturbation theory about a scale invariant point gives in fact only the perturbative expansion for the structure function associated to the identity operator $C^{\mathbf{1}}_{\Phi\Phi}(x)$. This formulation of conformal perturbation theory will be used in Part I of this dissertation in order to obtain the short distance expansion of the two-point function of topologically charged fields in the sine-Gordon model.

Conformal perturbation theory will also be used in Part II in the Ising field theory on a negatively curved space; the curvature will provide an explicit infrared cutoff. This is a particular infrared cutoff, different from the one considered above: it does not put an actual boundary, but modifies the bulk theory. Hence the structure functions are modified by this cutoff. However, they still satisfy a nontrivial requirement. Denote by R the infrared length scale introduced by this cutoff. Then every coefficient in an expansion of the structure function in the perturbation parameters should independently have a well-defined limit $R \to \infty$. This requirement will be used in Section 5.7 in order to derive appropriate initial conditions for the solution to the Painlevé VI equation which determines two-point functions of spin fields in the Ising field theory.

iv Free fermion models

Lastly, in models admitting a free-fermion description (that is, models where the on-shell description is that of free particles with scattering matrix -1), much more information can be obtained concerning correlation functions. The interest is usually in particular "interacting" fields, called *twist fields* (or sometimes "disorder fields") [69, 53]. They are generically not local with respect to the fundamental free Fermi fields, but are self-local and local with respect to the Hamiltonian density. They usually represent the scaling limit of local degrees of freedom in some genuinely interacting underlying model scaling to the free fermion quantum field theory. Twist fields are in fact present in any model (which might be described in terms of possibly interacting fermionic of bosonic "fundamental" local degrees of freedom) which possess a global internal symmetry: for every element of a symmetry group of the model, there is an associated twist field.

Let me now describe in complete generality a twist field σ_g associated to an element g of a symmetry group. It is essentially defined by the fact that when any other local field, inside a correlation function, is brought around the twist field counterclockwise back to its initial position, it is transformed according to the element g of the symmetry group. Since the transformation g is that of a symmetry group, this insures that the twist field is local with respect to the Hamiltonian density (hence it is a field of the theory).

From this property, we can obtain a path integral definition of twist fields. A correlation function containing a twist field $\sigma_g(x)$ at a point x is defined by the path integral over field configurations with monodromy conditions according to which the fundamental field is affected by g counterclockwise around the point x:

$$\langle \sigma_g(x) \cdots \rangle = \int_{\Psi \to \hat{g}\Psi \text{ around } x} [\mathcal{D}\Psi]\, e^{-S[\Psi]}\, \cdots \tag{iv.1}$$

with obvious definitions when more than one twist fields are present. In order for the fundamental field to be defined on the plane, where the integration is taken in the action $S[\Psi]$, we must chose a cut, starting at x (in massive models, this cut can be taken to end at infinity, otherwise it should end at the position of a conjugate twist field $\sigma_{g^{-1}}$), across which the fundamental field is discontinuous. The correlation function as defined above is independent of the shape of this cut, apart from transformations of fields by the element g when the cut is brought through them. Indeed, consider a set of field configurations where the cut is taken on a path $\mathcal{C}_1$. It is related in a simple way to the set of field configurations where the cut is taken on a path $\mathcal{C}_2$ if $\mathcal{C} = \mathcal{C}_1 \bigcup \mathcal{C}_2$ is a closed path. One only need apply on each field configuration of the former set a transformation where inside $\mathcal{C}$ (inside is the region covered by going counterclockwise from $\mathcal{C}_2$ to $\mathcal{C}_1$) the field is affected by g, whereas outside it is unaffected. Since g is an element of a symmetry group, the action is unchanged under such a transformation, except for a contribution on the path $\mathcal{C}$:

$$\langle\sigma_g(x)\cdots\rangle_{\mathcal{C}_2} = \langle\mathcal{P}e^{\int_{\mathcal{C}} ds^\mu j^\nu \epsilon_{\mu\nu}}\sigma_g(x)\cdots\rangle_{\mathcal{C}_1} \tag{iv.2}$$

where ds^μ is the length element along $\mathcal{C}$ and j^ν is a local field. This local field is in fact exactly the current associated to the charge generating the element g. Hence, the only effect of this contribution is to apply the symmetry transformation g on the fields inside the path $\mathcal{C}$, as asserted.

If two elements g_1 and g_2 commute with each other, then the associated twist fields σ_{g_1} and $sigma_{g_2}$ are local with respect to each other, otherwise they are not. Indeed, the symmetry transformation g_1, say, acts on the twist field σ_{g_2} by producing $\sigma_{g_2 g_1 g_2^{-1}}$, as can be seen by making a global symmetry transformation g_1 on the configurations in the path integral (iv.1). Hence formula (iv.2) shows that $sigma_{g_1}$ and σ_{g_2} are local with respect to each other only if $g_2 g_1 g_2^{-1} = g_1$. In particular, a twist field is self-local.

For twist fields associated to the $U(1)$ symmetry in a free-fermion theory, it is possible to carry out further the form factor programme: usually one can re-sum the form factor expansion in terms of Fredholm determinants [112, 111, 6, 12]. It is also possible to calculate exactly the one-point functions [141, 91, 37] (see also the derivation in Section 4.6 of this dissertation) and to have the complete local expansion of correlation functions.

In fact, one of the most important characteristics of correlation functions of twist fields associated to the $U(1)$ symmetry in free-fermion models is that they can be described by particular non-linear differential equations of Painlevé type. The first result of this sort was found in the context of a lattice theory in [132], for correlation functions of spin variables in the Ising model. The Fredholm determinant description, which can be obtained from the form factor expansion or by other means, also provides a way of obtaining a description in terms of non-linear differential equations [79, 6, 12]. A powerful viewpoint is the one relating the correlation functions of twist fields to problems of isomonodromic deformations [112, 111]. This is based on considering the linear equations of motion for the fermion fields inside correlation functions with twist fields. The solutions to these linear differential equations are such that their monodromy properties around the positions of the twist fields are unaffected by moving the twist fields. This imposes constraints on the dependence of the solution on the positions of the twist fields, from which it is possible to deduce non-linear differential equations for correlation functions of the twist fields. Finally, methods involving Ward identities [52, 39] provide very simple ways of obtaining such non-linear differential equations. Supplemented with appropriate asymptotic conditions, coming from the large-distance or from the short-distance analysis, correlation functions can then be evaluated very efficiently.

The problem of calculating correlation functions is still in general not solved in integrable models. Examples with a free-fermion description and conformally invariant models offer essentially the only nontrivial exceptions. It is still not known if the methods in these two classes of examples are related, and if they can be generalized to models of interacting massive particles. It would certainly be very interesting to understand more deeply their meaning.

In Part II of this dissertation, methods for exact calculations in free fermion models will be developed and explained to some extent. Hence, I will not elaborate further on this subject in this section.

v Plan of the dissertation

In this work, I study correlation functions and their applications in two particular integrable models using the techniques mentioned above.

In Part I, I study the fermion two-point correlation functions in the $SU(2)$-Thirring model (or $O(4)$ chiral Gross-Neveu model). As I said, this is a model for the low-energy behavior of electrons in a Mott insulator. The $SU(2)$-Thirring model is integrable, and its spectrum contains particles and anti-particles with a nontrivial scattering matrix. Using techniques of form factor expansions and conformal perturbation theory, the two-point function of Fermi fields is evaluated accurately at all distance scales. This part is based on the joint work [42] in collaboration with my dissertation director Prof. S. Lukyanov.

In Part II, I study the correlation functions of spin fields in the Ising field theory on a curved space of constant negative curvature. The aim of this study is to understand the effects of a space curvature on the critical properties of statistical systems. The Ising field theory is integrable, and in fact its spectrum is that of free fermions. Hence correlation functions and their long and short distance expansions can be evaluated exactly, and I generalize the methods used in flat space to the space of constant negative curvature. This part is based on my works [37, 38] and on the joint work [39] with Dr. P. Fonseca.

Part I

One-dimensional correlated electron systems

Chapter 1
Introduction

1.1 Fermi Liquids

Understanding the behavior of strongly correlated electrons is one of the main problems of condensed matter physics. In some situations in two or more space dimensions, the behavior of electrons in metals is well understood through the Fermi liquid theory of Landau (see [106]). Although this is very standard material, I will explain briefly the main points of this theory, in order to emphasize afterwards the differences that occur in one-dimensional systems.

In a system of non-interacting spin-1/2 fermions, the spectrum of the Hamiltonian can be described by giving a set of allowed energy levels, characterized by the momentum $\mathbf{p}$ and by a spin s, with an associated energy $\epsilon_{\mathbf{p},s}$, and by describing the ground state and the excitations by a distribution function $n_{\mathbf{p},s}$ specifying the occupancy of the allowed levels, which can be 0 or 1. In a system with a fixed number of fermions, for instance, the ground state is obtained by filling the energy levels up the Fermi energy ϵ_F determined by the number of particles; this is the "Fermi sea." The excitations are then obtained by putting particles above the Fermi surface and by putting holes below it.

In the Fermi liquid theory, one essentially assumes that the low-lying excitations of a theory of interacting fermions can be described by the same picture: the system possesses a well-defined Fermi surface, and elementary excitations can be accurately described, at low temperatures, by long-lived particles, or quasiparticles, above the surface, and holes below it. More formally, the elementary quasi-particle excitations are defined by starting with a system without interactions and by adiabatically turning on the interactions between fermions. By this process, if there is no level crossing[1], energy eigenstates of the noninteracting system are mapped to energy eigenstates of the interacting system, so that the latter can be labelled by the distribution function of the noninteracting state, say $n_{\mathbf{p}}$ (spin indices are implicit from now on). In particular, one assumes that the ground state of the interacting theory can be reached from some (possibly excited) state of the noninteracting theory, call it $n^0_{\mathbf{p}}$ (it can be seen as describing the Fermi sea in the interacting theory). Low energy single quasi-particle and single quasi-hole states are obtained by starting from states with one particle or one hole near to the corresponding Fermi surface: $n^0_{\mathbf{p}} + \delta n_{\mathbf{p}}$.

Low-lying excitations with many quasi-particles and quasi-holes are then assumed to be described by quasi-particles and quasi-holes near to the Fermi surface in a fashion similar to the description in a free theory. Of course, the number of quasi-particles and of quasi-holes is not conserved, so that the lifetime of quasi-particles and quasi-holes is not infinite; it is in fact proportional to the inverse of their energy difference to the Fermi energy. Hence this description becomes valid only at low temperatures, for quasi-particles and quasi-holes located near to the Fermi surface. The excitations are then described by a rarefied gas of quasi-particles and quasi-holes near to the Fermi surface, and in some sense, the Fermi sea survives the interactions. The quasi-particles and quasi-holes also interact among themselves, and at low excitation energies, a consistent lowest-order description of the free energy only needs density-density interaction terms of the type $\delta n_{\mathbf{p}} \delta n_{\mathbf{p}'}$:

$$\delta F = \sum_{\mathbf{p}} (\epsilon_{\mathbf{p}} - \mu) \delta n_{\mathbf{p}} + \frac{1}{2} \sum_{\mathbf{p},\mathbf{p}'} f_{\mathbf{p},\mathbf{p}'} \, \delta n_{\mathbf{p}} \delta n_{\mathbf{p}'} \ .$$

For describing electrons in the conduction band of a metal, one needs to take

[1] In particular, if the interaction between the fermions is attractive there is level crossing because nontrivial bound states can form; hence we must assume repulsive (or no) interaction.

into consideration the Coulomb interaction which has a long range. The Fermi liquid theory can be adjusted to include the screening effect of the Coulomb interaction, giving a so-called charged Fermi liquid, and the considerations above stay valid. The charged Fermi liquid theory is then a phenomenological theory for electrons in metals, valid when the Fermi level is inside a conduction band (that is, in a metallic state) and at temperatures low compared to the Fermi energy and to the width of the conduction band. Such restrictions are in fact fulfilled in many materials of interest.

Most of the qualitative features of thermodynamical quantities of a free Fermi gas are not altered by the interacting quasi-particle description depicted above. This can be understood from the viewpoint of quantum field theory. The quasi-particles and quasi-holes near to the Fermi surface are fermionic excitations that can be represented by massless Fermi fields at low temperatures. For such fields, repulsive interactions of the type density-density, involving four-fermion terms, are irrelevant in two or more space dimensions, so that the low energy or long distance physics is that of free massless fermions: the quasi-particles and quasi-holes are asymptotically the true excitations. The features of Fermi liquid theory are in good agreement with many experimental observations.

Interesting situations occur when the Fermi liquid theory breaks down. Situations with the most interest for the present chapter are systems in one space dimension, which show drastically different behaviors.

1.2 Luttinger liquids

One-dimensional correlated electron systems in their metallic phase are believed to be Luttinger liquids, in some sense the one-dimensional counterpart of Fermi liquids [125]. The drastic difference from Fermi liquids that occurs in one dimension can be understood from the fact that the phase space around the Fermi surface of a free theory is very restricted: the Fermi surface in one dimension is composed solely of two disjoint points p_F and $-p_F$ (the Fermi momenta). Hence it is not possible to have a rarefied gas of quasi-particles or quasi-holes near to the Fermi surface. In interacting theories, no matter how weak the interaction, collective modes are stable and the Fermi surface does not survive (in higher dimensions, collective modes decay into quasi-particles).

Particular models of one-dimensional correlated electrons may have very different features, but as usual, their universal properties at long distances may be fruitfully described using quantum field theory. In a free theory, excitations near to the Fermi surface are described by right- and left-moving massless fields, for momenta around the points $p = p_F$ and $p = -p_F$, respectively, of the Fermi surface. For low-lying excitations, the dispersion relation around these momenta is linear, so that the resulting theory looks like a relativistically invariant free massless fermion theory, where the velocity of "light" is the Fermi velocity. In order to account for interactions, one can simply add certain density-density local interaction terms to the Hamiltonian, similar to density-density interactions among quasi-particles and quasi-holes in the Fermi liquid theory. In this fashion, one obtains the Tomanaga-Luttinger model. In its simplest form, the interaction couples the fermion density around p_F with itself (right-right coupling), the fermion density around $-p_F$ with itself (left-left coupling), and both densities among themselves (right-left coupling). The Luttinger liquid model of quantum field theory is then described by the second quantized Hamiltonian

$$\begin{aligned}\mathbf{H}_{LL} &= \int dx \left[v_F \left(-i\psi_R^\dagger \partial_x \psi_R + i\psi_L^\dagger \partial_x \psi_L \right) \right. \\ &\quad + g_1 \left(: j_R\, j_R : + : j_L\, j_L : \right) + \ g_2 : j_R\, j_L : \\ &\quad \left. + g_3 \left(: \vec{J}_R \cdot \vec{J}_R + : \vec{J}_L \cdot \vec{J}_L \right) \right] \end{aligned} \tag{1.2.1}$$

where v_F is the Fermi velocity, and where I wrote explicitly only interaction terms that preserve spin $SU(2)$-invariance. Here the $U(1)$ currents are $j_{R,L} =:$

$\psi^\dagger_{R,L}\psi_{R,L}$: and are associated to the electric charge of the excitations, and the $SU(2)$ currents are $\vec{J}_{R,L} =: \psi^\dagger_{R,L}\vec{\tau}\psi_{R,L}$: and are associated to their spin. The colons represent an appropriate normal ordering of the product of operators. Note that I do not consider here electron-phonon interactions.

The main difference compared to the higher dimensional situation is that the perturbing interactions are exactly marginal: the coupling g_1 and g_2 are exactly dimensionless[2]. Hence the interaction terms have non-negligible effects at low energies, and the true excitations are not the initial fermions, but collective modes. There are two main properties of Luttinger liquids. First, there is spin-charge separation. This phenomenon can be easily understood from the bosonized form of (1.2.1): the theory becomes a sum of two independent free massless boson models, one carrying the charge of the electrons, the other carrying the spin. The separation occurs because the coupling g_1 renormalizes the velocity in the charge sector, whereas the velocity in the spin sector is renormalized by g_3. Note that relativistic invariance is broken in a simple way, since two different velocities appear. The spin and charge excitations, called spinons and holons, are the elementary excitations replacing the quasi-particles of the Fermi liquid theory. Second, another characteristics of Luttinger liquids is that the fermion fields ψ_R, ψ_L acquire anomalous dimensions with respect to the free theory. More precisely, under bosonization the fermion fields are decomposed into spin and charge "factors," which are nonlocal with respect to the original fermion fields and which each have independent power-law correlation functions with independent anomalous dimensions. The anomalous dimension in the charge sector, for instance, is a function of g_2. The presence of anomalous dimensions destroys the Fermi surface, as they result in no sharp jump in the distribution function $n_{\mathbf{p}}$.

[2]In fact, by the definition of a Luttinger liquid as describing gapless excitations, all exactly marginal local four-fermion interaction terms may be added in general.

1.3 One-dimensional Mott insulators

There are situations, in one dimension or in higher numbers of dimensions, where the low-energy physics is not described by a Luttinger liquid or by a Fermi liquid. In some of these situations, the Mott insulators, the breakdown is due to electron interactions creating a spectral gap between the ground state and the first excited state. That is, in Mott insulators, the gap is not due to the band structure coming from the periodic potential created by the atoms in the material, but rather to a dynamical effect coming from the strong interaction between electrons. In one dimension, by spin-charge separation, the gap can be either in the spin or in the charge sector (or in both). One-dimensional materials with a dynamical gap in the charge sector are one-dimensional Mott insulators. Of course, the effect of the dynamical gap can be seen only if the ground state is far enough from the end of the conduction band: then sub-bands can dynamically form.

What processes make the formation of a dynamical gap possible? Again, it is instructive to analyze this question in its universal form, from the viewpoint of quantum field theory. A gap will be formed if we add to the universal model Hamiltonian (1.2.1) terms that are relevant or marginally relevant in the charge sector. In their simplest form, interaction terms must contain four fermion fields located at the same point. Hence a gap may appear as a result of dimensional transmutation from marginally relevant local four-fermion interaction terms. For instance, there will be a spin gap generated in this way if there is an anti-ferromagnetic isotropic coupling between the right and left spin currents: $g\,\vec{J}_R\cdot\vec{J}_L,\ g>0$ (whereas a ferromagnetic coupling, $g<0$, gives marginally irrelevant terms). In order to have a charge gap from a marginally relevant operator, we must consider so-called umklapp terms, of the type $\psi^\dagger_{L,\uparrow}\psi^\dagger_{L,\downarrow}\psi_{R,\uparrow}\psi_{R,\downarrow}$. They correspond to backward scattering of charge-carrying spin singlets of electrons, and essentially occur when there is commensurate filling, for instance half filling

$\langle \hat{n}(x) \rangle = 1/2$, where $\hat{n}(x)$ is the number operator at position x. Intuitively, it is clear that such backward scattering of singlets may impair on the propagation of charge, but not on that of spin; it is also clear that commensurate fillings should provide better opportunities for spin singlets to form. In fact, umklapp terms may be embedded into a structure very similar to that of the spin current-current interaction. Indeed, there is another $SU(2)$-current structure, the so-called isospin current, and umklapp terms producing a dynamical gap in the charge sector may be seen as coming from an "anti-ferromagnetic" isotropic coupling between right and left isospin currents. We will see that a fruitful model of interacting electrons in the conduction band of a metal, the Hubbard model, is correctly described at low energies by such interaction terms, which we will explicitly construct in the context of this model.

1.4 Experimental observation of one-dimensional physics

The past two decades have witnessed experimental work to identify and study quasi one-dimensional systems (for a review, see [59, 17]). The most striking and nontrivial prediction of the Luttinger liquid description of one-dimensional strongly correlated electrons is spin-charge separation. An experimental investigation of this phenomenon is of conceptual importance for several reasons. First, the direct observation of spin and charge excitations and the comparison with theoretical predictions is a nontrivial test for models widely used to describe one-dimensional correlated electron systems, and in particular for the properties of Luttinger liquids. Second, the understanding of the spin and charge excitations gained from their experimental study can be valuable to assess the structure of the excitations in two-dimensional systems. In connection to this point, there is evidence that high-temperature superconductivity can be a result of an effective low dimensionality of space. Understanding the main properties of one-dimensional systems, and in particular of their fundamental excitations, may shed light onto the subject of high-temperature superconductivity.

However, identifying spin and charge degrees of freedom in a material believed to be described by one-dimensional physics is a difficult task. In fact, it requires studying the dynamics of these degrees of freedom; thermodynamical properties, although they may be suggestive of one-dimensional physics, cannot be used to accurately identify spinons and holons and study their properties.

In recent years, angle-resolved photoemission spectroscopy (ARPES) has matured into a powerful experimental method for probing the electronic Green's functions in quasi one-dimensional systems [63]. Roughly speaking, incident photons illuminate the surface of a crystalline sample, and photoelectrons are ejected from the surface. The kinetic energy and the angle of the outgoing electrons are measured, from which the probability for an excitation to eject an electron can be deduced. This probability is simply related to the electronic spectral density, and to the Green's function. The observation of the features of the spectral density is the most direct observation of spinons and of holons, and allows for an extensive study of their properties [75]. It is worth mentioning that ARPES data also offers evidence for the relevance of the Luttinger liquid spectral functions in cuprates high-temperature superconductors [14].

In view of these new experimental techniques, it is an important task to calculate electronic Green's functions in one-dimensional systems of correlated electrons. Integrable models offer a unique opportunity to obtain accurate theoretical predictions. This is why there were recently collective efforts of many physicists to apply integrable quantum field theory to describe quasi one-dimensional systems [58]. For this purpose, a non-perturbative treatment of the correlation functions in integrable models seems to be valuable.

Chapter 2

The $SU(2)$-Thirring model

2.1 Introduction

In this chapter, I study the fermion two-point correlation functions in the $SU(2)$-Thirring model. This chapter is based on my joint work with my dissertation director Prof. S. Lukyanov [42]. We performed perturbative and renormalization group analyses to obtain the short-distance asymptotic behavior of the two-point function, and numerically evaluated the long-distance behavior by using the form factor expansion. The results presented here illustrate the use of bosonization and conformal perturbation theory in the renormalization group analysis of a fermionic theory, and numerically confirm the validity of the form factor expansion in the case of the $SU(2)$-Thirring model.

The results and methods are also of interest because the $SU(2)$-Thirring model is the scaling limit, at half filling, of a popular microscopic model of interacting electron gas in the conduction band of a metal, the Hubbard model. The Hubbard model at half filling shows the behavior of a Mott insulator, and as emphasized in the previous chapter, results concerning the electronic spectral function are of interest. The one-dimensional Hubbard model is also relevant to the study of high-temperature superconductivity. Indeed, it probably has common properties with the two-dimensional Hubbard model, which is believed to possess a superconducting phase characteristic of high-temperature superconductors.

2.2 The Hubbard model and the $SU(2)$-Thirring model

The Hubbard model [62] captures most of the important properties of electrons in a narrow conduction band, where features of both the electron gas and electron localization on the atoms are relevant. The effect of the atoms is implemented essentially by accounting for Hund's rule: electrons on the same site with aligned spins, which cannot come too close by anti-symmetry of their wave function, do not feel as much the Coulomb repulsion as electrons with opposite spins. The standard Hubbard model in fact describes electrons in an s-band, where two electrons on the same site must have opposite spins, so that the effect is taken into account by an extra potential energy for two electrons to be on the same site. Some of the main features should hold in other situations with a narrow conduction band. In one dimension, the model is described by the following Hamiltonian:

$$\begin{aligned} \mathbf{H}_{\mathrm{Hub}} &= \sum_{j=-\infty}^{\infty} \Big[-t \sum_{\sigma=\uparrow,\downarrow} (c^\dagger_{j,\sigma} c_{j+1,\sigma} + c^\dagger_{j+1,\sigma} c_{j,\sigma}) \\ &\qquad + U\, n_{j,\uparrow} n_{j,\downarrow} + \mu\, (n_{j,\uparrow} + n_{j,\downarrow}) \Big] \end{aligned} \tag{2.2.1}$$

with $n_{j,\sigma} = c^\dagger_{j,\sigma} c_{j,\sigma}$, and where t can be seen as a tunnelling amplitude between sites (the so-called transfer integral), $U > 0$ represents the difference of strength of the on-site Coulomb interaction between opposite- and aligned-spin configurations, and μ is the chemical potential.

The Hubbard model is in fact an integrable model, and can be solved by Bethe ansatz techniques [88, 44, 50, 129, 80]. In the charge sector, the spectrum of the Hubbard model shows two (sub-)bands, and exactly at half filling ($\mu = -U/2$) the ground state fills the first band: the model is a Mott insulator. At half filling, the theory possesses some extra symmetries which are worth describing briefly. Consider the Hubbard model on a finite chain of L (even) sites with periodic boundary conditions; the model (2.2.1) is obtained by taking the limit

$L \to \infty$. The model on the finite chain possesses an $SO(4) \equiv SU(2) \times SU(2) / \mathbb{Z}_2$ symmetry, which is realized by the spin $SU(2)$ algebra satisfied by the charges

$$S^- = \sum_{j=1}^{L} c_{j,\uparrow}^\dagger c_{j,\downarrow}, \quad S^+ = \sum_{j=1}^{L} c_{j,\downarrow}^\dagger c_{j,\uparrow}, \quad S^z = \frac{1}{2} \sum_{j=1}^{L} (n_{j,\downarrow} - n_{j,\uparrow}) \tag{2.2.2}$$

and by the isospin $SU(2)$ algebra satisfied by the charges

$$\eta^- = \sum_{j=1}^{L} (-1)^j \, c_{j,\uparrow} c_{j,\downarrow}, \quad \eta^+ = \sum_{j=1}^{L} (-1)^j \, c_{j,\downarrow}^\dagger c_{j,\uparrow}^\dagger, \quad \eta^z = \frac{1}{2} \sum_{j=1}^{L} (n_{j,\downarrow} + n_{j,\uparrow} - 1) \; . \tag{2.2.3}$$

Because L is even, half-integer representations of $SU(2) \times SU(2)$ are projected out, giving $SO(4)$.

From the exact solution, one finds that at half-filling, the correlation length, which is inversely proportional to the gap, is given by [99]

$$R_c = \frac{\pi}{2} \sqrt{\frac{t}{U}} e^{\frac{2\pi t}{U}} \; .$$

In particular, the correlation length diverges as $U/t \to 0$, so that there is a quantum phase transition at this point (at zero temperature). The scaling limit, describing the universal behavior of the model near to its phase transition, is then obtained by sending U/t to 0 from above and by considering correlation functions at lattice separations proportional to the correlation length. In this limit, the leading behavior of correlation functions assumes certain scaling forms. In particular, the equal-time fermion correlation function can be written as

$$\langle c_{j',\sigma'} \, c_{j,\sigma}^\dagger \rangle \to \delta_{\sigma',\sigma} \frac{\sin\left(\frac{\pi}{2}(j'-j)\right)}{\pi(j'-j)} \, F(|j'-j|/R_c) \tag{2.2.4}$$

for some scaling function F.

Since the scaling limit is obtained by sending U/t to 0, the quantum field theory that describes it can be obtained directly from the Hamiltonian (2.2.1) by taking a naive continuous limit. We simply keep modes of the electron operator $c_{j,\sigma}$ in the vicinity of the Fermi wave numbers $\pm k_F = \pm\pi/(2a)$ (with the lattice spacing a):

$$c_j = \sqrt{a} \left[\, e^{ik_F x} \psi_R(x) + e^{-ik_F x} \psi_L(x) \right] \; ,$$

where spin indices are implicit, and replace sums over indexes $j = x/a$ by integrals over positions x in the Hamiltonian (2.2.1). We find

$$\begin{aligned}
\mathbf{H}_{\mathrm{Hub}}^{\mathrm{scaling}} &= \int dx \Big[v_F \left(-i\psi_R^\dagger \partial_x \psi_R + i\psi_L^\dagger \partial_x \psi_L \right) \\
&\quad + \frac{\tilde{g}}{6} \left(: \vec{I}_R \cdot \vec{I}_R : + : \vec{I}_L \cdot \vec{I}_L : - : \vec{J}_R \cdot \vec{J}_R : - : \vec{J}_L \cdot \vec{J}_L : \right) \\
&\quad + \tilde{g} \left(\vec{I}_R \cdot \vec{I}_L - \vec{J}_R \cdot \vec{J}_L \right) \Big] \qquad (2.2.5)
\end{aligned}$$

where $v_F = 2ta$ and $\tilde{g} = Ua/2$. The spin currents are, as usual, $\vec{J}_{R,L} = \psi_{R,L}^\dagger \vec{\tau} \psi_{R,L}$, and the components of the isospin currents $\vec{I}_{R,L}$ are

$$I_{R,L}^3 =: \psi_{R,L}^\dagger \psi_{R,L} : \ , \quad I_{R,L}^+ = (\psi_{R,L}^\dagger)_\uparrow (\psi_{R,L}^\dagger)_\downarrow \ , \quad I_{R,L}^- = (\psi_{R,L})_\downarrow (\psi_{R,L})_\uparrow \ . \quad (2.2.6)$$

(Here, according to our normalization, we have for instance $: \vec{J}_R \cdot \vec{J}_R : = : J_R^3 J_R^3 : + 2 : J_R^+ J_R^- : + 2 : J_R^- J_R^+ :$.) These spin and isospin currents are local currents in the model (2.2.5) of quantum field theory, and they correspond to the spin and isospin charges (2.2.2) and (2.2.3) of the lattice Hubbard model. The $U(1)$ currents $j_{R,L}$ involved in the Luttinger liquid Hamiltonian (1.2.1) are just the components $I_{R,L}^3$ of the isospin currents. The term $\vec{I}_R \cdot \vec{I}_L$ contains the umklapp terms.

In fact, it is a simple matter to observe spin-charge separation and to understand more precisely the effects of the various terms in the Hamiltonian (2.2.5) by applying the techniques of bosonization [58]. Consider two mutually commuting chiral operators, $\phi_R(x)$ and $\phi_L(x)$, satisfying the commutation relations

$$\begin{aligned}
[\phi_R(x), \phi_R(x')] &= \frac{i}{4}(\Theta(x - x') - \Theta(x' - x)) \\
[\phi_L(x), \phi_L(x')] &= -\frac{i}{4}(\Theta(x - x') - \Theta(x' - x))
\end{aligned}$$

where $\Theta(x)$ is Heaviside's step function. Assume that these commutation relations are represented on a Hilbert space, where the two-point vacuum expectation values are formally given by

$$\begin{aligned}
\langle \mathrm{vac} | \phi_R(x), \phi_R(x') | \mathrm{vac} \rangle &= -\frac{1}{4\pi} \ln(-i(x - x')) \\
\langle \mathrm{vac} | \phi_L(x), \phi_L(x') | \mathrm{vac} \rangle &= -\frac{1}{4\pi} \ln(+i(x - x'))
\end{aligned}$$

and other vacuum expectation values are calculated using Wick's theorem. The logarithmic function $\ln(z)$ is taken on its principal branch, with a cut along $\Re e(z) < 0$, $\Im m(z) = 0$. Then we can represent the fermion operators by exponential operators on this Hilbert space:

$$\psi_{R;\sigma}(x) = \epsilon_{R;\sigma}\frac{1}{\sqrt{2\pi}}\, e^{i\sqrt{4\pi}\phi_{R;\,\sigma}(x)} \;, \qquad \psi_{L;\sigma}(x) = \epsilon_{L;\sigma}\frac{1}{\sqrt{2\pi}}\, e^{-i\sqrt{4\pi}\phi_{L;\,\sigma}(x)} \tag{2.2.7}$$

where $\epsilon_{R,L;\sigma}$ are Klein factors, which anti-commute with each other and square to 1. We can then define canonical bosonic operators $\varphi_\sigma = \phi_{R;\sigma} + \phi_{L;\sigma}$ and $\Pi_\sigma = \partial_x(\phi_{R;\sigma} - \phi_{L;\sigma})$ satisfying $[\Pi_\sigma(x), \varphi_{\sigma'}(x')] = i\delta(x-x')\delta_{\sigma,\sigma'}$. Changing basis to a charge and spin basis:

$$\varphi_c = \frac{1}{\sqrt{2}}(\varphi_\uparrow + \varphi_\downarrow) \;, \quad \varphi_s = \frac{1}{\sqrt{2}}(\varphi_\uparrow - \varphi_\downarrow) \;,$$

with similar relations for Π_c, Π_s, the kinetic part of the Hamiltonian (2.2.5) becomes

$$\frac{v_F}{2}\int dx\, \left(\Pi_c^2 + (\partial_x\varphi_c)^2 + \Pi_s^2 + (\partial_x\varphi_s)^2\right) \;.$$

The operator $:\vec{I}_R\cdot\vec{I}_R: + :\vec{I}_L\cdot\vec{I}_L:$ in the Hamiltonian contributes to the charge part of the kinetic term, $(3/\pi)(\Pi_c^2 + (\partial_x\varphi_c)^2)$, up to a term $\partial_x^2\varphi_c$ which does not contribute in the Hamiltonian. The operator $:\vec{J}_R\cdot\vec{J}_R: + :\vec{J}_L\cdot\vec{J}_L:$ contributes to the spin part of the kinetic term, $(3/\pi)(\Pi_s^2 + (\partial_x\varphi_s)^2)$, up to a term $\partial_x^2\varphi_s$. Hence the second line of (2.2.5) only renormalizes the velocities in the charge and spin sectors; this separates spin and charge excitations:

$$v_F^{(c)} = v_F + \frac{\tilde{g}}{\pi} \;, \qquad v_F^{(s)} = v_F - \frac{\tilde{g}}{\pi} \;.$$

The third line of (2.2.5) is the most interesting. The operator $\tilde{g} I_R^3 I_L^3$ is equal to

$$\frac{\tilde{g}}{2\pi}(\Pi_c^2 - (\partial_x\varphi_c)^2) \;.$$

This has two effects. First, it modifies further the charge velocity: $v_F^{(c)} = \sqrt{v_F(v_F + 2\tilde{g}/\pi)}$. But most importantly, it changes the dimensions of exponential operators. For instance, the operator $e^{i\sqrt{8\pi}\varphi_c}$, which has dimension 2 at $\tilde{g}=0$, acquires dimension

$$\frac{2}{\sqrt{1+\frac{2\tilde{g}}{\pi v_F}}}$$

which is smaller than 2 (since $\tilde{g}>0$). But the operator $J_R^+J_L^- + J_R^-J_L^+$ is proportional to $\cos(\sqrt{8\pi}\varphi_c)$, hence it becomes relevant. This indicates that $\tilde{g}\vec{I}_R\cdot\vec{I}_L$ is marginally relevant, and it opens a gap in the charge sector. On the other hand, a similar analysis shows that the term $-\tilde{g}\vec{J}_R\cdot\vec{J}_L$ is marginally irrelevant and it does not contribute to the large distance physics. That is, the spin sector is a purely massless bosonic field theory in the scaling limit.

Apart from the different velocities in the charge and spin sectors and dropping the marginally irrelevant interaction, the quantum field theory above is exactly the $SU(2)$-Thirring model [58, 147], with Euclidean action

$$\mathcal{A}=\int d^2x\left[\bar{\Psi}_\sigma\gamma^\mu\partial_\mu\Psi_\sigma+\frac{\pi g}{8}\left(\bar{\Psi}\gamma_\mu\vec{\tau}\Psi\right)^2\right] \tag{2.2.8}$$

(with implied summation over repeated indices) for $g>0$. A more accurate relation between the scaling limit of the Hubbard model and the $SU(2)$-Thirring model is given in Refs [99, 130]. Here Ψ_σ is a doublet of Dirac Fermi fields, and the Pauli matrices $\vec{\tau}=(\tau^1,\tau^2,\tau^3)$ act on the indices $\sigma=\uparrow,\downarrow$. The "spin" of the Dirac fermion, labelled by its index σ, is in fact an *isospin* index in the connection to the Hubbard model (and will be referred to as the isospin index from now on). The interaction term has a similar effect to that of the "anti-ferromagnetic" right-left isospin current interaction $\vec{I}_R\cdot\vec{I}_L$ in the Hamiltonian (2.2.5); it opens a gap in the isospin sector. As is clear from the discussion above, this gap appears by dimensional transmutation. The scaling function appearing in (2.2.4) can be obtained from fermion Schwinger's function (Green's function in the Euclidean region) in the $SU(2)$-Thirring model:

$$\langle\Psi_{\sigma'}(x)\bar{\Psi}_\sigma(0)\rangle=\frac{\delta_{\sigma',\sigma}}{2\pi}\frac{\gamma_\mu x^\mu}{|x|^2}F(M|x|)\ . \tag{2.2.9}$$

The $SU(2)$-Thirring model present a number of interesting properties. For instance, it is an asymptotically free theory (for $g > 0$) with unbroken chiral symmetry. Of course, like the Hubbard model, it is also an integrable model, and the corresponding Hamiltonian was diagonalized by Bethe ansatz techniques in [3, 9]. In Ref. [45] the Green's function was studied and it was used in a model of weakly coupled one-dimensional chains in order to study Mott metal-insulator transitions. In Ref. [29], the optical conductivity was calculated in this model and compared with experimental ARPES results on Bechgaard salts, which are believed to be quasi one-dimensional systems at temperatures above a "critical" scale.

In this chapter, I will be interested in the Schwinger's function of the $SU(2)$-Thirring model. The analysis is based, on the one hand, on expressions for the form factors of soliton-creating operators (or topologically charged fields) in the sine-Gordon model proposed some time ago [92][1], and on the other hand, on a conformal perturbative analysis of two-point correlation functions involving such fields. The form factor expressions can be used to obtain the long-distance behavior of these two-point functions, whereas conformal perturbation theory gives their short-distance expansion [144]. The interest in some of these topological fields stems from their role in fermionic theories; examples of such topological fields are given in (2.2.7) for a free massless theory. For instance, it is well-known that the sine-Gordon model is equivalent to the massive Thirring model [27]. The components of the Thirring fermion field are then associated with soliton-creating operators of topological charge ± 1 and Lorenz spin $\pm\frac{1}{2}$, and correlators of these operators in the sine-Gordon model are related to fermion correlators in the massive Thirring model [96]. More interestingly, the sine-Gordon theory is closely related to a model which is an integrable deformation of (2.2.8) [147]. This

[1]Without taking normalization into consideration, some of such form factors were considered previously in Refs. [7, 33]

"deformed" (or anisotropic) $SU(2)$-Thirring model also exhibits the spin-charge separation. The fermion field is likewise "factorized;" the spin part of the fermion field, which represents the charge part of the electrons from the connection to the Hubbard model, corresponds to soliton-creating operators of topological charge ± 1 and Lorenz spin $\pm\frac{1}{4}$ in the sine-Gordon model. The charge part of the fermion field, which is the spin part of the electrons, is related to similar operators in a free massless bosonic theory.

Although form factor expansions and conformal perturbation theory are very effective tools for the study of, respectively, the long-distance and the short-distance asymptotics of Schwinger's functions [13, 144, 8, 1], one usually gets into trouble when trying to compare both predictions in a region where they are expected to be accurate enough. Indeed, in general, one has the freedom of choosing the overall multiplicative normalization in the expansion arising from conformal perturbation theory as well as in the form factor expansion, and there is no systematic way of relating both normalizations. For the case of the soliton-creating operators, the constant relating both normalizations was conjectured in [92]. It allows one to make unambiguous numerical predictions on the correlation functions of soliton-creating fields on the whole distance scale using the combined conformal perturbation theory and form factor data. In this chapter, this calculation is performed for the case of the $SU(2)$-Thirring Fermi fields.

2.3 Bosonization of the anisotropic $SU(2)$-Thirring model

The $SU(2)$-invariant Thirring model admits an integrable generalization such that the underlying $SU(2)$ symmetry is explicitly broken down to $U(1) \otimes \mathbf{Z}_2$:

$$\mathcal{A}_{ATM} = \int d^2x \left\{ \sum_{\sigma=\uparrow,\downarrow} \bar{\Psi}_\sigma \gamma_\mu \partial^\mu \Psi_\sigma + \frac{\pi g_{||}}{8} J^3_\mu J^3_\mu + \frac{\pi g_\perp}{8} \left(J^1_\mu J^1_\mu + J^2_\mu J^2_\mu \right) \right\}, \quad (2.3.1)$$

where

$$J^A_\mu = \bar{\Psi} \gamma_\mu \tau^A \Psi \quad (2.3.2)$$

are vector currents (and, as before, τ^A are Pauli matrices). The model (2.3.1) is renormalizable, and its coupling constants $g_{||}$, $g_\perp$ should be understood as "running" ones. In particular, in the RG-invariant domain $g_{||} \geq |g_\perp|$, all RG trajectories originate from the line $g_\perp = 0$ of UV stable fixed points, and (2.3.1) indeed defines a quantum field theory[2]. Hence, in this domain (which is the only one that we discuss here), each RG trajectory is uniquely characterized by the limiting value

$$\rho = \frac{1}{2} \lim_{\ell \to 0} g_{||}(\ell) \tag{2.3.3}$$

of the running coupling $g_{||}(\ell)$ at extremely short distances (ℓ stands for the length scale), i.e. the theory (2.3.1) depends only on the dimensionless parameter ρ, besides the mass scale M appearing through dimensional transmutation.

As is well known (with similar arguments as in the previous section, see e.g. [58, 147]), the model (2.3.1) can be bosonized in terms of the sine-Gordon field $\varphi(x)$,

$$\mathcal{A}_{sG} = \int d^2x \left\{ \frac{1}{16\pi} (\partial_\nu \varphi)^2 - 2\mu \cos(\beta\varphi) \right\}, \tag{2.3.4}$$

with the coupling constant β in (2.3.4) related to ρ (2.3.3) by

$$\beta^2 = \frac{1}{1+\rho}, \tag{2.3.5}$$

and a free massless boson. Then the mass scale M is identified with the mass of the sine-Gordon solitons, which is related to the parameter μ by [142]

$$\mu = \frac{\Gamma(\frac{1}{1+\rho})}{\pi\Gamma(\frac{\rho}{1+\rho})} \left[M \, \frac{\sqrt{\pi}\Gamma(\frac{1}{2} + \frac{1}{2\rho})}{2\,\Gamma(\frac{1}{2\rho})} \right]^{\frac{2\rho}{1+\rho}} \tag{2.3.6}$$

under the "conformal normalization"

$$\langle \cos(\beta\varphi(x)) \cos(\beta\varphi(y)) \rangle \sim \frac{1}{2} |x-y|^{-4\beta^2} \quad \text{as} \quad x \to y \, . \tag{2.3.7}$$

---

[2]The Hamiltonians corresponding to opposite choices of the sign of $g_\perp$ are unitary equivalent, so the sign of this coupling does not affect the physical observables.

The precise operator relations between (2.3.1) and (2.3.3) can be found in [92]. In particular, for the two-point fermion correlator, the bosonization implies that

$$\langle\, \Psi_\sigma(x)\, \bar{\Psi}_{\sigma'}(0)\, \rangle = \frac{\delta_{\sigma'\sigma}}{2\pi}\, \frac{\gamma_\mu x^\mu}{|x|^{\frac{3}{2}}} \quad F^{(1)}_{1/4}(r)\ , \tag{2.3.8}$$

where we use the notation $F^{(n)}_\omega$ ($n = 1,\ \omega = 1/4$) for the real function which depends only on the distance $r = |x|$ (and implicitly on the the mass scale M and the parameter ρ), and which, in essence, coincides with the Euclidean correlator of nonlocal topologically charged fields in the model (2.3.4):

$$\langle\, \mathcal{O}^n_{-\omega\beta}(x)\, \mathcal{O}^{-n}_{\omega\beta}(0)\, \rangle = \left(e^{i\pi}\, \frac{\bar{\mathrm{z}}}{\mathrm{z}}\right)^{\omega n} \quad F^{(n)}_\omega(r)\,, \tag{2.3.9}$$

where $\mathrm{z} = \mathrm{x}^1 + i\mathrm{x}^2$, $\bar{\mathrm{z}} = \mathrm{x}^1 - i\mathrm{x}^2$. Again we refer the reader to the paper [92] for the precise definition of the field $\mathcal{O}^n_a$ ($a = \omega\beta$). Here we note that it carries an integer topological charge n, a scale dimension

$$d = \frac{2\omega^2}{1+\rho} + \frac{n^2}{8}\,(1+\rho)\ , \tag{2.3.10}$$

and a Lorentz spin ωn.

2.4 Short-distance expansion

In this section, the short-distance behavior of correlators (2.3.9) of the soliton-creating operators is examined by means of conformal perturbation theory. I then perform a renormalization group (RG) resummation of the perturbative expansion in the vicinity of the Kosterlitz-Thouless point which corresponds to the $SU(2)$ limit of the fermion theory.

2.4.1 Conformal perturbation theory

In general, one can examine the short-distance behavior of correlators via the operator product expansion, for instance:

$$F^{(n)}_\omega(r) = \mathbf{C}_{\mathbf{I}}(r) + \mathbf{C}_{\cos(\beta\varphi)}(r)\ \langle\, \cos(\beta\varphi)\, \rangle + \dots\ . \tag{2.4.1}$$

The structure functions ($\mathbf{C_I}(r)$, $\mathbf{C}_{\cos(\beta\varphi)}(r)$, etc.) admit power series expansions in μ^2, which can be obtained by using the standard rules of conformal perturbation theory [144, 60] (see also Section iii in the Introduction), whereas the vacuum expectation values of the associated operators are in general non-analytical at $\mu = 0$. In the perturbative treatment, we regard the sine-Gordon model (2.3.4) as a Gaussian conformal field theory

$$\mathcal{A}_{Gauss} = \int d^2x \, \frac{1}{16\pi} \, (\partial_\nu \varphi)^2 \tag{2.4.2}$$

perturbed by the relevant operator $\cos(\beta\varphi)$. Notice that in the limit $\mu \to 0$, the nonlocal topologically charged fields $\mathcal{O}^n_a$ can be expressed in terms of the right and left moving parts of a free massless field $\varphi = \varphi_R(\mathrm{z}) - \varphi_L(\bar{\mathrm{z}})$ governed by the action (2.4.2):

$$\mathcal{O}^n_a|_{\mu\to 0} \to \tilde{\mathcal{O}}^n_a = \exp\left\{ i\left(a - \frac{n}{4\beta}\right)\varphi_R(\mathrm{z}) - i\left(a + \frac{n}{4\beta}\right)\varphi_L(\bar{\mathrm{z}}) \right\} . \tag{2.4.3}$$

Conformal perturbation theory gives the structure function $\mathbf{C_I}$ (2.4.1) in the form

$$\left(e^{i\pi}\,\frac{\bar{\mathrm{z}}}{\mathrm{z}} \right)^{\omega n} \mathbf{C_I}(r) = \left\langle \tilde{\mathcal{O}}^{-n}_{\omega\beta}(x)\tilde{\mathcal{O}}^{n}_{-\omega\beta}(0) \exp\left(2\mu \int' d^2y \, \cos(\beta\varphi) \right) \right\rangle_{Gauss} , \tag{2.4.4}$$

where $\langle \ldots \rangle_{Gauss}$ is the expectation value in the Gaussian theory $\mathcal{A}_{Gauss}$ and the exponential is understood as a perturbative series in μ. In the perturbative series, the integrals will have *power law* IR divergences which should be thrown away [144]. Such a regularization prescription is indicated by the prime near the integral symbol. In the absence of logarithmic divergences, throwing away the divergences is equivalent to treating the integrals as analytical continuations in the field dimensions [144]. Considering only the part of $F^{(n)}_\omega$ perturbative in μ, it is a simple matter to obtain

$$F^{(n)}_\omega(r) = r^{-2d} \left\{ 1 + J_n(2\omega\beta^2, -2\beta^2)\, \mu^2\, r^{4-4\beta^2} + O\left(r^{8-8\beta^2}, r^2 \right) \right\} , \tag{2.4.5}$$

where d is given by (2.3.10) and

$$J_n(a,c) \;=\; \int' d^2x d^2y \; x^{a+\frac{n}{2}} \bar{x}^{a-\frac{n}{2}} \, (1-x)^{-a-\frac{n}{2}} (1-\bar{x})^{-a+\frac{n}{2}} \tag{2.4.6}$$

$$\times y^{-a-\frac{n}{2}} \bar{y}^{-a+\frac{n}{2}} \, (1-y)^{a+\frac{n}{2}} (1-\bar{y})^{a-\frac{n}{2}} \, |x-y|^{2c} \, . \tag{2.4.7}$$

Two comments are in order here. First, the next omitted term in the short-distance expansion (2.4.5) comes from either the next term in the perturbative series for $\mathbf{C_I}$ ($O(r^{8-8\beta^2})$) or from the leading contribution of $\cos(\beta\varphi)$ ($O(r^2)$) in (2.4.1). Therefore, the μ^2 term written in (2.4.5) is a leading correction to the scale invariant part of the correlation function for $\frac{1}{2} < \beta^2 < 1$ only. Second, in writing (2.4.5) we specify the overall multiplicative normalization of the nonlocal topologically charged field $\mathcal{O}^n_{\omega\beta}$ by the condition

$$F^{(n)}_\omega(r) \to r^{-2d} \qquad \text{as} \quad r \to 0 \ . \tag{2.4.8}$$

The integral (2.4.6) can be calculated using, for instance, techniques illustrated in [97]. The result can be expressed in terms of two generalized hypergeometric functions at unity:

$$\begin{aligned} A(q,c) &= {}_3F_2(-c,-c-1,1-q;-c-q,2;1) \\ B(q,c) &= {}_3F_2(q,q+1,c+2;c+q+2,c+q+3;1) \ . \end{aligned} \tag{2.4.9}$$

With $q = a + n/2$ and $\bar{q} = a - n/2$, we found:

$$J_n(a,c) = J^{(1)} + J^{(2)} + J^{(3)} + J^{(4)} \ , \tag{2.4.10}$$

where

$$\begin{aligned} J^{(1)} &= q\bar{q}\,\Gamma(1-q)\Gamma(1-\bar{q})\,\Gamma(1+c+q)\Gamma(1+c+\bar{q})\Gamma^2(-1-c)\,\times \\ &\quad (\,\cos(\pi(q-\bar{q})) - \cos(\pi c)\cos(\pi(q+\bar{q}+c)))\ A(q,c)A(\bar{q},c)\,, \\ J^{(2)} &= \frac{\pi^2 q\Gamma(1+c)\Gamma(1+\bar{q})\Gamma(1+c+q)\Gamma(-1-c-\bar{q})}{\Gamma(q)\Gamma(-\bar{q})\Gamma(3+c+\bar{q})}\ A(q,c)B(\bar{q},c)\,, \\ J^{(3)} &= \frac{\pi^2 \bar{q}\Gamma(1+c)\Gamma(1+q)\Gamma(1+c+\bar{q})\Gamma(-1-c-q)}{\Gamma(\bar{q})\Gamma(-q)\Gamma(3+c+q)}\ B(q,c)A(\bar{q},c)\,, \\ J^{(4)} &= -\frac{\pi^2\Gamma^2(1+q)\Gamma^2(2+c)\Gamma(-1-c-\bar{q})\Gamma(-2-c-\bar{q})}{\Gamma^2(-\bar{q})\Gamma^2(-c)\Gamma(2+c+q)\Gamma(3+c+q)}\ B(q,c)B(\bar{q},c)\,. \end{aligned}$$

Notice that for $n = 0$, the integral (2.4.6) was calculated previously in the work [36] (see also Ref. [61]).

2.4.2 Renormalization group resummation

The perturbative series in the previous subsection was obtained by expanding in powers of μ the action (2.3.4). This action is the bosonized form of the isospin sector of the model (2.3.1) (recall that the index σ in the Fermi field Ψ_σ is an isospin index in the connection to the Hubbard model that we consider).

In fact, the parameters μ and β can be seen as forming a convenient system of coordinates in the space of parameters of the model (2.3.1). As is clear from the calculation above, this system of coordinates is defined through the ultraviolet regularization procedure obtained by considering the model (2.3.1) as a perturbation by the term $(\pi\tilde{g}_\perp/8)\int d^2x\,\left(J^1_\mu J^1_\mu + J^2_\mu J^2_\mu\right)$ of the fix-point theory (Luttinger liquid)

$$\int d^2x\,\left\{\sum_{\sigma=\uparrow,\downarrow}\bar{\Psi}_\sigma\gamma_\mu\partial^\mu\Psi_\sigma + \frac{\pi\tilde{g}_{||}}{8}\,J^3_\mu J^3_\mu\right\} \tag{2.4.11}$$

(here we use the notation $\tilde{g}_{||}$ and $\tilde{g}_\perp$ for this system of coordinates, reserving $g_{||}$ and $g_\perp$ for other coordinates defined below). This fix-point theory is equivalent to a free massless boson $\frac{1}{16\pi}\int d^2x\,(\partial_\nu\varphi)^2$, and in this bosonic language the perturbing field $J^1_\mu J^1_\mu + J^2_\mu J^2_\mu$ is (up to positive normalization) the cosine field $-\cos(\beta\varphi)$ with $\beta^2 = 1/(1+\tilde{g}_{||}/2)$.

All needed ultraviolet regularization associated to the perturbation about a free massless Dirac theory by the field $J^3_\mu J^3_\mu$ are simply embedded into anomalous dimensions of fields in the fix-point theory (2.4.11). As for the remaining ultraviolet regularization associated to the perturbing field $J^1_\mu J^1_\mu + J^2_\mu J^2_\mu$, note that the field $\cos(\beta\varphi)$ has dimension $2\beta^2$ in the fix-point theory. Hence for $\tilde{g}_{||} > 0$, the perturbing field has dimension less than 2, so that the theory is super-renormalizable. Then one only need normal-order (with respect to the ground state of the fix-point theory) the perturbing operator $\cos(\beta\varphi)$ in the corresponding Hamiltonian in order to get rid of all remaining ultraviolet divergencies (the normalization of the cosine operators (2.3.7) completely fixes this normal ordering). That is, in this system of coordinates, $\tilde{g}_{||} = 2(\beta^{-2} - 1)$ does not flow and $\tilde{g}_{\perp} \propto \mu$ flows trivially according to the scale dimension $2 - 2\beta^2$:

$$\ell \frac{d}{d\ell} \tilde{g}_{||} = 0 \quad \ell \frac{d}{d\ell} \tilde{g}_{\perp} = \frac{\tilde{g}_{||} \tilde{g}_{\perp}}{1 + \tilde{g}_{||}/2} \tag{2.4.12}$$

where ℓ is a distance scale.

We are interested in the short-distance expansion of the correlator (2.3.9) for β^2 sufficiently close to unity. As will become clear below, at $\beta^2 = 1$ one recovers the $SU(2)$-invariant model (2.2.8). In this case there are new (logarithmic) ultraviolet divergences, and the system of coordinates formed by β and μ as described above is singular.

It is convenient to use the notation

$$\epsilon = 1 - \beta^2 \ll 1 \ . \tag{2.4.13}$$

Our previous short-distance analysis suggests the following expansion for the structure function $\mathbf{C_I}$:

$$\mathbf{C_I}(r) = r^{2d} \left\{ 1 + \sum_{k=1}^{\infty} c_k \left(\mu r^{2\epsilon} \right)^{2k} \right\} , \tag{2.4.14}$$

where the coefficients c_k are given by certain $4k$-fold Coulomb-type integrals. Evidently, this expansion cannot be directly applied in the limit $\epsilon \to 0$, where the perturbation $\cos(\beta\varphi)$ of the Gaussian action (2.4.2) becomes marginal. However, being expressed as a function of the scaling distance Mr, the structure function $\mathbf{C_I}(r)$ should admit the following form:

$$\mathbf{C_I}(r) = \ Z_{n,\omega} \ \mathbf{C_I^{(ren)}}(Mr) \ , \tag{2.4.15}$$

where the r-independent renormalization constant $Z_{n,\omega}$ absorbs all divergences at $\epsilon = 0$ and renders the renormalized structure function $\mathbf{C_I^{(ren)}}$ finite in this limit. The divergences of the renormalization constant $Z_{n,\omega}$ should be directly related to the singularities of $\mathbf{C_I^{(ren)}}$ at $Mr = 0$; they point out that the power law asymptotic behavior (2.4.5) is modified by logarithmic corrections at $\epsilon = 0$.

Note that the use of the deformation (2.3.1) of the model (2.2.8) along with a singular system of coordinates where the renormalization group flow is that of a super-renormalizable theory can be understood as a *nonperturbatively defined* dimensional regularization for the model (2.2.8). The parameter ϵ is the small parameter of this dimensional regularization, and the form (2.4.15) for the structure function is the expected form where the renormalization constant $Z_{n,\omega}$ absorbs all ultraviolet ($\epsilon \to 0$) divergences.

In order to explore the short-distance behavior for $\epsilon \ll 1$, it is convenient to return to the fermion description, and to define an appropriate system of coordinates in the coupling space. Being essentially the corresponding structure function in the renormalizable QFT (2.3.1), $\mathbf{C_I}(r)$ obeys the Callan-Symanzik equation. Therefore it can be written in the form:

$$\mathbf{C_I}(r) = r^{-2d} \ \exp\Big\{ -2 \int_0^r \frac{dr}{r} \, (\, \Gamma_g - d \,) \Big\} \, . \tag{2.4.16}$$

Here the function Γ_g is supposed to have a regular power series expansion in terms of the running coupling constants $g_{||,\perp} = g_{||,\perp}(r)$:

$$\Gamma_g = \sum_{l,k=0}^{\infty} \gamma_{lk} \; g_{||}^{l} \, g_{\perp}^{2k} \tag{2.4.17}$$

where γ_{lk} are constant coefficients. Notice that only even powers of the coupling $g_\perp$ appear in this expansion (see footnote #3). In writing (2.4.16), we use the normalization condition (2.4.8), and take into account that the UV limiting value of Γ_g coincides with the scale dimension (2.3.10),

$$\lim_{r\to 0} \Gamma_g = d \, . \tag{2.4.18}$$

We have also assumed that there is no resonance mixing of the operator $\mathcal{O}^n_{\omega\beta}$ with other fields, so it is renormalized as a singlet. One can easily check that this is indeed the case for the operators with $|\omega| < \frac{1}{2} + \frac{|n|}{4}$.

Condition (2.4.18) already encloses an important restriction on the series (2.4.17). Indeed, using Eqs. (2.3.3) and (2.3.10) along with the condition that the line of UV stable fixed points corresponds to $g_\perp = 0$, one obtains

$$\Gamma_g = \Gamma^{(0)}(g_{||}) + \Gamma^{(1)}(g_{||})\, g_\perp^2 + \Gamma^{(2)}(g_{||})\, g_\perp^4 + O(g_\perp^6)\ , \tag{2.4.19}$$

where

$$\Gamma^{(0)}(g_{||}) = \frac{2\omega^2}{1+\frac{g_{||}}{2}} + \frac{n^2}{8}\left(1+\frac{g_{||}}{2}\right)\ . \tag{2.4.20}$$

The values of the other coefficients $\gamma_{l,k\geq 1}$ appearing in (2.4.17) essentially depend on the choice of a renormalization scheme, i.e on the precise specification of the running coupling constants. The latter obey the RG equations

$$r\,\frac{dg_{||}}{dr} = \frac{g_\perp^2}{f_{||}(g_{||},g_\perp)}\,, \qquad r\,\frac{dg_\perp}{dr} = \frac{g_{||}g_\perp}{f_\perp(g_{||},g_\perp)}\ . \tag{2.4.21}$$

Perturbatively, $f_{||}(g_{||},g_\perp)$ and $f_\perp(g_{||},g_\perp)$ admit loop expansions as power series in $g_{||}$ and $g_\perp$. In this work, we will use the scheme introduced by Al.B. Zamolodchickov [142, 140]. First note that from arbitrary power series $f_{||}(g_{||},g_\perp)$ and $f_\perp(g_{||},g_\perp)$, it is always possible to find a diffeomorphism in $g_{||}$ and $g_\perp$ such that in (2.4.21) we have

$$f_{||}(g_{||},g_\perp) = f_\perp(g_{||},g_\perp) = f(g_{||})\ .$$

In such a system of coordinates the line $g_{||} = g_\perp$ is preserved under the renormalization group and corresponds to the $SU(2)$-invariant situation. Next observe that at very short distance scales $r \to 0$, we have $g_\perp \to 0$ and $g_{||} \to 2\rho$ so that in that region,

$$r\,\frac{dg_{||}}{dr} = O(g_\perp^2)\ , \qquad r\,\frac{dg_\perp}{dr} = \frac{2\rho\, g_\perp}{f(2\rho)} + O(g_\perp^3) \qquad (r\to 0)\ . \tag{2.4.22}$$

This renormalization group flow should be the same, in the region $r \to 0$, as that for any system of coordinates sharing the same limiting values 2ρ of $g_{||}$ and 0 of $g_\perp$ as $r \to 0$. In particular, comparing with the renormalization group flow (2.4.12) for the system of coordinates formed by $\tilde g_{||}$, $\tilde g_\perp$ discussed above, we find

$$f(g_{||}) = 1 + \frac{g_{||}}{2}\ . \tag{2.4.23}$$

Note that the fact that we have a line of fixed points allowed us to determine a regularization scheme where the beta function is known to all orders.

It is in fact possible to exhibit quite explicitly a transformation of coordinates that brings (2.4.21) with (2.4.23) to coordinates $\tilde{g}_{||}$, $\tilde{g}_{\perp}$ obeying (2.4.12). Indeed, this can be achieved by

$$g_{||} = \tilde{g}_{||} + h_2(\tilde{g}_{||})\,\tilde{g}_{\perp}^2 + O(\tilde{g}_{\perp}^4)\,, \quad g_{\perp} = h_1(\tilde{g}_{||})\,\tilde{g}_{\perp} + h_3(\tilde{g}_{||})\,\tilde{g}_{\perp}^3 + O(\tilde{g}_{\perp}^5)$$

where all h_n are determined once we fix $h_1 = O(\tilde{g}_{||})$. For instance,

$$2\tilde{g}_{||}h_2 = h_1^2\,, \quad 2\tilde{g}_{||}h_3 = \frac{h_1 h_2}{1+\tilde{g}_{||}/2}\,, \ldots$$

and we can choose $h_1 = \tilde{g}_{||}$. Since we know that there exists a regularization scheme leading to coordinates described by the flow (2.4.12) in the model (2.3.1), then the change of coordinates above shows that there exists another regularization scheme leading to coordinates described by the flow (2.4.21) with (2.4.23).

With the choice (2.4.23) for the β-function, the RG equations (2.4.21) can be integrated. To do this, we note that this system of differential equations has a first integral, the numerical value of which is determined through the condition (2.3.3),

$$g_{||}^2 - g_{\perp}^2 = (2\rho)^2\,. \tag{2.4.24}$$

Using (2.4.24), (2.4.13) and (2.3.5), equations (2.4.21) are solved as

$$g_{||} = 2\rho\,\frac{1+q}{1-q}\,, \quad g_{\perp} = \rho\,\frac{4\sqrt{q}}{1-q}\,, \tag{2.4.25}$$

where

$$q\left(\frac{1-q}{\rho}\right)^{-2\epsilon} = (r\Lambda)^{4\epsilon}\,. \tag{2.4.26}$$

The normalization scale Λ is another integration constant of the system (2.4.21). It is of the order of the physical mass scale and supposed to have a regular loop expansion,

$$\Lambda = M\;\exp\left(\tau_0 + \tau_1\rho + \tau_2\rho^2 + \ldots\right)\,. \tag{2.4.27}$$

It should be noted that the even coefficients τ_0, $\tau_2, \ldots$ are essentially ambiguous and can be chosen at will. A variation of these coefficients corresponds to a smooth redefinition of the coupling constants which does not affect the β-function. By contrast, the odd constants τ_{2k+1} are unambiguous and precisely specified once the form of the RG equations is fixed. It is possible to show [142, 140] that the odd constants vanish in Zamolodchikov's scheme:

$$\tau_{2k+1} = 0 \qquad (k = 0, 1 \ldots) \ .$$

Once the coefficients τ_{2k} in (2.4.27) are chosen, the running coupling constants are completely specified, and all coefficients in the power series expansion (2.4.17) are determined unambiguously. They can be explicitly calculated by comparing the conformal perturbative result (2.4.5) with the form (2.4.16). From (2.4.16),

$$\Gamma_g = -\frac{1}{2}\, r\, \partial_r \log\left(\mathbf{C_I}\right) \tag{2.4.28}$$

and, as it follows from the general conformal perturbative expansion (2.4.14) and the definition (2.4.26) of q, the function Γ_g can be expanded in powers of q. Explicitly, using the conformal perturbative result (2.4.5),

$$\Gamma_g = d - 2\,\epsilon \left(\frac{\sqrt{\rho}}{\Lambda}\right)^{4\epsilon} \mu^2\ J_n(2\omega(1-\epsilon), 2\epsilon - 2)\ q + O(q^2)\ . \tag{2.4.29}$$

Moreover, the coefficients in this expansion are power series in ρ. For example, using Eqs. (2.3.6) and (2.4.27), it is easy to show that

$$\frac{\pi\mu}{\epsilon}\left(\frac{\sqrt{\rho}}{\Lambda}\right)^{2\epsilon} = \exp\left\{-2\bar{\tau}_0\rho + \left(2\bar{\tau}_0 - \frac{1}{2}\right)\rho^2 - \left(2\tau_2 + 2\bar{\tau}_0 - \frac{2}{3}\,\zeta(3) - \frac{1}{2}\right)\rho^3 + O(\rho^4)\right\}. \tag{2.4.30}$$

Here and after, we set for convenience

$$e^{\tau_0} = \sqrt{\frac{\pi}{8}}\ \ e^{\gamma_E + \bar{\tau}_0}\ , \tag{2.4.31}$$

where $\gamma_E = 0.5772\ldots$ is the Euler constant. The integral $J_n(2\omega(1-\epsilon), 2\epsilon - 2)$ appearing in (2.4.29) can also be expanded in powers of ρ, using $\epsilon = \rho/(1 +$

ρ). In Appendix A.1, we quote the first few terms in the expansion of $J_n(a,c)$ (2.4.6) around $c=-2$, which are obtained through the use of (2.4.10). From this expansion, it is easy to obtain the expansion of $J_n(2\omega(1-\epsilon), 2\epsilon-2)$ in powers ρ. Then, one can compare the conformal perturbative expansion of Γ_g in q and ρ (2.4.29) with the corresponding expansion (2.4.17) coming from the RG analysis (where of course one should expand $g_{||}$ and $g_\perp^2$ in q and ρ from (2.4.25)). This determines the coefficients $\gamma_{l,1}$ for $l=0,1,2$. If we want an expression valid to order g^4, we need one more coefficient: $\gamma_{0,2}$. In principle, it can be obtained from the expansion in ρ of the coefficients c_2 in the series (2.4.14). In Section 5, we describe a way to find $\gamma_{0,2}$ without the cumbersome calculation beyond the lowest order in conformal perturbation theory.

In order to simplify the form of the structure function (2.4.16), it is convenient, instead of using the coefficients $\gamma_{l,k}$, to parametrize the first few terms of the power series expansions $\Gamma^{(1,2)}(g_{||})$ (2.4.19) as:

$$\begin{aligned}
\Gamma^{(1)}(g_{||}) &= -\frac{1}{1+\frac{g_{||}}{2}}\ \left\{ \frac{n^2}{32} - \frac{u_1}{2} + v_1\, g_{||} + \left(v_2 - \frac{3u_2}{2}\right) g_{||}^2 + O(g_{||}^3) \right\} , \\
\Gamma^{(2)}(g_{||}) &= -\frac{v_2}{2} + O(g_{||}) \; . \qquad (2.4.32)
\end{aligned}$$

The explicit values of the coefficients u_1, u_2, v_1 and v_2 in (2.4.32) are given in Appendix B.

Let us substitute (2.4.19) and (2.4.32) into Eq. (2.4.16). The RG flow equations (2.4.21) allow one to evaluate the integral and to write the structure function in the form (2.4.15) with

$$\begin{aligned}
\mathbf{C}_{\mathbf{I}}^{(ren)} &= (Mr)^{-4\omega^2 - n^2(1+\rho^2)/4}\ \ (g_\perp^2)^{\omega^2 - n^2(1-\rho^2)/16} \\
&\quad \times e^{-u_1 g_{||} - u_2 g_{||}^3}\ \Big(1 + g_\perp^2 (v_1 + v_2 g_{||}) + O(g^4)\Big) \ , \qquad (2.4.33)
\end{aligned}$$

and

$$Z_{n,\omega} = M^{2d}\ \left(2^{\rho+1}\sqrt{\rho}\, e^{\tau_0\rho + \tau_2\rho^3 + \dots}\right)^{n^2/2-2d}\quad e^{2\rho u_1 + (2\rho)^3 u_2 + \dots} \ . \qquad (2.4.34)$$

Notice that the transformation

$$Z_{n,\omega} \to e^{w_0+w_1(2\rho)^2+w_2(2\rho)^4+\dots} \;\; Z_{n,\omega} \ , \tag{2.4.35}$$

where the series contains only even powers of ρ with arbitrary coefficients w_k, accompanied by the transformation

$$\mathbf{C}_{\mathbf{I}}^{(ren)} \to e^{-w_0-w_1(g_{\|}^2-g_{\perp}^2)-w_2(g_{\|}^2-g_{\perp}^2)^2+\dots} \;\; \mathbf{C}_{\mathbf{I}}^{(ren)}$$

does not affect the structure function $\mathbf{C_I}$ (2.4.15) due to relation (2.4.24).

Our prime interest in this work is the correlation function (2.3.8). For $n = 1$ and $\omega = \frac{1}{4}$, the relations obtained above lead to the following perturbative expansion for the two-point fermion correlator in the anisotropic $SU(2)$-Thirring model:

$$\begin{aligned} \langle\, \Psi_{\sigma'}(x)\, \bar{\Psi}_\sigma(0)\,\rangle &= \frac{Z_\Psi \delta_{\sigma'\sigma}}{2\pi}\ \frac{\gamma_\mu x^\mu}{|x|^{2+\frac{\rho^2}{4}}}\ (g_\perp^2)^{\frac{\rho^2}{16}}\ \exp\Big\{-\frac{3}{16}\,g_{\|}-\frac{\bar{\tau}_0}{32}\,g_{\|}^3\Big\} \\ &\times \exp\Big\{\frac{3}{16}\Big(\bar{\tau}_0-\frac{1}{4}\Big)\,g_\perp^2-\frac{3}{16}\Big(\bar{\tau}_0^2-\frac{1}{6}\,\bar{\tau}_0-\frac{1}{16}\Big)\,g_{\|}g_\perp^2+O(g^4)\Big\}\ , \end{aligned} \tag{2.4.36}$$

where

$$Z_\Psi = (4\rho)^{-\frac{\rho^2}{8(1+\rho)}}\ \Big(M\sqrt{\frac{\pi}{2}}\Big)^{-\frac{\rho^3}{4(1+\rho)}}\ \exp\Big\{\frac{3\rho}{8}-\frac{\gamma_E}{4}\,\rho^3+O(\rho^4)\Big\}\ .$$

In Eq. (2.4.36), we use the notation $\bar{\tau}_0$ defined by (2.4.31).

We now set $\rho = 0$ and $g_{\|} = g_\perp = g$ in (2.4.36) to obtain the perturbative expansion of the scaling function F (2.2.9) for the $SU(2)$-Thirring model,

$$F^{(pert)} = \exp\Big\{-\frac{3}{16}\,g+\frac{3}{16}\Big(\bar{\tau}_0-\frac{1}{4}\Big)\,g^2-\frac{3}{16}\Big(\bar{\tau}_0^2-\frac{1}{16}\Big)\,g^3+O(g^4)\Big\}\ . \tag{2.4.37}$$

Here the running coupling constant g solves the equation

$$-g^{-1}+\frac{1}{2}\,\ln(g) = \ln\Big(\sqrt{\frac{\pi}{2}}\,e^{\gamma_E+\bar{\tau}_0}\,Mr\Big)\ , \tag{2.4.38}$$

which is the limit $\rho = 0$ of Eqs. (2.4.25) and (2.4.26).

Let us stress here that, if the perturbation series could be summed, then the function F should not depend on the auxiliary parameter $\bar{\tau}_0$:

$$\frac{\partial F}{\partial \bar{\tau}_0} = 0 \ .$$

This is, however, not true if we truncate the series (2.4.37) at some order N (for instance, if one leaves only the terms explicitly written in (2.4.37)). In this case,

$$\frac{\partial}{\partial \bar{\tau}_0} F_N^{(pert)} = O(g^{N+1}) \ ,$$

where the truncated series is denoted by $F_N^{(pert)}$. In fitting numerical data with (2.4.37), we may treat $\bar{\tau}_0$ as an optimization parameter, allowing us to minimize or at least develop a feeling for the effects of the remainder of the series. Similar ideas have been discussed for QCD in Ref. [121].

2.5 Perturbative expansion in momentum-space

Perturbative calculations of fermion Green's functions in renormalizable 2D models with four-fermion interaction are widely covered in the literature (see [11, 2] and references therein). The results in this domain are usually expressed in momentum space. Hence it seems appropriate at this point to adapt the calculation of the previous section to the momentum-space fermion correlator, giving a large-momentum expansion. Here we give the two-point function in the $SU(2)$-Thirring model to third order in the running coupling.

The RG analysis performed in the previous section can be applied in essentially the same way to the Fourier transform of the fermion correlator (2.3.8):

$$\int d^2x \, e^{-ipx} \, \langle \, \Psi_\sigma(x) \, \bar{\Psi}_{\sigma'}(0) \, \rangle = -i \, \delta_{\sigma\sigma'} \, \frac{\gamma^\mu p_\mu}{p^2} \quad \tilde{F}(p^2) \ . \qquad (2.5.1)$$

Here and after we use the notation $p^2 = p^\mu p_\mu$. From the result of conformal perturbation theory, (2.4.5), one can immediately obtain the large momentum expansion of this Fourier transform. The RG analysis in momentum space goes as in the previous section. The perturbative part in μ of $\tilde{F}$ obeys the Callan-Symanzik equation, so it can be written as

$$\tilde{F}^{(pert)} = Q(d_\Psi)\ (p^2)^{d_\Psi - \frac{1}{2}}\ \exp\left\{ - \int_{p^2}^{\infty} \frac{ds}{s} (\tilde{\Gamma}_g - d_\Psi) \right\} , \tag{2.5.2}$$

where the function $\tilde{\Gamma}_g$ admits a power series expansion in terms of the momentum-space running coupling constants $g_{||,\perp} = g_{||,\perp}(p^2)$ depending on the Lorentz invariant p^2:

$$\tilde{\Gamma}_g = \sum_{l,k=0}^{\infty} \tilde{\gamma}_{l,k}\ g_{||}^l g_{\perp}^{2k} . \tag{2.5.3}$$

Here

$$Q(a) = 2^{1-2a}\ \frac{\Gamma(\frac{3}{2} - a)}{\Gamma(\frac{1}{2} + a)} ,$$

and

$$d_\Psi = \frac{1}{2} + \frac{\rho^2}{4(1+\rho)}$$

is the scale dimension of the fermion field. Notice that, with some abuse of notations, we use here the same symbols $g_{||,\perp}$ for the momentum-space running couplings as we used for the coordinate-space running couplings. In order to fix the coefficients in (2.5.3), we have to choose a renormalization scheme. Substituting r by $1/\sqrt{p^2}$ in (2.4.26) defines Zamolodchikov's scheme in momentum space. It is a simple matter to repeat the steps of the previous section in order to determine the first few coefficients $\tilde{\gamma}_{l,1}$ in (2.5.3). As for the coefficients $\tilde{\gamma}_{l,2}$, one would in principle need the next order in conformal perturbation theory. However, again as in the previous section, it is possible to determine $\tilde{\gamma}_{0,2}$ without this calculation, as described in the next section. From these coefficients, and from the form of the RG flow equation, one can evaluate the integral in (2.5.2) and obtain the asymptotic behavior of the two-point function in the Euclidean region at $p^2 \to +\infty$. I quote here the result in the case of the $SU(2)$-Thirring model,

$$\tilde{F}^{(pert)} = \exp\left\{ -\frac{3}{16} g + \frac{3}{16}\left(\tilde{\tau}_0 - \frac{1}{4}\right) g^2 - \frac{3}{16}\left(\tilde{\tau}_0^2 - \frac{1}{16}\right) g^3 + O(g^4) \right\} . \tag{2.5.4}$$

Here

$$-g^{-1} + \frac{1}{2}\ln(g) = \ln\left(\sqrt{2\pi} M\, e^{\tilde{\tau}_0} / \sqrt{p^2} \right) , \qquad (2.5.5)$$

and $\tilde{\tau}_0$ is an arbitrary parameter which can be chosen at will. Notice the strong similarity between (2.5.4) and (2.4.37).

I also quote here the corresponding function $\tilde{\Gamma}_g$ (2.5.3) in the case $g_{||} = g_{\perp}$:

$$\tilde{\Gamma}_g = \frac{1}{2} + \frac{3}{32} g^2 - \frac{3}{16} \tilde{\tau}_0\, g^3 + \frac{3}{32} \left(3\, \tilde{\tau}_0^2 + \tilde{\tau}_0 - \frac{3}{16} \right) g^4 + O(g^5) \; . \qquad (2.5.6)$$

In [2], the anomalous dimension for the fermion field in the $\overline{\mathrm{MS}}$ scheme was found to fourth order for a general non-abelian Thirring model (see also [11] and references therein for a discussion of various aspects of dimensional regularization in the non-abelian Thirring model and for results to lower order). In contrast, we have calculated, in coordinate space, the two-point functions of more general operators, including the fermion fields, in the particular case of the $SU(2)$-Thirring model (and an anisotropic deformation of it), and we have sketched the equivalent calculation in momentum space for the fermion fields. We would now like to compare Eq. (2.5.4) with the $SU(2)$ case of the Ali-Gracey result [2]. In order to perform the comparison, we need to find the relation between our running coupling constant g and theirs, which will be denoted $g_{AG} = -\lambda$ [3], and then find the relation between our function $\tilde{\Gamma}_g$ (2.5.2) and their anomalous dimension, which we will denote γ_λ.

The coupling λ corresponds to the $\overline{\mathrm{MS}}$ scheme; the associated β-function was found in [11] to fourth order:

$$2\, p^2 \frac{d\lambda}{dp^2} = \beta_\lambda = -\frac{\lambda^2}{\pi} + \frac{\lambda^3}{2\pi^2} - \frac{83}{128\pi^3}\, \lambda^4 + O(\lambda^5) \; . \qquad (2.5.7)$$

By comparison, in the scheme that we use, the β-function (2.4.21), (2.4.23) is

$$2\, p^2 \frac{dg}{dp^2} = \beta_g = -\frac{g^2}{1 + g/2} = -g^2 + \frac{g^3}{2} - \frac{g^4}{4} + O(g^5) \; . \qquad (2.5.8)$$

[3] Notice that in [2], the coupling constant g_{AG} is assumed to be negative, so $\lambda > 0$, which agrees with the sign of our coupling constant g.

The difference in the factor multiplying the square of the coupling in these two expressions results only from a different normalization of the coupling in the action (see Eq. (2.2.8)). The relation between the couplings g and λ that corresponds to these different β-functions is

$$\frac{\lambda}{\pi} = g - \tau\, g^2 + \Big(\tau^2 + \frac{\tau}{2} + \frac{51}{128}\Big)\, g^3 + O(g^4)\ . \tag{2.5.9}$$

Here τ is some numerical factor which cannot be determined by comparing the β-functions: its variation modifies the choice of the normalization scale and doesn't affect the β-functions. The normalization scale for the $\overline{\mathrm{MS}}$ scheme is defined by imposing the following condition on the subleading asymptotics of the solution of the RG flow equation (2.5.7):

$$\frac{\lambda}{\pi} = \frac{1}{\ln\big(\sqrt{p^2}/\Lambda_{\overline{\mathrm{MS}}}\big)} + \frac{1}{2}\,\frac{\ln\ln(\sqrt{p^2}/\Lambda_{\overline{\mathrm{MS}}})}{\ln^2(\sqrt{p^2}/\Lambda_{\overline{\mathrm{MS}}})} + O\Big(\frac{\ln^2\ln(\sqrt{p^2}/\Lambda_{\overline{\mathrm{MS}}})}{\ln^3(\sqrt{p^2}/\Lambda_{\overline{\mathrm{MS}}})}\Big)\ . \tag{2.5.10}$$

(This implies that the term $O(1/\ln^2(\sqrt{p^2}/\Lambda_{\overline{\mathrm{MS}}}))$ does not appear in the expansion of λ.) From (2.5.5), (2.5.9) and (2.5.10), we find that

$$\Lambda_{\overline{\mathrm{MS}}} = \sqrt{2\pi} M\ e^{\tilde{\tau}_0 - \tau}\ . \tag{2.5.11}$$

In [2], the perturbative part of the function $\tilde{F}$ (2.5.1) was calculated up to the overall multiplicative normalization to third order in λ. The result can be written in the following form

$$\tilde{F}^{(pert)} \propto \frac{1}{h_\lambda}\ \exp\Big\{-\frac{1}{2}\int^{p^2}\frac{ds}{s}\,\gamma_\lambda\Big\}\ ,$$

where the function h_λ and the anomalous dimension γ_λ were given in [2] to fourth order in λ for the Thirring model with a general non-abelian symmetry. In the particular case of the $SU(2)$-symmetry, they specialize to

$$h_\lambda = 1 + \frac{15}{128\pi^2}\,\lambda^2 - \frac{11}{512\pi^3}\,\lambda^3 + \frac{3(80\zeta(3) - 511)}{32768\,\pi^4}\,\lambda^4 + O(\lambda^5)\,, \tag{2.5.12}$$

and

$$\gamma_\lambda = -\frac{3}{16\pi^2}\,\lambda^2 + \frac{15}{64\pi^3}\,\lambda^3 + \frac{3}{1024\pi^4}\,\lambda^4 + O(\lambda^5)\ . \tag{2.5.13}$$

Comparing (2.5.2) in the case $\rho = 0$ with the above expressions, one has the following relation:

$$\tilde{\Gamma}_g = \frac{1}{2} - \frac{\gamma_\lambda}{2} - \frac{\beta_\lambda}{2} \frac{d}{d\lambda} \log(h_\lambda) \ . \qquad (2.5.14)$$

Using Eqs. (2.5.9)-(2.5.13), one can check that our result (2.5.6) agrees with (2.5.14), provided that

$$\tau = \tilde{\tau}_0 \ . \qquad (2.5.15)$$

Notice that the relation between the normalization scale $\Lambda_{\overline{\mathrm{MS}}}$ and M,

$$\Lambda_{\overline{\mathrm{MS}}} = \sqrt{2\pi} M \ , \qquad (2.5.16)$$

which is a consequence of (2.5.11) and (2.5.15), was previously found in Ref. [130].

2.6 Long-distance behavior

Here we concentrate on the long-distance behavior of Schwinger's function (2.3.9) for $n = 1$ and $\omega = 1/4$. Let us recall that for $\frac{1}{2} < \beta^2 \leq 1$, there are only solitons and antisolitons in the spectrum of the sine-Gordon model. We will denote them by A_- and A_+, respectively. The conservation of the topological charge,

$$\frac{\beta}{2\pi} \int_{-\infty}^{\infty} d\mathrm{x}\, \partial_{\mathrm{x}} \varphi \ ,$$

implies that the non-vanishing form factors of the operator $\mathcal{O}^{+1}_{\beta/4}$ are of the form

$$\langle \, \mathrm{vac} \mid \mathcal{O}^{+1}_{\beta/4}(0) \mid A_-(\theta_1) \cdots A_-(\theta_{N+1})\, A_+(\theta'_1) \cdots A_+(\theta'_N) \, \rangle \ , \qquad (2.6.1)$$

where θ_i and θ'_j denote rapidities of solitons and antisolitons, respectively. Up to an overall normalization, all these form factors can be written down in closed form, as certain N-fold integrals [118, 89, 90, 7]. The spectral decomposition for the correlation function (2.3.8) then gives

$$F^{(1)}_{1/4}(r) \ = \ \int_{-\infty}^{+\infty} \frac{d\theta}{2\pi} \, e^{-Mr\cosh(\theta)} \, |\langle \, \mathrm{vac} \mid \mathcal{O}^{+1}_{\beta/4}(0) \mid A_-(\theta) \, \rangle|^2 \qquad (2.6.2)$$

$$+\frac{1}{3!}\int_{-\infty}^{\infty}\frac{d\theta_1 d\theta_2 d\theta_3}{(2\pi)^3}e^{-Mr\sum_{k=1}^{3}\cosh(\theta_k)}$$
$$\times\sum_{\sigma_1+\sigma_2+\sigma_3=-1}|\langle\,\mathrm{vac}\mid\mathcal{O}^{+1}_{\beta/4}(0)\mid A_{\sigma_1}(\theta_1)A_{\sigma_2}(\theta_2)A_{\sigma_3}(\theta_3)\,\rangle|^2+\dots\ ,$$

where the dots stand for the five-particle and higher contributions, which are of the order of e^{-5Mr}. The long-distance asymptotic behavior of the correlation function is dominated by the contribution of the one-particle states,

$$\langle\ \mathrm{vac}\mid\mathcal{O}^{+1}_{\beta/4}(0)\mid A_-(\theta)\,\rangle=\sqrt{\mathbf{Z}_1(\beta/4)}\ e^{\frac{1}{4}(\theta+\frac{i\pi}{2})}\ ,$$

and has an especially simple form,

$$F^{(1)}_{1/4}(r)=\mathbf{Z}_1(\beta/4)\ \Big\{\frac{e^{-Mr}}{\sqrt{2\pi Mr}}+O(e^{-3Mr})\Big\}\ . \tag{2.6.3}$$

Here we use the notation $\mathbf{Z}_n(a)$ $(a=\omega\beta)$ from work [92] for the field-strength renormalization which controls the long-distance asymptotics of the correlation function (2.3.9). Let us stress here that the overall multiplicative normalization of the field $\mathcal{O}^1_{\beta/4}$ was already fixed by the condition (2.4.8), hence the constant $\mathbf{Z}_1(\beta/4)$ is totally unambiguous. In [92], the following explicit formula for $\mathbf{Z}_n(\omega\beta)$ was proposed:

$$\begin{aligned}\mathbf{Z}_n(\omega\beta)\ &=\ \Big(\frac{\mathcal{C}_2}{2\mathcal{C}_1^2}\Big)^{\frac{n}{2}}\ \Big(\frac{\mathcal{C}_2}{16\rho}\Big)^{-\frac{n^2}{4}}\ \Big[\frac{\sqrt{\pi}M\Gamma(\frac{3}{2}+\frac{1}{2\rho})}{2\,\Gamma(1+\frac{1}{2\rho})}\Big]^{2d}\\ &\times\exp\Big[\int_0^{\infty}\frac{dt}{t}\ \Big\{\frac{\cosh(4\omega t)\,e^{-(1+\rho)nt}-1}{2\,\sinh(t)\sinh\big((1+\rho)t\big)\,\cosh(t\rho)}\\ &\qquad+\frac{n}{2\,\sinh(t)}-2d\ e^{-2t}\Big\}\Big]\ .\end{aligned} \tag{2.6.4}$$

In this formula, d is the scale dimension (2.3.10) and the constants $\mathcal{C}_1$, $\mathcal{C}_2$ read

$$\begin{aligned}\mathcal{C}_1\ &=\ \frac{2^{-\frac{5}{12}}\,e^{\frac{1}{4}}\,\Gamma(\frac{1}{4})}{\sqrt{\pi}\,A_G^3}\ \exp\Big\{\int_0^{\infty}\frac{dt}{t}\ \frac{\sinh^2(\frac{t\rho}{2})\,e^{-t}}{2\,\cosh^2(t\rho)\,\sinh(t)}\Big\}\,,\\ \mathcal{C}_2\ &=\ \frac{\Gamma^4(\frac{1}{4})}{4\pi^3}\ \exp\Big\{-2\int_0^{\infty}\frac{dt}{t}\ \frac{\sinh^2(\frac{t\rho}{2})\,e^{-t}}{\cosh(t\rho)\,\sinh(t)}\Big\}\,,\end{aligned}$$

where $A_G=1.282427\ldots$ is the Glaisher constant.

We do not write down explicitly the general formula for the three-particle contribution in (2.6.3) because it is a rather mechanical substitution of relations presented in [92]. (For $\beta^2 = 1$ the corresponding formulas can be found in Appendix A.4.) Here we make the following observation concerning the $\beta^2 \to 1$ limit. The examination of (2.6.2) based on explicit formulas for the form factors shows that the function $[\mathbf{Z}_1(\beta/4)]^{-1}\, F^{(1)}_{1/4}$ admits an asymptotic power series expansion in terms of the variable ρ^2. In other words, all divergences at $\rho^2 \to 0$ of $F^{(1)}_{1/4}$, considered as a function of the variables ρ^2 and Mr, are absorbed by the normalization constant $\mathbf{Z}_1(\beta/4)$. Using Eq. (2.6.4), one can check that the constant $\mathbf{Z}_1(\beta/4)$ admits exactly the same type of singular behavior at $\rho^2 = 0$ as the constant $Z_{n,\omega}$ (2.4.34) for $n = 1$, $\omega = 1/4$, and also that

$$\frac{\mathbf{Z}_1(\beta/4)}{Z_{1,1/4}} = 2^{-\frac{1}{3}}\ \sqrt{\pi}\, e^{-\frac{1}{4}}\ A_G^3\ \exp\left(w_1\, \rho^2 + O(\rho^3)\right)\, . \tag{2.6.5}$$

The explicit form of the coefficient w_1 is not essential here. What is important is that the linear term in ρ does not appear in the expansion (2.6.5). This observation can be immediately generalized and checked for any n and ω. Furthermore, we expect that

$$\log\left(\frac{\mathbf{Z}_n(\omega\beta)}{Z_{n,\omega}}\right) = \sum_{k=0}^{\infty} w_k\ \rho^{2k} + O(\rho^\infty)\ , \tag{2.6.6}$$

where $\mathbf{Z}_n(\omega\beta)$ is the normalization constant (2.6.4). In other words, by means of the transformation (2.4.35) with properly chosen coefficients w_k, the constant $Z_{n,\omega}$ in (2.4.15) can be set to be equal (in a sense of formal power series) to $\mathbf{Z}_n(\omega\beta)$. At the moment, we do not have a rigorous proof of (2.6.6). But it leads to some interesting prediction to be checked. As was already mentioned, the calculations performed in the leading order in conformal perturbation theory determine only the combination $v_2 - 3u_2/2$, but do not fix the individual values of the coefficients u_2 and v_2 in the series (2.4.33). Accepting (2.6.6), one can immediately find the values of the coefficients u_2 (see Appendix A.2). In the case $n = 1$, $\omega = \frac{1}{4}$, it allows one to extend the perturbative expansion (2.4.37), as well as the equivalent expansion (2.5.4), to order g^3. As was discussed in Section 4, Eq. (2.5.4) is in a complete agreement with the result of four-loop perturbative calculations from [2].

This in fact shows that the ρ^3-term really is absent in the series (2.6.5). Equation (2.6.6) predicts an infinity of relations among the perturbative coefficients of the anomalous dimension Γ_g in Zamolodchikov's scheme (see Appendix A.3).

2.7 Spectral density

The spectral density is an important quantity related to the two-point function and its analytical structure in momentum space. It is often what is measured in actual condensed matter experiments [63, 45], and it allows one to completely reconstruct the two-point function. In this section, we discuss the properties of the spectral density in the $SU(2)$-Thirring model.

The spectral decomposition of the fermion Green's function yields the following form for the function $\tilde{F}$ (2.5.1):

$$\tilde{F}(p^2) = 1 - \int_{M^2}^{+\infty} ds\ \frac{\Delta\tilde{F}(s)}{p^2 + s}\ . \tag{2.7.1}$$

The notation $\Delta\tilde{F}$ for the spectral density reminds us that $\tilde{F}$, considered as a function of one complex variable p^2, has a branch cut in the Minkowski region $p^2 < 0$ starting at $p^2 = -M^2$, and that the spectral density can be recovered from the discontinuity along this cut:

$$\Delta\tilde{F}(s) = \frac{1}{2\pi i}\ (\ \tilde{F}(e^{i\pi}s) - \tilde{F}(e^{-i\pi}s)\ )\ . \tag{2.7.2}$$

The easiest way to obtain the large s asymptotics of the spectral density is to use the expansion (2.5.4) along with knowledge of the analytical properties of the coupling constant g (2.5.5) as a function of the complex variable p^2. Notice that g can be expressed in terms of the principal branch of the product log (or Lambert) function, which gives the solution for W in $W\,e^W = z$ (see e.g. [30]):

$$g = 2\ W^{-1}\Big(\frac{p^2 e^{-2\tilde{\tau}_0}}{\pi M^2}\Big)\ . \tag{2.7.3}$$

The principal branch of the W-function analytically maps the complex z-plane minus the branch cut $z \in]-\infty, -e^{-1}]$ to the part of the complex W-plane enclosing the real axis and delimited by the curve $\Re e\, W = -\Im m\, W \cot(\Im m\, W)$ for $-\pi < \Im m\, W < \pi$. The analyticity implies that the power series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(i\phi\, z\, \frac{d}{dz}\right)^n W(z)\Big|_{z=s}$$

converges for real positive $s > e^{-1}$ and $|\phi| \leq \pi$ and coincides with $W(e^{i\phi}s)$. Similar considerations are, of course, valid for the coupling constant g (2.7.3). In particular, for sufficiently large s,

$$g(e^{\pm i\pi}s) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\pm\, i\pi\, p^2\, \frac{d}{dp^2}\right)^n g(p^2)\Big|_{p^2=s} .$$

This then gives us, with (2.5.4) and the RG flow equation (2.5.8), the asymptotic expansion of the spectral density for large s. It can be written in the following form:

$$\Delta\tilde{F}(s) = -\frac{g^2}{2}\left\{1 - \frac{g}{2} - \frac{\pi^2-1}{4}\, g^2 + O(g^3)\right\} \frac{\partial \tilde{F}^{(pert)}}{\partial g}\Big|_{p^2=s} . \tag{2.7.4}$$

Here the function $\tilde{F}^{(pert)}$ is given by (2.5.4) and g is defined by the equation (2.5.5).

Now let us consider the threshold behavior of the spectral density. According to the analysis of the previous section, the long-distance asymptotic behavior of the scaling function F (2.2.9) is described by the expansion

$$F = F^{(1)} + F^{(3)} + O(e^{-5Mr}) , \tag{2.7.5}$$

where

$$F^{(1)} = C\, e^{-Mr} ,$$

with the constant

$$C = 2^{-\frac{5}{6}}\, e^{-\frac{1}{4}}\, A_G^3 = 0.921862\ldots .$$

The function $F^{(3)}$ in (2.7.5) gives the three-particle contribution to the correlation function. Using the definitions (2.2.9), (2.5.1) and the above relation, one can obtain:

$$\tilde{F}(p^2) = C\left\{1 - \frac{1}{\sqrt{1+p^2/M^2}}\right\} + \ldots . \tag{2.7.6}$$

Here the dots stand for contributions of the massive multiparticle intermediate states. The last relation implies that the spectral density (2.7.2) can be written as

$$\Delta\tilde{F}(s) = \frac{C}{\pi}\,\frac{\Theta(s-M^2)}{\sqrt{s/M^2-1}} + \Theta(s-9M^2)\;\Delta\tilde{F}^{(3)}(s)\;, \tag{2.7.7}$$

where

$$\Theta(s) = \begin{cases} 1 & \text{for } s \geq 0 \\ 0 & \text{for } s < 0 \end{cases}\;,$$

and $\Delta\tilde{F}^{(3)}$ is some function which contributes to the spectral density only above the threshold $s = 9M^2$.

2.8 Numerics

In Table 2.8 we present results of numerical evaluation of the function F (2.2.9) as a function of the scaling distance Mr ($r = |x|$). To estimate the short-distance behavior, we use the perturbative expansion (2.4.37). As was already mentioned, the parameter $\bar{\tau}_0$ allows one to have control over the accuracy of the truncated series, so we calculate (2.4.37) for two different values of $\bar{\tau}_0$: -0.25 and $+0.25$. To determine the long-distance behavior of the function F, we use the formula (2.7.5), where the three-particle contribution $F^{(3)}$ was obtained by means of Eq. (2.6.2) along with formulas for the three-particle form factors quoted in [92] (see Appendix A.4). It is interesting to see that the sum of the one- and three-particle contributions to F is very near to unity at $r = 0$ (to within 1%), which indicates that this three-particle computation of the correlation function is in fact accurate to about 1% for all distance scales (more accurate, of course, for larger r). Also, note that the crossover between the long- and short-distance asymptotics appears to be at the scaling distances $Mr \sim 0.001 - 0.01$, where both asymptotics coincide to within about 0.1%.

Mr	$F^{(1)}$	$F^{(3)}$	$F^{(1)}+F^{(3)}$	$F^{(pert)}$ $(\bar{\tau}_0 = -0.25)$	$F^{(pert)}$ $(\bar{\tau}_0 = 0.25)$
0	.921862	.068	.990	1.00000	1.00000
.00001	.921853	.0553	.9771	.980129	.980130
.00005	.921816	.0522	.9740	.976311	.976314
.0001	.921770	.0504	.9722	.974192	.974196
.0002	.921678	.0483	.9700	.971674	.971678
.001	.920941	.0415	.9624	.963508	.963520
.002	.920020	.0375	.9575	.958435	.958454
.01	.912689	.0252	.9379	.939386	.939460
.025	.899101	.0168	.9159	.919294	.919494
.05	.876902	.0106	.8875	.894050	.894547
.075	.855251	.00738	.86263	.871796	.872717
.1	.834135	.00541	.83955	.850520	.852013
.15	.793454	.00317	.79662	.808380	.811548
.2	.754757	.00200	.75676	.765139	.770842
.25	.717947	.00131	.71926	.719980	.729252
.3	.682932	.000889	.683822	.672640	.686654
.35	.649625	.000617	.650243	.623153	.643171
.4	.617942	.000436	.618379	.571774	.599063
.45	.587805	.000313	.588118	.518942	.554677
.5	.559137	.000227	.559365	.465257	.510405

Table 2.8. The scaling function F (2.2.4), (2.2.9). The first columns give the results of the long-distance expansion which includes contributions of the one-, three- and one+three-particle states. The data in the last two columns correspond to the perturbative expansion (2.4.37) for the two different values of the auxiliary parameter $\bar{\tau}_0$.

2.9 Conclusion

In this chapter, I studied the fermion Schwinger's function in the $SU(2)$-Thirring model and in an anisotropic integrable deformation of it. I used and compared two well-established methods: the form factor expansion, and conformal perturbation theory combined with a renormalization-group (RG) analysis, leading respectively to the long-distance and the short-distance behavior of the correlation function. A proper comparison of these opposite limit behaviors needs an adjustment of the normalization used in one method with respect to that used in the other method. The necessary formula for such an adjustment was proposed in [92]: there the exact form factors, with appropriate normalization constant, were conjectured under a "conformal" normalization of the fields. This allowed us to numerically compare both methods and to observe an agreement to within 1% in the region $0 < Mr < 0.05$. Moreover, using these exact form factors it was possible to conjecture an infinite set of relations between expansion coefficients of the fermion anomalous dimension arising in the RG treatment of the anisotropic model. This was done essentially by identifying the singular part in ρ^2 (see Eqs. (2.3.3) and (2.3.5)) of the normalization constant conjectured in [92] with the singular part of the normalisation constant obtained in the RG treatment (see (2.6.6)). Using one of these relations along with a first non-trivial order calculation in conformal perturbation theory, it was possible to obtained the desired fermion Schwinger's function to third order in the coupling of the $SU(2)$ model, and to observe agreement with what was obtained in [2] by standard perturbation theory. These results, numerical and analytical, suggest the validity of the conjectured exact form factors of [92] in the case of the $SU(2)$-Thirring model. It might be interesting to apply similar methods to Thirring-like models with other symmetries.

Part II

Statistical mechanics on curved space

Chapter 3
Introduction

Quantum field theory in curved space-time is a subject which has been studied from many viewpoints (see for instance [15, 126]). Naturally, understanding this subject is an important first step in the development of a theory that includes quantum gravity as well as the quantum dynamics of local fields, as one needs to probe the effects of a curvature on quantum fields. But, borrowing intuition from theories on flat space, we can expect other applications of quantum field theory in curved space, in particular in a space with Euclidean signature: those related to the study of classical statistical systems on curved space[1].

A classical statistical system on a manifold is composed of a large number of classical local degrees of freedom interacting locally on this manifold. Such classical systems on flat space have been used to describe many physical situations where quantum fluctuations are negligible compared to thermal fluctuations. Probably the most important outcome of the study of classical statistical models is the understanding of second order phase transitions occurring at finite temperatures. The importance of this understanding resides in the universality of the descriptions of phase transitions: models with very different physical constituents may behave in very similar ways near to a second order phase transition. This universality can be understood in terms of quantum field theory, and the universality classes of phase transitions are essentially identified with the symmetries of

[1]Although in the case of flat space, models of quantum field theory with Euclidean signature are related by analytical continuation to models with Minkowsky signature, generally there is not such relation on a curved space.

the order parameters describing the phase transitions.

On curved space, the problem of understanding classical statistical systems has much room for development. An effective curved space induced by the microscopic physics of a "classically described" material may occur, for instance, because of the systematic presence of *defects* in the material (locations where the local interaction is modified), or because of smooth variations in space of the strength of local interactions. Defects are certainly present in real materials, and in many models they are studied in random distributions (quenched disorder). Studying their effects when they are distributed in more systematic fashions can lead to interesting new insights into the problem of defects in materials. In addition, there may be situations where such a more systematic distribution of defects naturally appears. Some materials can be described effectively by two-dimensional statistical models: crystalline planes do not interact strongly among each other. It is natural to inquiry as to the effects of smoothly "bent" planes, where the bending induces defects, on the statistical properties. In fact, any three-dimensional material where the interaction between the degrees of freedom is restricted, by some means (like by application of a magnetic field), to a curved surface, will have a description near "phase transition" in terms of a statistical model on a curved space.

Defects can also be considered as dynamical degrees of freedom. If they are much slower than the degrees of freedom of the statistical model, then one can consider the thermodynamical laws of the statistical model as inducing a dynamics on the defects. Glasses are examples of materials where the structure is a very slow dynamical degree of freedom, and the understanding of the thermodynamics of statistical systems on curved space may shed some light onto systems of this kind. From this point of view, there might also be an interest in quantum gravity: a statistical system that thermalizes faster than natural fluctuations of the metric can lead in this way to an approximate dynamics of space-time geometry.

Finally, there is a possible *a posteriori* reason to study statistical mechanics on curved space. In systems with boundaries, one usually observes, in the renormalization group flow of quantum field theory, a cross-over behavior from a model of a given dimensionality to a model of a lower dimensionality. This is very natural, since the short distance physics does not see the boundaries, whereas the long distance physics sees only the directions (if any) parallel to the boundaries. It would be very interesting to obtain a model where the renormalization group goes in the opposite direction. Indeed, one is sometimes able to solve models of quantum field theory of lower dimensionality, and a model is usually described by its local degrees of freedom, which is essentially an ultraviolet description. Solving a model that has low dimensionality in the ultraviolet and that flows to a higher dimensional model would give information about this higher dimensional model. As we will argue from our exact results, it is possible to have a model that flows to "infinite" dimensionality by considering its local description on a two-dimensional manifold of constant negative curvature. Although "infinite" dimensionality is not the most interesting case and can usually be studied by mean field theory, it is tempting to suggest that there may be ways, using similar ideas, of finding fixed points of intermediate dimensionality.

3.1 From lattices on curved space to quantum field theory

The most natural question in the study of statistical models on curved space is probably about the modification of their critical properties due to a nonzero curvature, which introduces an additional scale. In analogy to models on flat space, I expect that this question can be studied using Euclidean quantum field theory. More precisely, the "program" for such a study can be divided into three steps:

The first step in our program would be to construct an actual statistical model on a curved space: to give an explicit description of the microscopic degrees of freedom of a model on a given curved space. There are many ways of doing that. One can start with a given statistical system on a regular lattice on flat space, and implement the curvature by assuming that the sites are separated by a constant geodesic distance and by modifying the *number of neighbors* around some sites: more neighbors around a site implements a local negative curvature, whereas less neighbors implements a local positive curvature. There is a precise relation between the number of neighbors for a given regular lattice and the local curvature. This procedure is useful, because it directly gives a curved-space equivalent of any statistical system where the interaction between sites is function of "neighborhood," as is often the case in statistical models. For instance, the nearest-neighbor interaction in the Ising model is easily adapted to this procedure.

One could also imagine implementing a curvature by modifying the *distance* between neighbors: a greater distance implements a local negative curvature, whereas a smaller distance implements a local positive curvature. From the viewpoint of lattice statistical models, thought, it seems slightly harder to realize this method. One would have to change the strength of the interaction between neighbors in a position-dependent way, according to the inter-site distance; but in many statistical models, like in the Ising model, there is usually no *a priori* way of relating the interaction strength to the inter-site distance.

In the second step, in analogy with the situation on flat space, a scaling limit has to be defined so as to keep only the universal properties of the model: the parameters (temperature and curvature) should be adjusted so that the system is around a critical point, with a correlation length much larger than microscopic distances. This is in fact a very nontrivial step, and little is done in the literature. Let me describe in more precise terms how the scaling limit should be defined, assuming that most of the basic concepts underlying it in the flat-space situation carry over to the curved-space situation. Consider for definiteness a statistical model on a space of constant negative curvature, the pseudosphere; this space will be the subject of my study in the following. Assume that we first embed the statistical model on a lattice with a finite number of sites into a given patch of the pseudosphere, in one of the ways described in the previous paragraphs, for instance. There should be some sites that can be identified as "boundary" sites on this finite lattice; they lay on the boundary of the patch. We prescribe some boundary condition on the degrees of freedom of these boundary sites. There should also be a prescribed ratio between the "radius of curvature" of the pseudosphere, denoted R below and defined such that the Gaussian curvature is $-1/R^2$, and some microscopic distance characterizing the lattice, playing the role of a lattice spacing.

Now, consider the operation that modifies the lattice in such a way that the curvature is brought towards zero in microscopic units, and that the patch on which the lattice sits is kept invariant on the scale of the radius of curvature R; during this operation, the number of sites certainly increases. Consider doing this operation in such a way that the lattice embedded into the patch of the pseudosphere resembles more and more, in any region of fixed radius in microscopic units, a prescribed flat-space regular lattice. Assume that the statistical model we are considering, when put on that prescribed flat-space regular lattice at infinite volume, possesses a critical temperature T_c^{flat}, with an associated correlation length ξ^{flat}. We define the finite-volume scaling limit on the pseudosphere by the operation described above along with bringing the temperature towards T_c^{flat}, keeping the ratio of the radius of curvature to the flat-space correlation length R/ξ^{flat} constant. The resulting theory should be a quantum field theory on a finite patch of the pseudosphere. Correlation functions of local variables on the lattice, when evaluated at distances proportional to the radius of curvature R in the scaling limit, should be described by correlation functions of local fields in the quantum field theory. Then we may take the thermodynamic limit by taking the size of the patch to infinity in units of the radius of curvature. This step is also nontrivial: there is no guarantee that the resulting theory will be independent of the conditions that we impose on the boundary degrees of freedom as we send the volume of the patch to infinity in units of the radius of curvature, or even that it will be independent of the shape of the patch.

Finally, the last step is the one which I will develop below: analyzing the expected model of quantum field theory on a curved space with Euclidean signature, obtained after taking the scaling and thermodynamic limit.

It is important to note that the validity of this picture for the scaling limit may depend on the way the lattice is embedded into the pseudosphere. For instance, putting defects on the lattice in order to implement a curvature will cause the lattice to effectively have a curvature concentrated at points separated by finite distances in units of R. Then, the quantum field theory can only be expected to describe correlation functions at very large or very short distances compared to R in the scaling limit. One way of circumventing this problem would be to smooth out the curvature by statistical fluctuations of the lattice defects. This, however, introduces an additional randomness akin to random lattice systems, and require further studies.

3.2 The Ising model on the pseudosphere

A model of statistical mechanics which has been studied extensively for a long time is the classical two-dimensional Ising model (see [98]). It is a model where two-state variables representing semiclassically spin-1/2 quantum degrees of freedom are placed on the sites of a regular lattice, with nearest-neighbor ferromagnetic interactions, at some temperature and in some magnetic field. It is a very rough model of a ferromagnet, an in fact is used fruitfully to model other statistical systems like binary alloys. But probably its most prominent virtue is that it is a simple model, where many exact results can be obtained in two dimensions, with a nontrivial critical behavior representing a large universality class of statistical systems. Its universal critical and near-critical behavior is most adequately studied using Euclidean quantum field theory; in this case, it is the so-called Ising field theory. At zero magnetic field, this is the massive Majorana free fermion theory (where the mass represents the separation of the temperature from its critical value), and a magnetic field add to the free Majorana action a term nonlocal with respect to the fermion degrees of freedom, proportional to the Ising spin field[2].

I will be interested in studying the Ising field theory at zero magnetic field on a two-dimensional curved space of constant negative curvature, the pseudosphere[3]. In parallel to the case on flat space, I expect that this model represent a large universality class of critical (or near-critical) behaviors on curved space: $\mathbb{Z}_2$-symmetric statistical models with short-range interaction on a pseudosphere should have a near-critical behavior represented by the Ising field theory on the pseudosphere. Note that an Ising model on a regular lattice embedded on the Poincaré disk was studied in [110]. It will be possible to compare some results of [110] to results in Chapter 5; this is done in Section 5.8.

The interest in the pseudosphere is in part technical: since a space of constant curvature is maximally symmetric, known techniques developed for the case of flat space can be extended to the Ising field theory on such a space. But the pseudosphere also has unusual characteristics, for instance it has an infinite (two-dimensional) volume while providing an infrared regularization (as was argued in [20]), and it can be expected to have nontrivial effects on the thermodynamics. The study of these effects should in fact throw light on the main properties of the thermodynamics on any negatively curved spaces.

I will present exact results concerning the magnetization and the two-point

[2]With a magnetic field, the model is non-integrable, except at zero mass (at criticality).

[3]The pseudosphere is also called the Poincaré disk when realized on the unit disk.

correlation functions of spin fields. The magnetization, which is the average of the order parameter, gives a rough description of the various thermodynamical phases, and the way it vanishes at a critical point is a universal characteristics, in principle accessible to experiments. The two-point correlation function of order fields gives direct access to the magnetic susceptibility, which also presents universal singular behaviors, and which gives information about the phase structure of the system with a magnetic field.

These results, and in fact more general results, are derived in the three papers [37, 38, 39]. The development in these papers goes as follows. In [37], I calculate the form factors of a set of local spinless scaling fields in the free Dirac theory on the pseudosphere. In particular, I obtain the vacuum averages of these scaling fields, and the form factors give the large (geodesic) distance behaviors of their two-point correlation functions. The Dirac theory is just two copies of the Ising field theory at zero magnetic field, and on flat space, particular linear combinations of scaling fields in the Dirac theory factorize into products of order and of disorder fields [148, 113, 132]. On a curved space, a similar relation should exist. If so, then vacuum averages of particular scaling fields in the Dirac theory give the magnetization in the Ising field theory, and their form factors give the large distance expansion of two-point correlation functions of spin fields. In [38], I verify this relation between the scaling fields in the Dirac theory and order and disorder fields in the Ising field theory on the pseudosphere. Finally, in [39], with Dr. P. Fonseca, we obtain nonlinear differential equations for describing the two-point correlation functions of order and of disorder fields in the Ising field theory, and we briefly discuss the thermodynamics of the model.

It is worth noting that these results have interesting applications in the theory of Painlevé equations. Indeed, correlation functions of scaling fields in the Dirac theory on the pseudosphere were suggested in [104] to be described by a family of Painlevé VI differential equations. Then, my results about the form factors in [37] give a solution to the so-called connection problem in the theory of Painlevé equations: I found an explicit relation between the asymptotic forms of the Painlevé transcendent around two of the singular points of the Painlevé VI equation. Likewise, the nonlinear differential equations that we found in [39] are a particular degenerate case of the equations in [104], and results about the form factors of order and disorder fields as well as our short-distance analysis using conformal perturbation theory give a solution to the associated connection problem.

In Chapter 4, I reproduce the results and a slightly extended version of the derivations of the paper [37] concerning scaling fields in the Dirac theory on the Poincaré disk. Some of the results will then be used in Chapter 5 in order to study the Ising field theory on the Poincaré disk, reproducing the results of the papers [38] and [39].

Chapter 4

Two-point function of scaling fields in the Dirac theory on the Poincaré disk

4.1 Description in terms of Painlevé VI transcendents

The free massive Dirac action with fermion mass m in the system of coordinates of the Poincaré disk with a Gaussian curvature $-1/R^2$ is

$$\mathcal{A} = \int d\mathrm{x}d\mathrm{y}\ \bar{\Psi}\left(\gamma^{\mathrm{x}}\partial_{\mathrm{x}} + \gamma^{\mathrm{y}}\partial_{\mathrm{y}} + \frac{2r}{1-|z|^2}\right)\Psi, \tag{4.1.1}$$

where we use

$$r = mR\ , \tag{4.1.2}$$

and $\Psi = \begin{pmatrix} \Psi_R \\ \Psi_L \end{pmatrix}$, $\bar{\Psi} = \Psi^\dagger\gamma^{\mathrm{y}}$. We choose the Dirac matrices as

$$\gamma^{\mathrm{x}} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\ ,\ \gamma^{\mathrm{y}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The field Ψ_R has $SU(1,1)$ dimension $(1/2,0)$, and the field Ψ_L has dimension $(0,1/2)$ (see Appendix B.1).

The two-point function that we are interested in can be represented by the appropriately regularized Euclidean functional integral

$$\langle\mathcal{O}_\alpha(x)\mathcal{O}_{\alpha'}(y)\rangle = \int_{\mathcal{F}_{\alpha,\alpha'}}[\mathcal{D}\Psi\mathcal{D}\bar{\Psi}]e^{-\mathcal{A}}. \tag{4.1.3}$$

The integration is over the space $\mathcal{F}_{\alpha,\alpha'}$ of field configurations vanishing on the boundary of the disk and such that Ψ, $\bar{\Psi}$ acquire phases when continued counterclockwise around the points x and y:

$$\mathcal{F}_{\alpha,\alpha'}:\begin{cases}\text{around } x:\ \Psi\to e^{2\pi i\alpha}\Psi,\ \bar{\Psi}\to e^{-2\pi i\alpha}\bar{\Psi}\\ \text{around } y:\ \Psi\to e^{2\pi i\alpha'}\Psi,\ \bar{\Psi}\to e^{-2\pi i\alpha'}\bar{\Psi}\end{cases}. \tag{4.1.4}$$

It is easy to generalize this definition to correlation functions involving many fields $\mathcal{O}_\alpha(x)$. The fields thus defined, $\mathcal{O}_\alpha=\mathcal{O}^\dagger_{-\alpha}$, $-1<\alpha<1$, are so-called twist fields associated to the $U(1)$ symmetry of the Dirac theory (see Section iv in the Introduction). They are spinless, $U(1)$-neutral and have scaling dimension α^2. As is clear from their definition, they are not mutually local with respect to the Dirac field. Their mutual locality index with the Dirac field is α, that is, the Dirac field Ψ takes a factor, $\Psi\to e^{2\pi i\alpha}\Psi$, when continued counterclockwise around the field $\mathcal{O}_\alpha$. Their self-locality, on the other hand, follows from $U(1)$ invariance of (4.1.1).

In the free massive Dirac theory on two-dimensional flat space, correlation functions analogous to (4.1.3) can be studied by their relation to the problem of isomonodromic deformations [112]. They are tau functions [112, 68, 67, 103] for the isomonodromic deformation problem associated to some Painlevé equations: they can be expressed in terms of Painlevé transcendents.

In [104], the authors generalized the method of isomonodromic deformation to the study of determinants of the Dirac operator on the Poincaré disk. Their results suggest that the two-point function (4.1.3) can be expressed in terms of Painlevé VI transcendents. The general form of the Painlevé VI differential equation is:

$$\begin{aligned}w''-\frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-s}\right)(w')^2+\left(\frac{1}{s}+\frac{1}{s-1}+\frac{1}{w-s}\right)w'\\ =\frac{w(w-1)(w-s)}{s^2(1-s)^2}\left(\frac{(1-4r^2)s(s-1)}{2(w-s)^2}-\frac{(\tilde{\lambda}-1)^2s}{2w^2}+\frac{\gamma(s-1)}{(w-1)^2}+\frac{\lambda^2}{2}\right)\end{aligned} \tag{4.1.5}$$

where $w=w(s)$ and the primes mean derivatives with respect to s. There are four parameters: r, γ, λ, $\tilde{\lambda}$. For the description of the two-point function (4.1.3), the parameters are fixed to $r = mR$ as in (4.1.2), $\lambda = \alpha - \alpha'$, $\tilde{\lambda} = \alpha + \alpha'$ and $\gamma = 0$. The two-point function is then identified with the associated tau function $\tau(s)$, with s simply related to the geodesic distance $\mathrm{d}(x, y)$ between the points x and y:

$$\langle \mathcal{O}_\alpha(x)\mathcal{O}_{\alpha'}(y)\rangle = \tau(s)$$

with

$$s = \tanh^2\left(\frac{\mathrm{d}(x,y)}{2R}\right). \tag{4.1.6}$$

Up to normalization, the tau function is given by [104]:

$$\begin{aligned} \frac{d}{ds}\ln\tau(s) &= \frac{s(1-s)}{4w(1-w)(w-s)}\left(w' - \frac{1-w}{1-s}\right)^2 - \frac{r^2}{w-s} \\ &\quad + \frac{\tilde{\lambda}^2}{4(1-s)w} + \frac{\lambda^2 w}{4s(1-s)} - \frac{\lambda^2}{4s} + \frac{4r^2 - \tilde{\lambda}^2 - \lambda^2}{4(1-s)}. \end{aligned} \tag{4.1.7}$$

4.2 How to specify the Painlevé VI solution?

This description for the correlation function needs to be completed and to be compared with a direct calculation in the Dirac theory on the Poincaré disk. In order to complete it, one must supply appropriate integration constants specifying the Painlevé transcendent that describes the two-point function.

First, the exponent $2\alpha\alpha'$ in the short distance power law of the two-point function,

$$\langle \mathcal{O}_\alpha(x)\mathcal{O}_{\alpha'}(y)\rangle \sim \langle \mathcal{O}_{\alpha+\alpha'}\rangle \mathrm{d}(x,y)^{2\alpha\alpha'} \quad \text{as } x \to y\ , \tag{4.2.1}$$

specifies the exponent in the asymptotic behavior of $w(s)$ near the critical point $s = 0$:

$$w \sim Bs^{\alpha+\alpha'} \quad \text{as } s \to 0, \tag{4.2.2}$$

where B is some constant. A solution of the Painlevé equation (4.1.5) (with $\gamma = 0$) that has the form $w \sim s^\nu$ as $s \to 0$ must have $0 < \nu < 1$. Then we must have $0 < \alpha + \alpha' < 1$.

Second, the cluster property of the two-point function,

$$\langle \mathcal{O}_\alpha(x)\mathcal{O}_{\alpha'}(y)\rangle = \langle \mathcal{O}_\alpha\rangle\langle \mathcal{O}_{\alpha'}\rangle F\left(\frac{\mathrm{d}(x,y)}{2R}\right), \quad \lim_{t\to\infty} F(t) = 1, \tag{4.2.3}$$

fixes the form $1-w\sim(1-s)^\mu$ as $s\to 1$, and further imposes $\mu = 1\pm 2r$. But a solution that has the form $1-w\sim(1-s)^\mu$ as $s\to1$ must have $0<\mu<1$ or else must have $\mu=1+2r$. For $r>1/2$, the only valid possibility is then $\mu = 1+2r$, which specifies the exponent in the asymptotic behavior of $w(s)$ near the critical point $s=1$:

$$1-w\sim A(1-s)^{1+2r} \quad \text{as } s\to 1, \tag{4.2.4}$$

where again, A is some constant. For $0<r<1/2$, the two possibilities are valid: $\mu = 1\pm 2r$. I will consider for now only the case $\mu=1+2r$, for $r>0$. The other case corresponds to a different regime of the quantum field theory, and will be analyzed in more detail in the framework of the Majorana theory in Chapter 5.

One can expect that the exponents in (4.2.2) and (4.2.4) form a set of integration constants fixing the Painlevé transcendent. However, one doesn't know *a priori* that there exists a solution to the Painlevé equation with both behaviors (4.2.2) and (4.2.4). In addition, even if such a solution exists, this set is not the most convenient. One cannot use it for instance to provide initial conditions for numerically solving the differential equation (4.1.5). It is more appropriate to fix the full expansion of the Painlevé transcendent near the singular point $s=1$, that is, to specify the constant A in (4.2.4). Fixing this constant gives a solution to the connection problem for the particular Painlevé VI equation that we are considering, that is, the problem of relating the behaviors of the Painlevé transcendent near its various critical points. The expansion of the Painlevé transcendent near $s=1$ is directly related to the long distance expansion of the two-point function:

$$F(t) = 1 - A\frac{(r+\alpha)(r+\alpha')}{(2r+1)^2}4^{2r+1}e^{-(4r+2)t} + O\left(e^{-(4r+4)t}\right). \tag{4.2.5}$$

To my knowledge, the theory of Painlevé VI equations (4.1.5) in the case $\gamma=0$ provides no expression for A in terms of the exponents in (4.2.2) and (4.2.4).

The asymptotics (4.2.2) can also serve as initial condition once B is known, although it is numerically not as efficient. Using results of Jimbo [66] there is an explicit expression for the constant B involved in the short distance behavior of w:

$$B = r\frac{\Gamma(\alpha)\,\Gamma(\alpha')\,\Gamma(1-\alpha-\alpha')^2\,\Gamma(\alpha+\alpha'+r)}{\Gamma(1-\alpha)\,\Gamma(1-\alpha')\,\Gamma(\alpha+\alpha')^2\,\Gamma(1-\alpha-\alpha'+r)}. \tag{4.2.6}$$

From this, one can obtain the full short distance expansion of w, valid for $0 < \alpha+\alpha' < 1$:

$$w = Bs^{\alpha+\alpha'}\sum_{p,q=0}^{\infty} C_{p,q}\; s^{p(\alpha+\alpha')+q(1-\alpha-\alpha')}$$

where $C_{0,0} = 1$. Once B is fixed, the other coefficients $C_{p,q}$ are uniquely determined by the differential equation (4.1.5).

Another quantity needs to be calculated in order to determine the two-point function completely: one must have an expression relating the normalization of the leading long distance asymptotics of the two-point function to that of its leading short distance asymptotics. The theory of Painlevé VI equations does not provide such constant. Comparing (4.2.1) and (4.2.3), one sees that the relation between normalizations can be obtained by calculating the one-point function $\langle\mathcal{O}_\alpha\rangle$. Note that the condition (4.2.1) fixes the normalization of the scaling fields $\mathcal{O}_\alpha$. This normalization being fixed, the one-point function $\langle\mathcal{O}_\alpha\rangle$ is unambiguous.

In the next sections I will develop a quantization scheme for the theory (4.1.1) whose Hilbert space is formed by the equivalent of the asymptotic states of theories on flat space. I will then calculate the long distance asymptotics of the two-point function by a "form factor" expansion, that is, by inserting between the two operators in a vacuum expectation value a resolution of the identity on this Hilbert space. This will give the following value of A:

$$A = \frac{\sin(\pi\alpha)\sin(\pi\alpha')\Gamma(r+\alpha)\Gamma(1+r-\alpha)\Gamma(r+\alpha')\Gamma(1+r-\alpha')}{\pi^2\Gamma(1+2r)^2}\ . \tag{4.2.7}$$

In order to calculate form factors, I use the angular quantization scheme where "time" is taken on orbits of compact subgroups of the $SU(1,1)$ isometry group of the Poincaré disk. The one-point function can similarly be calculated using this quantization scheme:

$$\langle \mathcal{O}_\alpha \rangle = (2R)^{-\alpha^2} \prod_{n=1}^{\infty} \left(\frac{1 - \frac{\alpha^2}{(r+n)^2}}{1 - \frac{\alpha^2}{n^2}} \right)^n . \tag{4.2.8}$$

This is in fact a simple generalization of calculations done on flat space in [141, 91]. The two constants (4.2.7) and (4.2.8), along with the Painlevé VI description, completely fix the two-point function (4.1.3).

4.3 Hilbert space I

An expansion of the functional integral (4.1.3) for large geodesic distance between the points x and y is most conveniently obtained by using the operator formalism. I will construct a Hilbert space $\mathcal{H}$ on which this functional integral is represented by a vacuum expectation value $\langle \mathrm{vac}|\mathcal{T}[\mathcal{O}_\alpha(x)\mathcal{O}_{\alpha'}(y)]|\mathrm{vac}\rangle$, where $|\mathrm{vac}\rangle$ is a $SU(1,1)$ invariant vacuum implementing vanishing boundary conditions and where the fields $\mathcal{O}_\alpha(x)$ and $\mathcal{O}_{\alpha'}(y)$ act on $\mathcal{H}$ as operators implementing the quasi-periodicity conditions (4.1.4). The symbol $\mathcal{T}$ here denotes an appropriate "time"-ordering, described below. The large distance expansion will be obtained by inserting between the fields $\mathcal{O}_\alpha(x)$ and $\mathcal{O}_{\alpha'}(y)$ a resolution of the identity in terms of a basis of states that diagonalize a non-compact subgroup of the $SU(1,1)$ isometry group of the Poincaré disk.

I define the Hilbert space $\mathcal{H}$ by quantizing the theory on curves that are orbits of the non-compact subgroup $\mathbb{K}$:

$$\mathbb{K} \,:\, g_q = \begin{pmatrix} \cosh(q) & \sinh(q) \\ \sinh(q) & \cosh(q) \end{pmatrix}, \; q \in \mathbb{R}. \tag{4.3.1}$$

Translations in the "time" direction, perpendicular to these curves, are not isometries, so that the Hamiltonian is not stationary. Translations in the "space" direction, along these curves, are isometries, and a basis for $\mathcal{H}$ will be obtained by

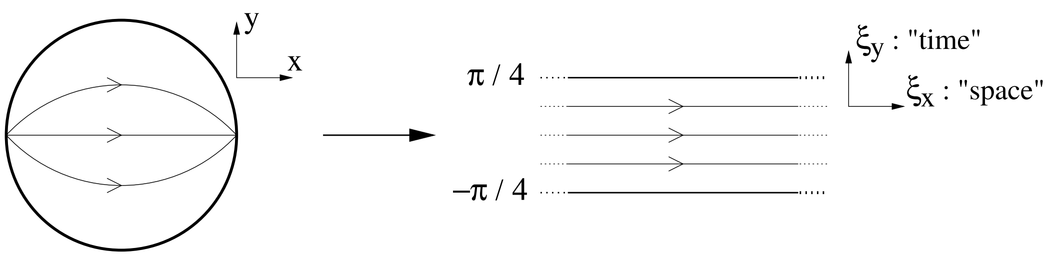

Figure 4.1: Mapping from the Poincaré disk to the strip described by $\mathrm{x} + i\mathrm{y} = \tanh(\xi_{\mathrm{x}} + i\xi_{\mathrm{y}})$, $\mathrm{x} - i\mathrm{y} = \tanh(\xi_{\mathrm{x}} - i\xi_{\mathrm{y}})$. Lines with an arrow represent orbits of a non-compact subgroup $\mathbb{K}$ of the isometry group of the Poincaré disk.

diagonalizing the generator of such translations. In order to construct $\mathcal{H}$, we first map the Poincaré disk onto the strip (see Figure 4.1):

$$z = \tanh(\xi),\ \ \bar{z} = \tanh(\bar{\xi})$$

with

$$\xi = \xi_{\mathrm{x}} + i\xi_{\mathrm{y}},\ \xi_{\mathrm{x}} \in \mathbb{R},\ -\pi/4 < \xi_{\mathrm{y}} < \pi/4.$$

The action (4.1.1) for the fermion fields $\Psi_s = \begin{pmatrix} (\Psi_s)_R \\ (\Psi_s)_L \end{pmatrix}$ on the strip is

$$\mathcal{A} = \int d\xi_{\mathrm{x}} d\xi_{\mathrm{y}}\, \bar{\Psi}_s \left(\gamma^{\mathrm{x}} \frac{\partial}{\partial \xi_{\mathrm{x}}} + \gamma^{\mathrm{y}} \frac{\partial}{\partial \xi_{\mathrm{y}}} + \frac{2r}{\cos(2\xi_{\mathrm{y}})} \right) \Psi_s,$$

where $\bar{\Psi}_s = \Psi_s^{\dagger}\gamma^{\mathrm{y}}$. Elements g_q of the subgroup $\mathbb{K}$ are translations parallel to the strip: $\xi_{\mathrm{x}} \to \xi_{\mathrm{x}} + q$, so that the quantization space is specified by the curves $\xi_{\mathrm{y}} = const$. The coordinate ξ_{y} is the Euclidean "time". Correlation functions of local operators are expressed as vacuum expectation values of "time"-ordered products, for instance

$$\langle \mathcal{O}_{\alpha_1}(\xi_{\mathrm{x}1}, \xi_{\mathrm{y}1}) \mathcal{O}_{\alpha_2}(\xi_{\mathrm{x}2}, \xi_{\mathrm{y}2}) \cdots \rangle = \langle \mathrm{vac} | \mathcal{T}[\mathcal{O}_{\alpha_1}(\xi_{\mathrm{x}1}, \xi_{\mathrm{y}1}) \mathcal{O}_{\alpha_2}(\xi_{\mathrm{x}2}, \xi_{\mathrm{y}2}) \cdots] | \mathrm{vac} \rangle,$$

where the "time"-ordering $\mathcal{T}$ means that operators have to be ordered form left to right in decreasing values of their variable ξ_{y}. The Hilbert space $\mathcal{H}$ is a module for the canonical equal-"time" anti-commutation relations

$$\{\Psi_s(\xi_{\mathrm{x}}, \xi_{\mathrm{y}}), \Psi_s^{\dagger}(\xi_{\mathrm{x}}', \xi_{\mathrm{y}})\} = \mathbf{1}\delta(\xi_{\mathrm{x}} - \xi_{\mathrm{x}}'),$$

where now $\Psi_s(\xi_x,\xi_y)$, $\Psi_s^\dagger(\xi_x,\xi_y)$ are fermion operators acting on $\mathcal{H}$, related on different "time" slices $\xi_y = const$ by the equations of motion. The vacuum state $|\mathrm{vac}\rangle \in \mathcal{H}$ is $SU(1,1)$ invariant, and satisfies the conditions that the fermion operators vanish on it at early "times" $\xi_y \to -\pi/4$ and on its dual $\langle\mathrm{vac}|$ at late "times" $\xi_y \to \pi/4$:

$$\begin{aligned}\lim_{\xi_y\to-\pi/4}\Psi_s(\xi_x,\xi_y)|\mathrm{vac}\rangle &= \lim_{\xi_y\to-\pi/4}\Psi_s^\dagger(\xi_x,\xi_y)|\mathrm{vac}\rangle = 0\\ \lim_{\xi_y\to\pi/4}\langle\mathrm{vac}|\Psi_s(\xi_x,\xi_y) &= \lim_{\xi_y\to\pi/4}\langle\mathrm{vac}|\Psi_s^\dagger(\xi_x,\xi_y) = 0.\end{aligned} \tag{4.3.2}$$

A basis for the Hilbert space $\mathcal{H}$ diagonalizing the subgroup $\mathbb{K}$ is obtained by considering a module for appropriate modes $A_\epsilon(\omega)$, $A_\epsilon^\dagger(\omega)$ of the Fermion operators $\Psi_s(\xi_x,\xi_y)$, $\Psi_s^\dagger(\xi_x,\xi_y)$, where $\epsilon=\pm$ represents the $U(1)$ charge. These modes appear in an expansion in terms of partial waves satisfying the equations of motion and diagonalizing the spin-$\frac{1}{2}$ action of the subgroup $\mathbb{K}$:

$$\begin{aligned}\Psi_s(\xi_x,\xi_y) &= \int d\omega\,\rho(\omega)\,\left(A_-^\dagger(\omega)e^{i\omega\xi_x}\bar{P}_-(\omega,\xi_y) + A_+(\omega)e^{-i\omega\xi_x}P_+(\omega,\xi_y)\right)\\ \Psi_s^\dagger(\xi_x,\xi_y) &= \int d\omega\,\rho(\omega)\,\left(A_+^\dagger(\omega)e^{i\omega\xi_x}\bar{P}_+(\omega,\xi_y) + A_-(\omega)e^{-i\omega\xi_x}P_-(\omega,\xi_y)\right)\ ,\end{aligned} \tag{4.3.3}$$

where the integrations are from $-\infty$ to ∞. We choose the measure

$$\rho(\omega) = \frac{\Gamma\left(\frac{1}{2}+r+i\frac{\omega}{2}\right)\Gamma\left(\frac{1}{2}+r-i\frac{\omega}{2}\right)}{2\pi\Gamma\left(\frac{1}{2}+r\right)^2} \tag{4.3.4}$$

in order for the partial waves to be entire functions of the spectral parameter ω; this analytical property will be used later. The solutions for the partial waves can be written:

$$P_+(\omega,\xi_y) = \tag{4.3.5}$$
$$\begin{pmatrix} 2^{-2r-\frac{1}{2}}e^{-i\frac{\pi}{2}\left(r+\frac{1}{2}-i\frac{\omega}{2}\right)}(1+e^{4i\xi_y})^r e^{\omega\xi_y}F\left(r,\frac{1}{2}+r-i\frac{\omega}{2};1+2r;1+e^{4i\xi_y}\right)\\ 2^{-2r-\frac{1}{2}}e^{-i\frac{\pi}{2}\left(r+\frac{1}{2}+i\frac{\omega}{2}\right)}(1+e^{4i\xi_y})^r e^{-\omega\xi_y}F\left(r,\frac{1}{2}+r+i\frac{\omega}{2};1+2r;1+e^{4i\xi_y}\right)\end{pmatrix}$$

and

$$\bar{P}_\epsilon(\omega,\xi_y) = (P_\epsilon(\omega,-\xi_y))^\dagger,\ (P_-(\omega,\xi_y))^t = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}P_+(\omega,\xi_y), \tag{4.3.6}$$

where $(\cdot)^t$ means transpose, and where $F(a,b;c;z)$ is Gauss's hypergeometric function on a branch delimited by the branch cut $(-\infty,1]$, with

$$\lim_{z\to0,\ \Im m(z)>0} F(a,b;c;z) = 1\ .$$

With this choice of solutions, the partial waves $P_\pm(\omega,\xi_{\rm y})$ vanish on the upper boundary $\xi_{\rm y} = \pi/4$ of the strip, and $\bar{P}_\pm(\omega,\xi_{\rm y})$ vanish on the lower boundary $\xi_{\rm y} = -\pi/4$ of the strip.

The canonical anti-commutation relations for the fermion operators imply the following anti-commutation relations for the modes:

$$\{A_\epsilon(\omega), A^\dagger_{\epsilon'}(\omega')\} = \frac{1}{\rho(\omega)}\delta(\omega-\omega')\delta_{\epsilon,\epsilon'}\ ,\ \{A_\epsilon(\omega), A_{\epsilon'}(\omega')\} = \{A^\dagger_\epsilon(\omega), A^\dagger_{\epsilon'}(\omega')\} = 0. \tag{4.3.7}$$

The Hilbert space is the Fock space over this algebra, and the asymptotic conditions (4.3.2) specify the vacuum $|{\rm vac}\rangle$:

$$A_\pm(\omega)|{\rm vac}\rangle = 0$$

which we normalize to $\langle{\rm vac}|{\rm vac}\rangle = 1$. A complete basis is given by

$$|\omega_1,\ldots,\omega_n\rangle_{\epsilon_1,\ldots,\epsilon_n} = A^\dagger_{\epsilon_1}(\omega_1)\cdots A^\dagger_{\epsilon_n}(\omega_n)|{\rm vac}\rangle \tag{4.3.8}$$

for a given ordering of the ω_j's, for instance the "in-ordering" $\omega_1 < \cdots < \omega_n$.

The states constructed diagonalize the subgroup $\mathbb{K}$:

$$\hat{g}_q|\omega_1,\ldots,\omega_n\rangle_{\epsilon_1,\ldots,\epsilon_n} = e^{-iq(\omega_1+\cdots+\omega_n)}|\omega_1,\ldots,\omega_n\rangle_{\epsilon_1,\ldots,\epsilon_n}$$

as well as the $U(1)$ charge, with eigenvalue $\epsilon_1+\cdots+\epsilon_n$. Here and below we use the notation $\hat{g}$ for representing the action of the group element g on the space $\mathcal{H}$. These states can be used to obtain a resolution of the identity on $\mathcal{H}$:

$$\mathbf{1}_{\mathcal{H}} = \sum_{n=0}^\infty \frac{1}{n!}\sum_{\epsilon_1,\ldots,\epsilon_n}\int\left(\prod_{j=1}^n d\omega_j\ \rho(\omega_j)\right)\ |\omega_1,\ldots,\omega_n\rangle_{\epsilon_1,\ldots,\epsilon_n}\ {}_{\epsilon_n,\ldots,\epsilon_1}\langle\omega_n,\ldots,\omega_1|,$$

where states with different orderings of ω_j's than the in-ordering $\omega_1 < \cdots < \omega_n$ are given by the same expression (4.3.8) in terms of modes. They differ from states with in-ordering by a sign through the anti-commutation relations (4.3.7).

On the Hilbert space $\mathcal{H}$, the scaling fields $\mathcal{O}_\alpha(x)$ act as appropriately regularized exponentials of line integrals of the $U(1)$ current:

$$\mathcal{O}_\alpha(x) \mapsto \pi_{\mathcal{H}} \left(\exp\left[2\pi i\alpha \int_{\mathcal{C}_x} dx^\mu \epsilon_{\mu,\nu} \bar{\Psi}\gamma^\nu \Psi \right] \right), \tag{4.3.9}$$

where $\mathcal{C}_x$ is a path from the position x to the boundary of the disk and $\pi_{\mathcal{H}}$ is the representation map on the space $\mathcal{H}$ (for a precise definition of the action of similar scaling fields on the standard Hilbert space of a Dirac theory on flat background, see for instance [113]). The resolution of the identity on $\mathcal{H}$ then gives the long geodesic distance expansion of the two-point function $\langle \mathcal{O}_\alpha(x)\mathcal{O}_{\alpha'}(y)\rangle = \langle \mathrm{vac}|\mathcal{T}[\mathcal{O}_\alpha(x)\mathcal{O}_{\alpha'}(y)]|\mathrm{vac}\rangle$. Using the fact that the one-point function $\langle \mathcal{O}_\alpha\rangle = \langle \mathrm{vac}|\mathcal{O}_\alpha|\mathrm{vac}\rangle$ is non-zero and defining the function

$$F_\alpha(\omega_1,\ldots,\omega_n)_{\epsilon_1,\ldots,\epsilon_n} \equiv \frac{\langle \mathrm{vac}|\mathcal{O}_\alpha(0)|\omega_1,\ldots,\omega_n\rangle_{\epsilon_1,\ldots,\epsilon_n}}{\langle \mathrm{vac}|\mathcal{O}_\alpha|\mathrm{vac}\rangle}, \tag{4.3.10}$$

we have

$$\begin{aligned}\langle \mathcal{O}_\alpha(x)\mathcal{O}_{\alpha'}(y)\rangle = \langle \mathcal{O}_\alpha\rangle\langle \mathcal{O}_{\alpha'}\rangle \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\epsilon_1,\ldots,\epsilon_n} \int \left(\prod_{j=1}^n d\omega_j \rho(\omega_j) \right) \\ \times F_\alpha(\omega_1,\ldots,\omega_n)_{\epsilon_1,\ldots,\epsilon_n} (F_{-\alpha'}(\omega_1,\ldots,\omega_n)_{\epsilon_1,\ldots,\epsilon_n})^* e^{-i(\omega_1+\cdots+\omega_n)\frac{\mathrm{d}(x,y)}{2R}}.\end{aligned} \tag{4.3.11}$$

In order to obtain this formula, one first brings y to the origin and x to the real axis inside the Poincaré disk; this can always be done by $SU(1,1)$ invariance of the correlator. Since the subgroup $\mathbb{K}$ generates geodesic translations along the real axis inside the Poincaré disk, matrix elements of the operator $\mathcal{O}_\alpha(x)$ are related to those of $\mathcal{O}_\alpha(0)$ by an exponential factor involving the geodesic distance $\mathrm{d}(x,0)$. Using $SU(1,1)$ invariance again, one can replace this by $\mathrm{d}(x,y)$ for the correlator of fields at arbitrary points x and y. In the next section we calculate the matrix elements $\langle \mathrm{vac}|\mathcal{O}_\alpha(0)|\omega_1,\ldots,\omega_n\rangle_{\epsilon_1,\ldots,\epsilon_n}$, which we call "form factors" of the scaling fields $\mathcal{O}_\alpha(0)$.

4.4 Angular quantization

In order to construct form factors of local fields in quantum integrable models, in [89, 18] the authors used the idea of embedding the Hilbert space $\mathcal{H}$ of a quantum field theory in two-dimensional flat space-time into a tensor product

$$\mathcal{H}_A \otimes \mathcal{H}_A^*$$

of the Hilbert space of angular quantization $\mathcal{H}_A$ and its dual $\mathcal{H}_A^*$. In angular quantization, the Hamiltonian is taken as the generator of rotations around a given point. Form factors of local fields can then be constructed as traces on the angular Hilbert space $\mathcal{H}_A$. We will make a similar construction in order to obtain the matrix elements $\langle \text{vac}|\mathcal{O}_\alpha(0)|\omega_1,\ldots,\omega_n\rangle_{\epsilon_1,\ldots,\epsilon_n}$. The angular Hamiltonian will be taken as the generator of rotations around the center of the Poincaré disk; it is the generator for the compact subgroup of the $SU(1,1)$ isometry group of the Poincaré disk.

We first briefly develop the formalism of angular quantization [91, 18, 74] for our theory. Angular quantization is done in conformal polar coordinates (η,θ), where θ is the Euclidean "time" and $-\infty<\eta<0$:

$$z = e^{\eta+i\theta},\ \bar{z} = e^{\eta-i\theta}. \tag{4.4.1}$$

The fermion fields $\Psi_{pol} = \begin{pmatrix} (\Psi_{pol})_R \\ (\Psi_{pol})_L \end{pmatrix}$ in these coordinates enter the Dirac action as

$$\mathcal{A} = \int_0^{2\pi} d\theta \int_{-\infty}^0 d\eta\, \bar{\Psi}_{pol}\left(\gamma^\eta\partial_\eta + \gamma^\theta\partial_\theta - \frac{r}{\sinh(\eta)}\right)\Psi_{pol}, \tag{4.4.2}$$

where $\bar{\Psi}_{pol} = \Psi_{pol}^\dagger\gamma^\theta$ and γ^η, γ^θ are, respectively, the same matrices as γ^{x}, γ^{y}. The angular Hamiltonian derived from this action is

$$H_A = \int_{-\infty}^0 d\eta : \Psi_{pol}^\dagger\gamma^\theta\left(\gamma^\eta\partial_\eta - \frac{r}{\sinh(\eta)}\right)\Psi_{pol} :, \tag{4.4.3}$$

where $\Psi_{pol}(\eta)$ and $\Psi_{pol}^\dagger(\eta)$ are now operators on the angular Hilbert space $\mathcal{H}_A$

satisfying the canonical anti-commutation relation

$$\{\Psi_{pol}(\eta), \Psi^\dagger_{pol}(\eta')\} = \mathbf{1}\delta(\eta - \eta').$$

The η-dependent mass term in the Hamiltonian (4.4.3) produces a "mass barrier" effect somewhat similar to the effect of the mass term in the theory on flat space [91]; it prevents the fermions from approaching too much the boundary of the disk. More precisely, it imposes vanishing asymptotic conditions for the fermion fields when $r > \frac{1}{2}$ (such asymptotic conditions are in fact allowed for all $r > 0$). With these asymptotic conditions, the Hamiltonian is diagonalized by the decomposition

$$\Psi_{pol}(\eta,\theta) = \int_{-\infty}^{\infty} \frac{d\nu}{\sqrt{2\pi}} c_\nu \mathcal{U}_\nu(\eta) e^{-\nu\theta} \ , \ \Psi^\dagger_{pol}(\eta,\theta) = \int_{-\infty}^{\infty} \frac{d\nu}{\sqrt{2\pi}} c^\dagger_\nu \mathcal{U}^\dagger_\nu(\eta) e^{\nu\theta} \qquad (4.4.4)$$

in terms of partial waves

$$\mathcal{U}_\nu = \begin{pmatrix} u_\nu \\ v_\nu \end{pmatrix},$$

with

$$\begin{aligned} u_\nu &= \frac{\Gamma(1+r)\Gamma\left(\frac{1}{2}+r-i\nu\right)}{\Gamma(1+2r)\Gamma\left(\frac{1}{2}-i\nu\right)} e^{i\nu\eta}(1-e^{2\eta})^r F(r,\frac{1}{2}+r+i\nu;1+2r;1-e^{2\eta}) \\ v_\nu &= -i\frac{\Gamma(1+r)\Gamma\left(\frac{1}{2}+r-i\nu\right)}{\Gamma(1+2r)\Gamma\left(\frac{1}{2}-i\nu\right)} e^{-i\nu\eta}(1-e^{2\eta})^r F(r,\frac{1}{2}+r-i\nu;1+2r;1-e^{2\eta}), \end{aligned} \qquad (4.4.5)$$

where $F(a,b;c;z)$ is Gauss's hypergeometric function on its principal branch. In (4.4.4), the operators c_ν, $c^\dagger_\nu$ satisfy the canonical anti-commutation relations

$$\{c^\dagger_\nu, c_{\nu'}\} = \delta(\nu - \nu').$$

The angular Hilbert space $\mathcal{H}_A$ is the fermionic Fock space over this algebra, with vacuum vector $|0\rangle_A$ defined by

$$c_\nu|0\rangle_A = 0 \quad (\nu > 0), \qquad c^\dagger_\nu|0\rangle_A = 0 \quad (\nu < 0).$$

With an appropriate normal-ordering, the Hamiltonian takes the form

$$H_A = \int_0^\infty d\nu\, \nu(c^\dagger_\nu c_\nu + c_{-\nu}c^\dagger_{-\nu}).$$

4.5 Form factors from angular quantization

We now consider the embedding $\mathcal{H} \hookrightarrow \mathcal{H}_A \otimes \mathcal{H}_A^*$ that will allow us to calculate form factors of scaling fields $\langle \mathrm{vac}|\mathcal{O}_\alpha(0)|\omega_1, \ldots, \omega_n\rangle_{\epsilon_1,\ldots,\epsilon_n}$. The embedding is described by identifying vectors in the Hilbert space $\mathcal{H}$ with endomorphisms on the angular Hilbert space $\mathcal{H}_A$:

$$|\omega_1, \ldots, \omega_n\rangle_{\epsilon_1,\ldots,\epsilon_n} \equiv a_{\epsilon_1}(\omega_1) \cdots a_{\epsilon_n}(\omega_n) e^{-\pi H_A}, \tag{4.5.1}$$

where the operators $a_\epsilon(\omega) \in \mathrm{End}(\mathcal{H}_A)$ are to be determined. Notice that the vacuum $|\mathrm{vac}\rangle$ is identified with $e^{-\pi H_A}$. The scalar product on $\mathcal{H}$ is identified with the canonical scalar product on the space $\mathrm{End}(\mathcal{H}_A)$, which coincides with the expression of correlation functions as traces on $\mathcal{H}_A$:

$$\langle u|v\rangle \equiv \frac{\mathrm{Tr}\left(U^\dagger V\right)}{\mathrm{Tr}\left(e^{-2\pi H_A}\right)} \;\text{ if } |u\rangle \equiv U,\ |v\rangle \equiv V. \tag{4.5.2}$$

The representation of a field on $\mathcal{H}$ is identified with its representation on $\mathcal{H}_A$:

$$\pi_{\mathcal{H}}(\mathcal{O})|u\rangle \equiv \pi_A(\mathcal{O})U \;\text{ if } |u\rangle \equiv U,$$

where π_A is the representation map on the space $\mathcal{H}_A$.

The operators $a_\epsilon(\omega)$ can be fixed by imposing two conditions. First, the operators on $\mathcal{H}_A$ representing fields at the center of the disk that are mutually local with the fermion field must commute (if they are bosonic) or anti-commute (if they are fermionic) with the operators $a_\epsilon(\omega)$. It is sufficient to impose this condition with the fermion operators themselves:

$$\{a_\epsilon(\omega), \Psi_{pol}(\eta \to -\infty)\} = \{a_\epsilon(\omega), \Psi^\dagger_{pol}(\eta \to -\infty)\} = 0. \tag{4.5.3}$$

Second, the embedding (4.5.1) must reproduce the form factors of fermion fields $\langle \mathrm{vac}|\Psi_s(\xi_\mathrm{x}, \xi_\mathrm{y})|\omega\rangle_+ = e^{-i\omega\xi_\mathrm{x}} P_+(\omega, \xi_\mathrm{y})$ and $\langle \mathrm{vac}|\Psi_s^\dagger(\xi_\mathrm{x}, \xi_\mathrm{y})|\omega\rangle_- = e^{-i\omega\xi_\mathrm{x}} P_-(\omega, \xi_\mathrm{y})$ obtained from the partial wave decomposition (4.3.3).

These two conditions are solved as follows. We first solve the locality condition (4.5.3) for a set of operators $Z_\epsilon(t)$ depending on a complex parameter t:

$$\{Z_\epsilon(t), \Psi_{pol}(\eta\to-\infty)\} = \{Z_\epsilon(t), \Psi_{pol}^\dagger(\eta\to-\infty)\} = 0,$$

with the following ansatz:

$$Z_+(t) = \int_{-\infty}^{\infty} d\nu\, f(\nu)\, c_\nu^\dagger\, t^{-i\nu}\ ,\ Z_-(t) = \int_{-\infty}^{\infty} d\nu\, f(\nu)\, c_{-\nu}\, t^{-i\nu}.$$

This leads to the equations:

$$\lim_{\eta\to-\infty}\int_{-\infty}^{\infty} d\nu\, e^{i\nu\eta} f(\nu) t^{-i\nu} = 0,\ \lim_{\eta\to-\infty}\int_{-\infty}^{\infty} d\nu\, S(\nu)(2R)^{-2i\nu} e^{-i\nu\eta} f(\nu) t^{-i\nu} = 0,$$

where $S(\nu)$ is given in (4.6.8). The first equation is satisfied if $f(\nu)$ is analytical in the lower half ν-plane and increases at most exponentially as $\Im m(\nu)\to-\infty$, and the second equation is satisfied if $S(\nu)f(\nu)$ is analytical in the upper-half ν-plane and increases at most exponentially as $\Im m(\nu)\to\infty$. Indeed, under such conditions it is possible, for η negative and large enough, to send the contour of integration to $\Im m(\nu)\to-\infty$ in the integral $\int_{-\infty}^{\infty} d\nu\, e^{i\nu\eta} f(\nu) t^{-i\nu}$, giving zero contribution, and similarly for the integral $\int_{-\infty}^{\infty} d\nu\, S(\nu)(2R)^{-2i\nu} e^{-i\nu\eta} f(\nu) t^{-i\nu}$ by sending the contour of integration to $\Im m(\nu)\to\infty$. This set of conditions on $f(\nu)$ forms a simple Riemann-Hilbert problem, a solution of which is:

$$f(\nu) = \sqrt{2\pi}\frac{\Gamma\left(\frac{1}{2}+r+i\nu\right)}{\Gamma\left(\frac{1}{2}+r\right)\Gamma\left(\frac{1}{2}+i\nu\right)}. \tag{4.5.4}$$

The operators $a_\epsilon(\omega)$ can then be formed by taking appropriate linear combinations of $Z_\epsilon(t)$. These linear combinations can be obtained by requiring that the states $|\omega\rangle_\epsilon \equiv a_\epsilon(\omega)e^{-\pi H_A}$ diagonalize the subgroup $\mathbb{K}$ and that they be correctly normalized. First consider states $|t\rangle_\epsilon \in \mathcal{H}$ embedded in $\mathcal{H}_A\otimes\mathcal{H}_A^*$ by the identification $|t\rangle_\epsilon \equiv Z_\epsilon(t)e^{-\pi H_A}$. Using this embedding, using integral representations of the hypergeometric functions involved in (4.4.5), and using the cyclic properties of the traces and the anti-commutation relations for the modes $c_\nu, c_\nu^\dagger$. we can calculate the following matrix elements:

$$\begin{aligned}
\langle \text{vac}|\pi_{\mathcal{H}}(\Psi_R(z,\bar{z}))|t\rangle_+ &= -i\frac{\Gamma(1+r)}{\Gamma\left(\frac{1}{2}+r\right)}(1-z\bar{z})^r t^{-\frac{1}{2}}(1-\bar{z}r)^{-r}(1-zt^{-1})^{-r-1} \\
\langle \text{vac}|\pi_{\mathcal{H}}(\Psi_L(z,\bar{z}))|t\rangle_+ &= \frac{\Gamma(1+r)}{\Gamma\left(\frac{1}{2}+r\right)}(1-z\bar{z})^r t^{\frac{1}{2}}(1-zt^{-1})^{-r}(1-\bar{z}t)^{-r-1} \ .
\end{aligned} \tag{4.5.5}$$

From this and from the transformation properties of the fermion fields one can infer the transformation properties of the states $|t\rangle_\epsilon$ under the isometry group $SU(1,1)$. It is apparent that the variable t should transform like a holomorphic coordinate. Then t^{-1} transforms like an anti-holomorphic coordinate and the factors $1-zt^{-1}$, $1-\bar{z}t$ and t all are covariant according to (B.1.10), (B.1.11) and (B.1.12). We then find that

$$\hat{g}\left|\frac{at+\bar{b}}{bt+\bar{a}}\right\rangle_\epsilon = H_{\frac{r}{2}+\frac{1}{4},g}(t)\bar{H}_{\frac{r}{2}+\frac{1}{4},g}(t^{-1})|t\rangle_\epsilon,$$

where the functions $H_{s,g}$ and $\bar{H}_{s,g}$ are the automorphic factors (B.1.7). Under the subgroup $\mathbb{K}$, holomorphic and anti-holomorphic coordinates transform in the same way. Then, by multiplying appropriate powers of $1-t^2$ and of t by the measure dt, it is easy to construct a covariant measure whose transformation properties under $\mathbb{K}$ exactly cancel those of the states $|t\rangle_\epsilon$. Using the fact that the ratio $(1-t)/(1+t)$ diagonalizes the action of $\mathbb{K}$ and integrating between fixed points of $\mathbb{K}$, we obtain the operators $a_\epsilon(\omega)$ that diagonalize $\mathbb{K}$:

$$a_\epsilon(\omega) = \frac{2^{-r-\frac{1}{2}}e^{-i\frac{\pi}{2}\left(r+\frac{1}{2}+i\frac{\omega}{2}\right)}\Gamma\left(\frac{1}{2}+r\right)^2}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}+r+i\frac{\omega}{2}\right)\Gamma\left(\frac{1}{2}+r-i\frac{\omega}{2}\right)}\int_{-1}^{1}dt(1-t)^{r-\frac{1}{2}+i\frac{\omega}{2}}(1+t)^{r-\frac{1}{2}-i\frac{\omega}{2}}t^{-r-\frac{1}{2}}Z_\epsilon(t), \tag{4.5.6}$$

where the integral is performed in the region $-\pi < \arg(t) < 0$ (other integration contours are possible but equivalent).

We can verify the orthonormality of the states $|\omega\rangle_\epsilon$ associated to the operators (4.5.6). We want to verify that

$${}_{\epsilon'}\langle\omega|\omega'\rangle_\epsilon \;=\; \frac{\mathrm{Tr}\left(e^{-2\pi H_A}a^\dagger_\epsilon(\omega)a_{\epsilon'}(\omega')\right)}{\mathrm{Tr}\left(e^{-2\pi H_A}\right)} = \frac{1}{\rho(\omega)}\,\delta(\omega-\omega')\,\delta_{\epsilon,\epsilon'}\ , \tag{4.5.7}$$

$$\langle \text{vac}|\omega,\omega'\rangle_{\epsilon,\epsilon'} \;=\; \frac{\text{Tr}\left(e^{-2\pi H_A}a_\epsilon(\omega)a_{\epsilon'}(\omega')\right)}{\text{Tr}\left(e^{-2\pi H_A}\right)} = 0\;.$$

Since the operators $a_\epsilon(\omega)$, $a^\dagger_\epsilon(\omega)$ are linear combinations of free modes, traces of products of such operators can be calculated by using Wick's theorem. Hence the two relations above are sufficient to show the orthonormality of all multi-particle states. Also, these multi-particle states have free fermionic S-matrix, that is, it is shown in Section B.2 that the operators $a_\epsilon(\omega)$ anti-commute among themselves:

$$\{a_{\epsilon_1}(\omega_1), a_{\epsilon_2}(\omega_2)\} = 0. \tag{4.5.8}$$

To show (4.5.7), change coordinates by setting $t=\tanh(\theta)$:

$$a_\epsilon(\omega) = \frac{e^{-i\frac{\pi}{2}\left(r+\frac{1}{2}+i\frac{\omega}{2}\right)}\Gamma\left(\frac{1}{2}+r\right)^2}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}+r+i\frac{\omega}{2}\right)\Gamma\left(\frac{1}{2}+r-i\frac{\omega}{2}\right)}\int_{-\infty}^{\infty} d\theta\;\sinh^{-r-\frac{1}{2}}(2\theta)\,e^{-i\omega\theta}Z_\epsilon(\tanh(\theta)). \tag{4.5.9}$$

It is easy to invert this relation:

$$\int_{-\infty}^{\infty} d\omega\, a_\epsilon(\omega)\,e^{i\omega\theta}\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}+r+i\frac{\omega}{2}\right)\Gamma\left(\frac{1}{2}+r-i\frac{\omega}{2}\right)}{e^{-i\frac{\pi}{2}\left(r+\frac{1}{2}+i\frac{\omega}{2}\right)}\Gamma\left(\frac{1}{2}+r\right)^2} = 2\pi\;\sinh^{-r-\frac{1}{2}}(2\theta)\,Z_\epsilon(\tanh(\theta)). \tag{4.5.10}$$

It is possible to check that this equation implies both equations of (4.5.7) using the result

$$\begin{aligned}(\sinh(2\theta_1)\sinh(2\theta_2))^{-r-\frac{1}{2}}\;&\frac{\text{Tr}\left(e^{-2\pi H_A}Z_\epsilon(\tanh(\theta_1))Z_{\epsilon'}(\tanh(\theta_2))\right)}{\text{Tr}\left(e^{-2\pi H_A}\right)} \\ &= \delta_{\epsilon,-\epsilon'}\sqrt{\pi}\frac{\Gamma(1+r)}{\Gamma\left(\frac{1}{2}+r\right)}\;\cosh^{-1-2r}\left(\theta_1-\theta_2-i\frac{\pi}{2}\right)\end{aligned} \tag{4.5.11}$$

and the integral

$$\int_{-\infty}^{\infty} d\omega\,\Gamma\left(r+\frac{1}{2}+i\frac{\omega}{2}\right)\,\Gamma\left(r+\frac{1}{2}-i\frac{\omega}{2}\right)\,e^{-i\omega\gamma} = 2^{1-2r}\pi\,\Gamma(1+2r)\;\cosh^{-1-2r}(\gamma)\;. \tag{4.5.12}$$

In terms of the modes c_ν, the expression (4.5.6) gives

$$a_+(\omega) = \int_{-\infty}^{\infty} d\nu\, g(\nu;\omega)\,c^\dagger_\nu\,,\quad a_-(\omega) = \int_{-\infty}^{\infty} d\nu\, g(\nu;\omega)\,c_{-\nu}, \tag{4.5.13}$$

$$
\begin{aligned}
g(\nu;\omega) &= \sqrt{\pi}2^{-r}e^{i\frac{\pi}{2}\left(r+\frac{1}{2}-i\frac{\omega}{2}\right)}\frac{e^{-\pi\nu}\Gamma\left(\frac{1}{2}+r+i\nu\right)}{\Gamma\left(1+r\right)\Gamma\left(\frac{1}{2}+i\nu\right)} \\
&\quad\times F\left(r+\frac{1}{2}+i\nu, r+\frac{1}{2}-i\frac{\omega}{2}; 1+2r; 2-i0\right) .
\end{aligned}
$$

Here and below we use the notation

$$
F(a,b;c;2\pm i0) = \lim_{\varepsilon\to 0^+} F(a,b;c;2\pm i\varepsilon),
$$

where on the right hand side $F(a,b;c;z)$ is Gauss's hypergeometric function on its principal branch. The solution (4.5.13) to the two conditions above is unique. One can verify for instance that

$$
\frac{\mathrm{Tr}\left(e^{-2\pi H_A}\pi_A(\Psi_s(\xi_{\mathrm{x}},\xi_{\mathrm{y}}))a_+(\omega)\right)}{\mathrm{Tr}\left(e^{-2\pi H_A}\right)} = e^{-i\omega\xi_{\mathrm{x}}}P_+(\omega,\xi_{\mathrm{y}}) .
$$

Using the contractions (4.5.7) and the fact that any form factor of fermion fields in $\mathcal{H}$ can be evaluated by Wick's theorem, the embedding (4.5.1) reproduces all form factors of fermion fields.

From the expression (4.5.13) one can verify the following identity:

$$
a_\epsilon^\dagger(\omega) = e^{\pi H_A}a_{-\epsilon}(\omega)e^{-\pi H_A}.
$$

This is the analogue of crossing symmetry present in theories on flat space. For the scaling fields $\mathcal{O}_\alpha$, which are scalar and have the property $\mathcal{O}_\alpha^\dagger = \mathcal{O}_{-\alpha}$, this leads for instance to

$$
\left(F_\alpha(\omega_1,\ldots,\omega_n)_{\epsilon_1,\ldots,\epsilon_n}\right)^* = F_{-\alpha}(\omega_n,\ldots,\omega_1)_{-\epsilon_n,\ldots,-\epsilon_1} \tag{4.5.14}
$$

using the notation (4.3.10).

From the representation (4.3.9) of the scaling fields on $\mathcal{H}$, we have the following embedding:

$$
\mathcal{O}_\alpha(0)|u\rangle \equiv e^{2\pi i\alpha Q}U \ \text{ if } |u\rangle \equiv U,
$$

where Q is the $U(1)$ charge in angular quantization:

$$
Q = \int_0^\infty d\nu\,(c_\nu^\dagger c_\nu - c_{-\nu}c_{-\nu}^\dagger). \tag{4.5.15}
$$

Using this embedding, form factors of the scaling fields $\mathcal{O}_\alpha$ are given heuristically by the following traces:

$$\langle \text{vac}|\mathcal{O}_\alpha(0)|\omega_1,\ldots,\omega_n\rangle_{\epsilon_1,\ldots,\epsilon_n} = \frac{\text{Tr}\left(e^{-2\pi H_A+2\pi i\alpha Q}a_{\epsilon_1}(\omega_1)\cdots a_{\epsilon_n}(\omega_n)\right)}{\text{Tr}\left(e^{-2\pi H_A}\right)}.$$

Both traces in the ratio above are ill-defined and need some regularization Tr $\to$ Tr$_\varepsilon$. In the next subsection we will regularize such traces by doing the angular quantization on a Poincaré disk from which a small disk of radius ε around the origin has been removed. As the regularization parameter disappears $\varepsilon \to 0$, the resulting ratio of traces above is then singular and goes as ε^{α^2}. We can cancel out this singularity by considering normalized form factors:

$$\frac{\langle \text{vac}|\mathcal{O}_\alpha(0)|\omega_1,\ldots,\omega_n\rangle_{\epsilon_1,\ldots,\epsilon_n}}{\langle \text{vac}|\mathcal{O}_\alpha|\text{vac}\rangle} = \frac{\text{Tr}\left(e^{-2\pi H_A+2\pi i\alpha Q}a_{\epsilon_1}(\omega_1)\cdots a_{\epsilon_n}(\omega_n)\right)}{\text{Tr}\left(e^{-2\pi H_A+2\pi i\alpha Q}\right)}. \quad (4.5.16)$$

These can be calculated without explicit reference to a regularization procedure, simply by using, as above, the cyclic properties of the trace and the anti-commutation relations for the modes c_ν, $c_\nu^\dagger$. The calculation of the one-point function requires the use of an explicit regularization procedure:

$$\langle \mathcal{O}_\alpha\rangle = \langle \text{vac}|\mathcal{O}_\alpha|\text{vac}\rangle = \lim_{\varepsilon\to 0}\varepsilon^{-\alpha^2}\frac{\text{Tr}_\varepsilon\left(e^{-2\pi H_A+2\pi i\alpha Q}\right)}{\text{Tr}_\varepsilon\left(e^{-2\pi H_A}\right)}, \quad (4.5.17)$$

and will be done in the Section 4.6.

The traces (4.5.16) are calculated in Appendix B.2. In particular, the "two-particle" form factors are given in (B.2.5). Other "multi-particle" form factors of scaling fields $\mathcal{O}_\alpha(0)$ can be constructed from these "two-particle" form factors by Wick's theorem, as in (B.2.10). I also verified that the expression (B.2.5) for the "two-particle" form factors specializes to the known expression for form factors in the flat-space limit.

4.6 One-point function

I now calculate the one-point function $\langle \mathcal{O}_\alpha\rangle$. This is a simple generalization of the calculation done in [141, 91] for similar vacuum expectation values in flat space.

Consider regularizing the trace $\mathrm{Tr} \to \mathrm{Tr}_\varepsilon$ by cutting a small disk of radius ε around the origin and considering the angular quantization of the theory (4.1.1) on the resulting annulus with the boundary conditions

$$[(\Psi_{pol})_R(\eta) - (\Psi_{pol})_L(\eta)]_{\eta=\ln\tanh\left(\frac{\varepsilon}{2R}\right)} = [(\Psi_{pol})_R^\dagger(\eta) - (\Psi_{pol})_L^\dagger(\eta)]_{\eta=\ln\tanh\left(\frac{\varepsilon}{2R}\right)} = 0. \tag{4.6.1}$$

Then the one-point function $\langle \mathcal{O}_\alpha \rangle$ can be expressed as (4.5.17), where the $U(1)$ charge Q in the regularized theory is

$$Q = \int_{\ln\tanh\left(\frac{\varepsilon}{2R}\right)}^{0} d\eta : \Psi_{pol}^\dagger \Psi_{pol} :,$$

which specializes to (4.5.15) in the limit $\varepsilon \to 0$. It was shown in [141, 91] that for the theory on flat space, the definition (4.5.17) with the boundary conditions (4.6.1) are in accordance with the conformal normalization (4.2.1). Since the leading behavior at short distances of the two-point function is not affected by the curvature, the expression (4.5.17) with the boundary conditions (4.6.1) lead to the same conformal normalization (4.2.1) for the theory on the Poincaré disk.

Using the Poincaré disk as an infrared regulator, an argument for the validity of (4.5.17) with the boundary conditions (4.6.1) goes as follows. The trace expression in (4.5.17) can be written in radial quantization as the overlap:

$$\frac{\mathrm{Tr}_\varepsilon\left(e^{-2\pi H_A + 2\pi i\alpha Q}\right)}{\mathrm{Tr}_\varepsilon\left(e^{-2\pi H_A}\right)} = {}_{\mathrm{radial}}\langle \mathrm{vac}|\varepsilon\rangle_\alpha\ . \tag{4.6.2}$$

Here the state $|\varepsilon\rangle_\alpha$ implements the boundary conditions (4.6.1) along with the condition that the fermion fields take a phase, $\Psi \to e^{2\pi i\alpha}\Psi$, $\bar\Psi \to e^{-2\pi i\alpha}\bar\Psi$, when brought counterclockwise around (and outside of) the circle of geodesic radius ε concentric with the disk. Of course, this monodromy condition in principle requires two singular points delimiting a branch cut of the field configurations on the Poincaré disk, but since the fields vanish at infinite geodesic distances (on the boundary of the disk), the vacuum state $\langle \mathrm{vac}|$ automatically supports the end of a branch cut. Remember also that the one-point function can be written as

$$\langle \mathcal{O}_\alpha \rangle = {}_{\mathrm{radial}}\langle \mathrm{vac}|\mathcal{O}_\alpha(0)|\mathrm{vac}\rangle_{\mathrm{radial}}\ . \tag{4.6.3}$$

In (4.6.2), the state $|\varepsilon\rangle_\alpha$ can be expressed as an infinite series of local operators at the center of the disk acting on the vacuum state $|\mathrm{vac}\rangle$ as follows:

$$|\varepsilon\rangle_\alpha = C\,\varepsilon^{\alpha^2}\mathcal{O}_\alpha(0)|\mathrm{vac}\rangle_{\mathbf{radial}} + O\left(\varepsilon^{\alpha^2+1}\right) \tag{4.6.4}$$

where C is a number. The contributions that are not written come from positive powers of $m\epsilon$ and from descendants of the field $\mathcal{O}_\alpha(0)$. The number C does not depend on the product $r = mR$ since it is a short-distance characteristics and is unaffected by the curvature. It can be calculated by taking the limit $m \to 0$ of the theory (4.1.1). This is a theory of free massless Dirac fermions on the disk of radius $2R$ with conformal boundary conditions. For this boundary conformal field theory (BCFT), the explicit calculation of the trace (4.6.2) gives (see below):

$${}_{\mathbf{radial}}\langle \mathrm{vac}|\varepsilon\rangle_\alpha^{BCFT} = \left(\frac{\varepsilon}{2R}\right)^{\alpha^2}. \tag{4.6.5}$$

On the other hand, the conformal normalization (4.2.1) of the operator $\mathcal{O}_\alpha$ gives

$${}_{\mathbf{radial}}\langle \mathrm{vac}|\mathcal{O}_\alpha(0)|\mathrm{vac}\rangle_{\mathbf{radial}}^{BCFT} = (2R)^{-\alpha^2}. \tag{4.6.6}$$

Replacing the state $|\varepsilon\rangle_\alpha^{BCFT}$ in (4.6.5) by its expansion given by (4.6.4) (recall that the term explicitly written is the same in the boundary conformal field theory as it is in the massive theory), one finds

$$C = 1.$$

Putting the expansion (4.6.4) in (4.6.2) and comparing with (4.6.3), we then obtain

$$\frac{\mathrm{Tr}_\varepsilon\left(e^{-2\pi H_A + 2\pi i\alpha Q}\right)}{\mathrm{Tr}_\varepsilon\left(e^{-2\pi H_A}\right)} \sim \varepsilon^{\alpha^2}\langle\mathcal{O}_\alpha\rangle \quad \text{as} \quad \varepsilon \to 0 \tag{4.6.7}$$

which gives the result (4.5.17).

In order to evaluate the limit $\varepsilon \to 0$ in the expression (4.5.17), we need the density of angular quantization states $\partial_\nu \ln(S(\nu))$, where the S-matrix $S(\nu)$ is associated with the scattering off the "mass barrier" described in the previous section:

$$\left.\begin{pmatrix} u_\nu \\ v_\nu \end{pmatrix}\right|_{\eta\to-\infty} = \begin{pmatrix} e^{i\nu\eta} \\ -ie^{-i\nu\eta}S(\nu)(2R)^{-2i\nu} \end{pmatrix}.$$

Here u_ν and v_ν are the partial waves (4.4.5). The S-matrix is given by:

$$S(\nu) = (2R)^{2i\nu}\frac{\Gamma(1/2+i\nu)\Gamma(1/2-i\nu+r)}{\Gamma(1/2-i\nu)\Gamma(1/2+i\nu+r)}. \tag{4.6.8}$$

Then one finds

$$\langle\mathcal{O}_\alpha\rangle = \exp\left[\int_0^\infty \frac{d\nu}{2\pi i}\ln\left(\frac{(1+e^{-2\pi\nu+2\pi i\alpha})(1+e^{-2\pi\nu-2\pi i\alpha})}{(1+e^{-2\pi\nu})^2}\right)\partial_\nu \ln S(\nu)\right]. \tag{4.6.9}$$

The result of the integration using (4.6.8) can be expressed in various ways:

$$\begin{aligned} \langle\mathcal{O}_\alpha\rangle &= (2R)^{-\alpha^2}\exp\left[\int_0^\infty \frac{dt}{t}(1-e^{-2rt})\frac{\sinh^2(\alpha t)}{\sinh^2(t)}\right] \\ &= (2R)^{-\alpha^2}\prod_{n=1}^{\infty}\left(\frac{1-\frac{\alpha^2}{(r+n)^2}}{1-\frac{\alpha^2}{n^2}}\right)^n \\ &= (2R)^{-\alpha^2}\frac{G(1+r-\alpha)G(1+r+\alpha)}{G(1+r)^2G(1-\alpha)G(1+\alpha)}, \end{aligned} \tag{4.6.10}$$

where $G(z)$ is Barnes' G-function, characterized mainly by the properties $G(z+1)=\Gamma(z)G(z)$ and $G(1)=1$ (cf. [122]).

Solving numerically the differential equation (4.1.5) with $\gamma=0$ by using the appropriate initial conditions, it is possible to verify the consistency of our result (4.2.7) for the constant involved in the long distance asymptotics of w, our result (4.6.10) for the one-point function, and the value (4.2.6) for the constant involved in the short distance asymptotics of w. We used as initial condition the long distance asymptotics (4.8.5) with non-zero coefficients $D_{0,q}$ given in (4.8.6) and with the value (4.2.7) for the normalization constant. We numerically verified that the behavior (4.2.2) with the constant (4.2.6) is recovered, and that the equation (4.2.1) is satisfied to a high accuracy.

4.7 Hilbert space II

As argued in Section B.2, the "two-particle" form factors (B.2.5), like the partial waves (4.3.5), are entire functions of the spectral parameters ω_1 and ω_2 (hence all form factors of scaling fields $\mathcal{O}_\alpha$ are entire functions of their spectral parameters). Moreover, they have the following behavior as the real part of ω_1 or ω_2 goes to positive or negative infinity:

$$\sqrt{\rho(\omega_1)}F_\alpha(\omega_1,\omega_2)_{+,-} \sim |\omega_1|^{-1\pm\alpha} \text{ as } \Re e(\omega_1) \to \pm\infty$$
$$\sqrt{\rho(\omega_2)}F_\alpha(\omega_1,\omega_2)_{+,-} \sim |\omega_2|^{-1\mp\alpha} \text{ as } \Re e(\omega_2) \to \pm\infty$$

up to proportionality factors, and other "multi-particle" form factors have similar behaviors. Hence for $-1 < \alpha - \alpha' < 1$, the integrals over ω_j's in (4.3.11) are absolutely convergent. They can be evaluated by deforming the contours in their lower half planes and summing over the residues at the poles of the measure $\rho(\omega_j)$. The measure $\rho(\omega)$ has poles on the imaginary axis; in the lower half plane, they are at positions $\omega = -i(1+2r+2k)$ for $k \in \mathbb{N}$.

It is instructive to interpret the poles in the measure $\rho(\omega)$, and the evaluation of the integrals in (4.3.11) by contour deformation as described above, in terms of a different quantization scheme. Let us summarize our construction. We choose an isometry $\mathbb{K}$ (4.3.1), subgroup of the full isometry group $SU(1,1)$, and we quantize the theory (4.1.1) on orbits of this isometry. That is, the generator K of the subgroup $\mathbb{K}$ generates "space" translations. A basis for the corresponding Hilbert space $\mathcal{H}$ is taken as a set of states diagonalizing K. They are normalized so that all form factors of local fields are entire functions of its eigenvalues ω. With this normalization, the resolution of the identity on $\mathcal{H}$ in terms of this basis involves a measure $\rho(\omega)$ (4.3.4) with a specific analytical structure, in particular with singularities on the imaginary axis. Now, these singularities should give information about the spectrum in a quantization scheme where K is taken as the Hamiltonian, that is, as the generator of "time" translations.

This last assertion is the analogue of what is well known to happen for instance in a free massive theory on flat space: the singularity structure of the invariant measure $\rho(p) = (m^2+p^2)^{-\frac{1}{2}}$ as function of the momentum p gives the energy spectrum. Momentum operator and energy operator (or Hamiltonian) can be seen as representations of the same translation generator in two different quantization scheme, one where this translation is along the "space" direction, the other where it is along the "time" direction.

In a free theory on flat space, the invariant measure exhibits in particular a branch cut starting at $p = -im$ and going to $p \to -i\infty$, corresponding to the continuous spectrum (from m to infinity) of the energy operator. In our free theory on the Poincaré disk, the position of the poles of the measure (4.3.4) on the imaginary axis are interpreted as the discrete eigenvalues of the generator K in the scheme where it is taken as the Hamiltonian. It is a simple matter to repeat the canonical quantization procedure of section 3 for this quantization scheme. One indeed finds a discrete set of eigenstates $|k_1,\ldots,k_n\rangle_{\epsilon_1,\ldots,\epsilon_n},\ k_j\in\mathbb{N},\ n=0,1,2,\ldots$ with eigenvalues $\lambda_1+\cdots+\lambda_n$, where

$$\lambda_j = 1+2r+2k_j.$$

Since these states diagonalize a generator of time translation, they can be more naturally interpreted as multi-particle states. Matrix elements of the operator $\mathcal{O}_\alpha$ between the vacuum $|\widetilde{\mathrm{vac}}\rangle$ and multi-particle states in this scheme can be found from similar matrix elements in the Hilbert space $\mathcal{H}$ by the following identification:

$$\begin{aligned}\langle\widetilde{\mathrm{vac}}|\mathcal{O}_\alpha|k_1,\ldots,k_n\rangle_{\epsilon_1,\ldots,\epsilon_n} &=\\ &\langle\mathcal{O}_\alpha\rangle\prod_{j=1}^{n}\left(i^{k_j}\sqrt{\frac{2\Gamma(1+2r+k_j)}{k_j!}}\frac{1}{\Gamma\left(\frac{1}{2}+r\right)}\right)F_\alpha(-i\lambda_1,\ldots,-i\lambda_n)_{\epsilon_1,\ldots,\epsilon_n}\end{aligned}$$

and ${}_{\epsilon_1,\ldots,\epsilon_n}\langle k_n,\ldots,k_1|\mathcal{O}_\alpha|\widetilde{\mathrm{vac}}\rangle = \langle\widetilde{\mathrm{vac}}|\mathcal{O}_{-\alpha}|k_1,\ldots,k_n\rangle^*_{\epsilon_1,\ldots,\epsilon_n}$.

4.8 Long distance expansion of the two-point function

The resolution of the identity on the Hilbert space introduced in the previous section is given by the sum

$$1 = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\epsilon_1,\ldots,\epsilon_n} \sum_{k_1,\ldots,k_n \geq 0} |k_1,\ldots,k_n\rangle_{\epsilon_1,\ldots,\epsilon_n} \; {}_{\epsilon_1,\ldots,\epsilon_n}\langle k_1,\ldots,k_n|,$$

which provides a long distance expansion for the two-point function:

$$\begin{aligned}\langle \mathcal{O}_\alpha(x)\mathcal{O}_{\alpha'}(y)\rangle = &\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\epsilon_1,\ldots,\epsilon_n} \sum_{k_1,\ldots,k_n=0}^{\infty} \times \\ &\times \langle \widetilde{\mathrm{vac}}|\mathcal{O}_\alpha(0)|k_1,\ldots,k_n\rangle_{\epsilon_1,\ldots,\epsilon_n} \; {}_{\epsilon_n,\ldots,\epsilon_1}\langle k_n,\ldots,k_1|\mathcal{O}_{\alpha'}(0)|\widetilde{\mathrm{vac}}\rangle e^{-(\lambda_1+\cdots+\lambda_n)\frac{\mathrm{d}(x,y)}{2R}}.\end{aligned}$$

This expansion coincides with the sum of residues coming from the evaluation of the integrals in (4.3.11) by contour deformation. Using crossing symmetry (4.5.14) and summing over the $U(1)$ charges, this can be written

$$\begin{aligned}\langle \mathcal{O}_\alpha(x)\mathcal{O}_{\alpha'}(y)\rangle = &\langle \mathcal{O}_\alpha\rangle\langle \mathcal{O}_{\alpha'}\rangle \\ &\times \sum_{N=0}^{\infty} \frac{1}{(N!)^2} \sum_{k_1,\ldots,k_{2N}=0}^{\infty} f_\alpha(k_1,\ldots,k_{2N}) f_{\alpha'}(k_{2N},\ldots,k_1) e^{-(\lambda_1+\cdots+\lambda_{2N})\frac{\mathrm{d}(x,y)}{2R}},\end{aligned} \tag{4.8.1}$$

where

$$\langle \mathcal{O}_\alpha\rangle\, f_\alpha(k_1,\ldots,k_{2N}) = \langle \widetilde{\mathrm{vac}}|\mathcal{O}_\alpha(0)|k_1,\ldots,k_{2N}\rangle_{\underbrace{+,+,\ldots,}_{N}\underbrace{-,-,\ldots}_{N}}. \tag{4.8.2}$$

The functions $f_\alpha(k_1,\ldots,k_{2N})$ are evaluated in closed form in Section B.2, formula (B.2.7). The first few terms of (4.8.1) give the following expansion for the two-point function:

$$\begin{aligned}F(t) - 1 = &\\ -4^{2r+1}&\frac{\sin(\pi\alpha)\sin(\pi\alpha')\Gamma(1+r+\alpha)\Gamma(1+r-\alpha)\Gamma(1+r+\alpha')\Gamma(1+r-\alpha')}{\pi^2\Gamma(2+2r)^2} \times \\ &\times e^{-(4r+2)t}\left(1 - 2\alpha\alpha'\frac{2r+1}{(r+1)^2}e^{-2t} + O(e^{-4t})\right) \; + \; O(e^{-(8r+4)t}),\end{aligned} \tag{4.8.3}$$

where the function $F(t)$ was defined in (4.2.3).

It is interesting to note that in the massless limit $r \to 0$, the two-point function still decreases exponentially at long distances. This is a signal of the infrared regulator properties of the negative curvature of the Poincaré disk [20].

From [104], the two-point function can be described in terms of a Painlevé VI transcendent:

$$F(t) = \exp\left[-\int_{\tanh^2 t}^{1} ds \frac{d}{ds} \ln \tau(s)\right], \tag{4.8.4}$$

where $\frac{d}{ds}\ln\tau(s)$ is given in (4.1.7) and the Painlevé transcendent $w(s)$ satisfies (4.1.5) with $\gamma = 0$.

The leading long distance asymptotics (4.2.4) of this Painlevé transcendent is fixed by specifying the proportionality constant A. This constant can be obtained by comparing our form factor result (4.8.3) with the expansion (4.2.5); this leads to the value (4.2.7). Since we must have $0 < w < 1$ for all $0 < s < 1$, the asymptotics (4.2.4) is valid only for $A > 0$, and the asymptotics (4.2.2) only for $B > 0$. This imposes $\alpha\alpha' > 0$. From the behavior (4.2.4) and using the differential equation, one can obtain the full long distance expansion of w:

$$1 - w = A(1-s)^{1+2r} \sum_{p,q=0}^{\infty} D_{p,q}\, (1-s)^{p(1+2r)+q} \tag{4.8.5}$$

with $D_{0,0} = 1$. For instance, the coefficients $D_{0,q}$ are given by

$$\sum_{q=0}^{\infty} D_{0,q}\, (1-s)^q = s^{\alpha-\alpha'} F(1-\alpha'+r, \alpha+r; 1+2r; 1-s)^2. \tag{4.8.6}$$

The long distance expansion (4.8.5) can also be obtained from the full form factors expansion (4.8.1). For instance, one can verify that the coefficient $D_{0,1}$ in (4.8.6) is consistent with the coefficient of e^{-2t} in the parenthesis in the form factor expansion (4.8.3).

4.9 Conclusion

I have fully characterized the two-point function $\langle \mathcal{O}_\alpha(x)\mathcal{O}_{\alpha'}(y)\rangle$ in the Dirac theory on the Poincaré disk, with fermion mass m and Gaussian curvature $-\frac{1}{R^2}$, in the region $0 < \alpha + \alpha' < 1$, $\alpha\alpha' > 0$ and $mR > \frac{1}{2}$. A comparison of the long distance expansion with the expansion obtained from the Painlevé VI differential equation strongly suggests both that the "form factor" expansion is correct and that the Painlevé VI differential equation indeed describes the two-point function of the scaling fields $\mathcal{O}_\alpha$ in the region of parameters above. Notice that the results and description above in terms of the differential equation are valid in the region $0 < mR < \frac{1}{2}$ as well; however, they do not describe all the possible behaviors in this region.

It would be very interesting to extend this description to a larger region of α, α'. A similar restriction on the validity of the description in terms of a nonlinear differential equation also occurs on flat space, and to my knowledge it is still an open problem to extend this description to a larger region of values of α and α'.

Also, it is interesting to fully analyze the region $0 < mR < \frac{1}{2}$ and the analytical continuation to negative values of mR. This will be done in the next chapter in the Ising field theory on the pseudosphere. We will find that the region $-\frac{1}{2} < mR < \frac{1}{2}$ is the most interesting from the viewpoint of the thermodynamics of the model, as it supports many stables regimes.

Chapter 5

Ising field theory on the Poincaré disk

5.1 Introduction

In this chapter I will describe results concerning the Ising field theory on the Poincaré disk at zero magnetic field. The lattice Ising model on flat space at zero magnetic field and at a temperature very near to its critical temperature (more precisely, in the scaling limit) is described by the quantum field theory of a free massive Majorana fermion [102, 73] (see also [131]). Strictly speaking, the Ising field theory is the Majorana theory in the sector composed of three primary fields: the identity field, the energy field, related to the energy density in the lattice Ising model, and a local, interacting fields, denoted by σ, corresponding to the spin or order variable in the lattice Ising model. It is in fact convenient to consider all fields of the Majorana theory, which includes also the disorder field, μ, corresponding to the disorder variable of the lattice Ising model. The order variable is the natural two-state degree of freedom on each site of the lattice, and the disorder variable is a variable defined non-locally [69] with respect to the order variable. Correlation functions of disorder variables are obtained by a duality transformation from correlation functions of order variables, which brings the system from its low-temperature to its high-temperature regime and vice versa. A nonzero expectation value of the order variable means that the system is in its ordered (low-temperature) regime, and a nonzero expectation value of the disorder variable means that it is in its disordered (high-temperature) regime. The order and disorder fields σ and μ are local fields, but they are not mutually local with respect to the Majorana fermion fields or with respect to each other.

It is natural to assume that the Ising quantum field theory on the pseudosphere represents the scaling limit of an Ising-like statistical system on a lattice embedded into the pseudoshpere. Although we do not have yet a precise construction of this statistical system and of its scaling limit, I will keep in mind, below, the situation depicted in Chapter 3.

In the next sections, I will first study the Hilbert space of the model and find some of its stable regimes. Then I will calculate the average of the order field, or magnetization, and the form factors of order and disorder fields from results of the previous chapter, and I will derive the nonlinear differential equations that specify the order-order and disorder-disorder correlation functions.

5.2 Definition of the model

The Ising field theory on the Poincaré disk can be described in terms of the boundary Ising conformal field theory on the disk [22] perturbed by the energy field $\varepsilon(x)$, with action

$$\mathcal{A} = \mathcal{A}_{\rm BI} - \frac{m}{2\pi}\int_{\rm disk} d^2x \; e^{\phi(x)}\,\varepsilon(x)\ . \tag{5.2.1}$$

Here, $\mathcal{A}_{\rm BI}$ stands for the action of the Ising conformal field theory on the unit disk and $d^2x\, e^{\phi(x)}$ is the volume element in the metric of the Poincaré disk (B.1.2):

$$e^{\phi(x)} = \frac{4R^2}{(1-z\bar{z})^2}\ . \tag{5.2.2}$$

The action $\mathcal{A}_{\rm BI}$ can be written explicitly in terms of free massless Majorana Fermi fields. Then the action (5.2.1) is the same as the action for the Majorana theory on the pseudosphere with mass m:

$$\mathcal{A} = \frac{1}{2\pi}\int_{|z|<1} d^2x \;\left[\psi\bar\partial\psi + \bar\psi\partial\bar\psi + \frac{2i\,r}{1-z\bar z}\bar\psi\psi\right]\ , \tag{5.2.3}$$

where we have used for the energy field

$$\varepsilon(x) = i(2R)^{-1}(1 - z\bar{z})\,(\psi\bar{\psi})(x)\ . \tag{5.2.4}$$

In (5.2.3), the parameter r is related to the mass parameter m and Gaussian curvature (B.1.1) as in (4.1.2), $(z,\bar{z})$ are complex coordinates on the unit disk $|z| < 1$, $\partial \equiv \partial_z = \frac{1}{2}(\partial_{\mathrm{x}} - i\partial_{\mathrm{y}})$, $\bar{\partial} \equiv \partial_{\bar{z}} = \frac{1}{2}(\partial_{\mathrm{x}} + i\partial_{\mathrm{y}})$ and $d^2x \equiv d\mathrm{x}\,d\mathrm{y}$; (x,y) are cartesian coordinates on the disk related in the usual way to the complex coordinates, $z = \mathrm{x} + i\mathrm{y}$, $\bar{z} = \mathrm{x} - i\mathrm{y}$.

The chiral components ψ and $\bar{\psi}$ obey the linear field equations

$$\bar{\partial}\,\psi(x) = \frac{i\,r}{1 - z\bar{z}}\,\bar{\psi}(x)\ , \qquad \partial\,\bar{\psi}(x) = \frac{-i\,r}{1 - z\bar{z}}\,\psi(x)\ , \tag{5.2.5}$$

and are normalized in (5.2.3) in accordance with the short-distance limit

$$(z - z')\,\psi(x)\psi(x') \to 1\ , \qquad (\bar{z} - \bar{z}')\,\bar{\psi}(x)\bar{\psi}(x') \to 1 \qquad \text{as} \qquad |x - x'| \to 0\ . \tag{5.2.6}$$

The action (5.2.1) is invariant under $SU(1,1)$ transformations described by (B.1.4). The Fermi fields ψ and $\bar{\psi}$ have dimensions $\left(\frac{1}{2},0\right)$ and $\left(0,\frac{1}{2}\right)$, respectively, and the energy field ε, as defined in (5.2.4), has dimension $(0,0)$. In order for the full quantum theory to be $SU(1,1)$-invariant, we have to impose appropriate $SU(1,1)$-invariant asymptotic conditions at the disk boundary $|z| \to 1$, as will be discussed below.

The zero curvature limit $R \to \infty$ corresponds to the familiar theory of free massive Majorana fermion in flat space (and mass m) after rescaling

$$z \mapsto z/(2R) \qquad \text{and} \qquad \psi(x) \mapsto (2R)^{1/2}\,\psi(x)\ , \tag{5.2.7}$$

with similar rescaling for $\bar{z}$ and $\bar{\psi}$.

5.3 Hilbert space and stable regimes

The boundary Ising conformal field theory $\mathcal{A}_{\mathrm{BI}}$ admits “free” and “fixed” boundary conditions [22], whereby the order field has, respectively, zero and nonzero vacuum expectation value[1] (and vice-versa for the disorder field). Likewise, by constructing the Hilbert space and demanding that the Hamiltonian be hermitian, I will show that when the energy perturbation is turned on ($m \neq 0$), the resulting theory (5.2.1) on the pseudosphere possesses in the region $-\frac{1}{2} < mR < \frac{1}{2}$ stable asymptotic conditions corresponding to "free" and "fixed" conditions: the order field has, respectively, zero and nonzero vacuum expectation value, and vice-versa for the disorder field. In the domain $mR > \frac{1}{2}$, only the "fixed" asymptotic condition is stable, whereas in the domain $mR < -\frac{1}{2}$, only the "free" asymptotic condition is stable (this description is in close connection with some results of, for instance, [19]). Situations with "free" and "fixed" asymptotic conditions can be obtained from one another by duality transformation, which interchanges the order and disorder fields $\sigma(x) \leftrightarrow \mu(x)$ and reverses the sign of the energy field $\varepsilon(x) \mapsto -\varepsilon(x)$. In the following sections, I will assume "fixed" asymptotic condition and $mR > -\frac{1}{2}$. Details regarding the construction of the Hilbert space can be found in Appendix B.3

A particularly convenient Hilbert space is the one obtained by associating time translations to isometry transformations corresponding to the non-compact subgroup $\mathbb{K}$ (4.3.1) of $SU(1,1)$. This Hilbert space is similar to the one developed briefly in Section 4.7 of the previous chapter for the Dirac theory. More precisely, we consider the system of "isometric" coordinates $(\xi,\bar\xi)$ related to coordinates $(z,\bar z)$ on the Poincaré disk by

$$z = \tan\xi\ , \qquad \bar z = \tan\bar\xi\ , \tag{5.3.1}$$

where $\xi = \xi_{\rm x} + i\,\xi_{\rm y}$ and $\bar\xi = \xi_{\rm x} - i\,\xi_{\rm y}$ (see Figure 5.1). The transformations

$$\xi_{\rm y} \mapsto \xi_{\rm y} + q\ , \qquad \text{for arbitrary real } q\ , \tag{5.3.2}$$

are the integrated versions of the second infinitesimal transformation of (B.1.13)

[1] With the "fixed" boundary condition, the order field can be fixed to a positive or a negative value at the boundary; we will choose to fix it to a positive value throughout the paper.

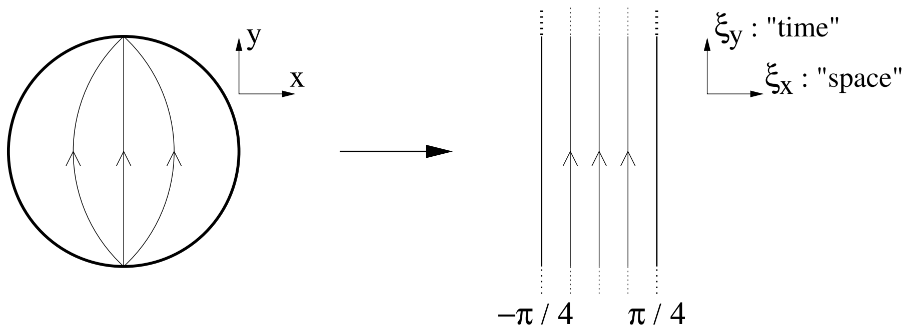

Figure 5.1: Mapping from the Poincaré disk to the strip described by $\mathrm{x} + i\mathrm{y} = \tan(\xi_{\mathrm{x}} + i\xi_{\mathrm{y}})$, $\mathrm{x} - i\mathrm{y} = \tan(\xi_{\mathrm{x}} - i\xi_{\mathrm{y}})$. Lines with an arrow represent orbits of a non-compact subgroup of the isometry group of the Poincaré disk.

and are elements of $\mathbb{K}$. We choose ξ_{x} to represent the space coordinate and ξ_{y} the Euclidean time coordinate, and the Hamiltonian is the integral of motion associated to (5.3.2).

The integrals of motion defined in (B.1.15) can be written as integrals over lines of constant ξ_{y},

$$\begin{aligned} \mathbf{P} &= \frac{1}{4\pi}\int_{\mathcal{C}}\left[\psi\,\mathcal{P}\psi\,dz - \bar{\psi}\,\mathcal{P}\bar{\psi}\,d\bar{z}\right] , \\ \bar{\mathbf{P}} &= \frac{1}{4\pi}\int_{\mathcal{C}}\left[\bar{\psi}\,\bar{\mathcal{P}}\bar{\psi}\,d\bar{z} - \psi\,\bar{\mathcal{P}}\psi\,dz\right] , \\ \mathbf{R} &= \frac{1}{4\pi i}\int_{\mathcal{C}}\left[\psi\,\mathcal{R}\psi\,dz - \bar{\psi}\,\mathcal{R}\bar{\psi}\,d\bar{z}\right] , \end{aligned} \tag{5.3.3}$$

where

$$\mathcal{C} = \{(z,\bar{z}) \mid -\pi/4 < \xi_{\mathrm{x}} < \pi/4 \ , \quad \xi_{\mathrm{y}} = \mathrm{const.}\} \ . \tag{5.3.4}$$

Inside correlation functions with some local fields, these integrals are independent of the value of ξ_{y} associated to the path $\mathcal{C}$, except for contributions at the positions of the fields, given by the corresponding Lie derivatives as in (B.1.15). The Hamiltonian is then simply

$$\mathbf{H} = \mathbf{P} + \bar{\mathbf{P}} \ , \tag{5.3.5}$$

and a basis can be obtained by diagonalizing it, as in Section 4.7. The corresponding states can be interpreted as “particle states”, since they form an irreducible representation of the $SU(1,1)$ symmetry group.

The Hilbert space is defined as a module for the canonical equal-time anti-commutation relations of the fermion operators $\psi^{(\mathrm{iso})}(\xi_{\mathrm{x}},\xi_{\mathrm{y}})$, $\bar{\psi}^{(\mathrm{iso})}(\xi_{\mathrm{x}},\xi_{\mathrm{y}})$ in the isometric system of coordinates[2]:

$$\begin{aligned} \{\psi^{(\mathrm{iso})}(\xi_{\mathrm{x}},\xi_{\mathrm{y}}),\,\psi^{(\mathrm{iso})}(\xi'_{\mathrm{x}},\xi_{\mathrm{y}})\} &= -2\pi i\,\delta(\xi_{\mathrm{x}}-\xi'_{\mathrm{x}})\ , \\ \{\bar{\psi}^{(\mathrm{iso})}(\xi_{\mathrm{x}},\xi_{\mathrm{y}}),\,\bar{\psi}^{(\mathrm{iso})}(\xi'_{\mathrm{x}},\xi_{\mathrm{y}})\} &= 2\pi i\,\delta(\xi_{\mathrm{x}}-\xi'_{\mathrm{x}})\ . \end{aligned} \tag{5.3.6}$$

Invariance under the subgroup described by (5.3.2) imposes that the Fermi fields vanish as $\xi_{\mathrm{y}}\to\pm\infty$, which gives the following conditions on the vacuum state:

$$\lim_{\xi_{\mathrm{y}}\to-\infty}\psi^{(\mathrm{iso})}(\xi_{\mathrm{x}},\xi_{\mathrm{y}})|\mathrm{vac}\rangle = 0\ ,\qquad \lim_{\xi_{\mathrm{y}}\to+\infty}\langle\mathrm{vac}|\psi^{(\mathrm{iso})}(\xi_{\mathrm{x}},\xi_{\mathrm{y}}) = 0\ , \tag{5.3.7}$$

with similar conditions for $\bar{\psi}^{(\mathrm{iso})}$. Correlation functions of local fields are then expressed as time-ordered vacuum expectation values of corresponding operators; the time ordering puts operators from left to right in decreasing values of their variable ξ_{y}.

The Hilbert space is further specified by the asymptotic conditions imposed on the Fermi fields at the disk boundary. In order to guarantee their stability, one has to choose such conditions giving a Hilbert space on which the Hamiltonian has real eigenvalues bounded from below. Among these, I shall only consider those which respect the $SU(1,1)$ symmetry. Let me note however that for $-\frac{1}{2}<r<\frac{1}{2}$ there are stable asymptotic conditions which are not $SU(1,1)$ invariant.

For $r>\frac{1}{2}$, finiteness of matrix elements of the Hamiltonian gives the asymptotic conditions imposing Fermi fields to vanish as

$$\psi\sim e^{-m\,\mathrm{d}} \tag{5.3.8}$$

when d, the geodesic distance between the origin, say, and the position of the Fermi field, goes to infinity. These asymptotic conditions correspond to “fixed”

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[2]The Fermi fields in isometric coordinates are related to the Fermi fields on the Poincaré disk by (B.1.5), that is, $\psi^{(\mathrm{iso})}(\xi_{\mathrm{x}},\xi_{\mathrm{y}})=\psi(x)/\cos(\xi)$, $\bar{\psi}^{(\mathrm{iso})}(\xi_{\mathrm{x}},\xi_{\mathrm{y}})=\bar{\psi}(x)/\cos(\bar{\xi})$.

asymptotic conditions on the order field σ. For $0 < r < \frac{1}{2}$, the above asymptotic conditions are also stable, and there is an additional set of stable asymptotic conditions, by which the Fermi fields diverge as

$$\psi \sim e^{m\mathrm{d}}\,(1 + O(e^{-\mathrm{d}/R})) \tag{5.3.9}$$

as the geodesic distance to the origin goes to infinity. This second set corresponds to "free" asymptotic conditions on the order field σ. The field theory with this second set of asymptotic conditions can be obtained by analytically continuing the field theory with the first set of asymptotic conditions from the region $0 < r < \frac{1}{2}$ to the region $-\frac{1}{2} < r < 0$, and vice versa. Hence it will be sufficient in what follows to consider only the conditions specifying the asymptotic behaviour $\sim e^{-m\,\mathrm{d}}$ of the Fermi fields for $r > 0$.

With these asymptotic conditions, the Fermi fields, obeying Eqs. (5.2.5), admit expansions in partial waves as

$$\begin{aligned}
\psi(\xi_{\mathrm{x}}, \xi_{\mathrm{y}}) &= \cos\xi \sum_{n=0}^{\infty} \left[e^{i\frac{\pi}{2}n}\, A_n^\dagger e^{\omega_n \xi_{\mathrm{y}}} g_n(\xi_{\mathrm{x}}) - i\, e^{-i\frac{\pi}{2}n}\, A_n e^{-\omega_n \xi_{\mathrm{y}}} \bar{g}_n(\xi_{\mathrm{x}}) \right] \qquad (5.3.10)\\
\bar{\psi}(\xi_{\mathrm{x}}, \xi_{\mathrm{y}}) &= \cos\bar{\xi} \sum_{n=0}^{\infty} \left[e^{i\frac{\pi}{2}n} A_n^\dagger e^{\omega_n \xi_{\mathrm{y}}} \bar{g}_n(\xi_{\mathrm{x}}) + i\, e^{-i\frac{\pi}{2}n}\, A_n e^{-\omega_n \xi_{\mathrm{y}}} g_n(\xi_{\mathrm{x}}) \right]
\end{aligned}$$

with discrete energy spectrum

$$\omega_n = 2r + 2n + 1 \qquad (n \geq 0)\ . \tag{5.3.11}$$

Partial waves are given by

$$\begin{aligned}
g_n(\xi_{\mathrm{x}}) &= \frac{2^{1-2r}\sqrt{\pi}}{\Gamma\left(r + \frac{1}{2}\right)} \frac{\Gamma(2r + n + 1)^{1/2}}{\Gamma(n+1)^{1/2}} \qquad (5.3.12)\\
&\quad \times e^{-i\omega_n \xi_{\mathrm{x}} - i\frac{\pi}{2}n} \left(1 + e^{4i\xi_{\mathrm{x}}}\right)^r F\left(-n, r; 1 + 2r; 1 + e^{4i\xi_{\mathrm{x}}}\right),
\end{aligned}$$

with $\bar{g}_n(\xi_{\mathrm{x}})$ denoting its complex conjugate, and $F(a, b; c; x)$ stands for the Gauss hypergeometric function, here specialized to polynomials; phases in the decomposition (5.3.10) were chosen for later convenience when introducing particle states.

The functions $g_n(\xi_{\rm x})$, $\bar{g}_n(\xi_{\rm x})$ satisfy the orthogonality relations

$$\begin{aligned} \int_{-\pi/4}^{\pi/4} d\xi_{\rm x} \Big[g_n(\xi_{\rm x})\bar{g}_{n'}(\xi_{\rm x}) + \bar{g}_n(\xi_{\rm x})g_{n'}(\xi_{\rm x}) \Big] &= 4\pi\, \delta_{n,n'} \ , \\ \int_{-\pi/4}^{\pi/4} d\xi_{\rm x} \Big[g_n(\xi_{\rm x})g_{n'}(\xi_{\rm x}) - \bar{g}_n(\xi_{\rm x})\bar{g}_{n'}(\xi_{\rm x}) \Big] &= 0 \ , \end{aligned} \tag{5.3.13}$$

as well as the relations

$$g_n(\xi_{\rm x})\Big|_{r=-\frac{1}{2}} = -i\, g_{n-1}(\xi_{\rm x})\Big|_{r=\frac{1}{2}} \ , \qquad \bar{g}_n(\xi_{\rm x})\Big|_{r=-\frac{1}{2}} = i\, \bar{g}_{n-1}(\xi_{\rm x})\Big|_{r=\frac{1}{2}} \ . \tag{5.3.14}$$

The creation and annihilation operators $A_n^\dagger$ and A_n ($n \geq 0$) in (5.3.10) satisfy canonical anti-commutation relations as consequence of (5.3.6):

$$\{A_n^\dagger, A_{n'}\} = \delta_{n,n'} \ , \qquad \{A_n^\dagger, A_{n'}^\dagger\} = \{A_n, A_{n'}\} = 0 \ , \tag{5.3.15}$$

with the vacuum state $|{\rm vac}\rangle$ obeying, from (5.3.7),

$$A_n|{\rm vac}\rangle = 0 \qquad \text{for all} \qquad n \geq 0 \ . \tag{5.3.16}$$

A basis of N-particle states is obtained from the set of states

$$|n_1 \, \ldots \, n_N\rangle \equiv A_{n_1}^\dagger \, \ldots \, A_{n_N}^\dagger |{\rm vac}\rangle \ , \tag{5.3.17}$$

which diagonalize the Hamiltonian,

$$\mathbf{H}\,|n_1 \, \ldots \, n_N\rangle = \left(\sum_{i=1}^{N} \omega_{n_i} \right) |n_1 \, \ldots \, n_N\rangle \tag{5.3.18}$$

with energy eigenvalues ω_n, Eq. (5.3.11). The discretization of the energy spectrum is essentially a consequence of requiring trivial monodromy of the hypergeometric functions involved in the partial waves (5.3.12) as $\xi_{\rm x} \to \xi_{\rm x} + \pi/2$, a necessary condition in order to ensure the proper vanishing asymptotic behaviour at the boundary of the disk.

The action of the operators $\mathbf{P}$, $\bar{\mathbf{P}}$ and $\mathbf{R}$, defined in (5.3.3), can be easily determined from the fact that the above Hilbert space provides a lowest weight module for $SU(1,1)$. The raising and lowering operators, $\mathbf{J}_+$ and $\mathbf{J}_-$ respectively, are given by

$$\mathbf{J}_\pm = \mathbf{P} - \bar{\mathbf{P}} \pm 2i\,\mathbf{R} \ , \tag{5.3.19}$$

and are related by hermitian conjugation, $\mathbf{J}_+^\dagger = \mathbf{J}_-$. Together with the Hamiltonian (5.3.5), they satisfy the algebra

$$[\mathbf{H}, \mathbf{J}_\pm] = \pm 2\,\mathbf{J}_\pm \ , \qquad [\mathbf{J}_-, \mathbf{J}_+] = 4\,\mathbf{H} \ , \tag{5.3.20}$$

from which the action of $\mathbf{J}_\pm$ on eigenstates of the Hamiltonian follows:

$$\mathbf{J}_+|n\rangle = \alpha_n |n+1\rangle \ , \qquad \mathbf{J}_-|n\rangle = \alpha_{n-1}|n-1\rangle \ , \tag{5.3.21}$$

with

$$\alpha_n = 2\sqrt{(n+1)(2\,r+n+1)} \ . \tag{5.3.22}$$

5.4 Order and disorder fields

Besides the Fermi and the energy fields, other local fields are present in the theory. Two spin fields associated to the $\mathbb{Z}_2$ symmetry $(\psi, \bar{\psi}) \mapsto (-\psi, -\bar{\psi})$ of the action (5.2.3) can be defined, the order field $\sigma(x)$ and the disorder field $\mu(x)$. They are not mutually local with respect to the Fermi fields, since the products

$$\psi(x)\sigma(x') \ , \quad \bar{\psi}(x)\sigma(x') \ , \quad \psi(x)\mu(x') \ , \quad \bar{\psi}(x)\mu(x') \tag{5.4.1}$$

acquire negative signs when the point x is brought around x'. This property does not define the fields σ and μ uniquely. Besides having this property, they are required to be "primary" with respect to the action of the Fermi fields in the operator algebra. This fixes operator product expansions (OPE) of the form

$$\begin{aligned}
&\psi(x)\sigma(x') = \\
&\qquad \sum_{n=0}^{\infty} c_n \left[\sqrt{\frac{i}{2}}\, s_n(x,x')\, u_n(s)\, \mathcal{D}^n \mu(x') + \sqrt{\frac{-i}{2}}\, \bar{t}_n(x,x')\, v_n(s)\, \bar{\mathcal{D}}^n \mu(x')\right] \ , \\
&\bar{\psi}(x)\sigma(x') = \\
&\qquad \sum_{n=0}^{\infty} c_n \left[-\sqrt{\frac{i}{2}}\, t_n(x,x')\, v_n(s)\, \mathcal{D}^n \mu(x') + \sqrt{\frac{-i}{2}}\, \bar{s}_n(x,x')\, u_n(s)\, \bar{\mathcal{D}}^n \mu(x')\right] \ .
\end{aligned} \tag{5.4.2}$$

The factors s_n and t_n are given by

$$s_n(x,x') = \left(\frac{1-z'\bar{z}'}{1-z\bar{z}'}\right)^{n+\frac{1}{2}} (z-z')^{n-\frac{1}{2}} , \qquad (5.4.3)$$
$$t_n(x,x') = \left(\frac{1-z'\bar{z}'}{1-z\bar{z}'}\right)^{n+\frac{1}{2}} \frac{(z-z')^{n+\frac{1}{2}}}{1-\bar{z}z'} ;$$

s_n transforms under the representation $\left(\frac{1}{2},0\right)$ in x and $(-n,0)$ in x', and t_n under $\left(0,\frac{1}{2}\right)$ in x and $(-n,0)$ in x' (see Appendix B.1); the factors $\bar{s}_n$, $\bar{t}_n$ are their complex conjugates; the projective invariant s is given by (4.1.6), (B.1.3), i.e.

$$s = \frac{(z-z')(\bar{z}-\bar{z}')}{(1-z\bar{z}')(1-\bar{z}z')} ; \qquad (5.4.4)$$

the functions $u_n(\eta)$ and $v_n(\eta)$,

$$u_n(\eta) = (1-\eta)^r F\left(r, r+\frac{1}{2}+n; \frac{1}{2}+n; \eta\right) , \qquad (5.4.5)$$
$$v_n(\eta) = \frac{ir}{n+1/2}(1-\eta)^r F\left(r+1, r+\frac{1}{2}+n; \frac{3}{2}+n; \eta\right) ,$$

are determined by the field equations (5.2.5); and $\mathcal{D}$, $\bar{\mathcal{D}}$ are the covariant derivatives introduced in (B.1.6). The constants c_n can be determined, say, from requiring associativity of the operator algebra on $\psi(x)\psi(x')\partial\sigma(0)$ (see Appendix B.4),

$$c_0 = 1 , \qquad c_n = 2 \Big/ \left(\frac{1}{2}\right)_n \qquad (n \geq 1) , \qquad (5.4.6)$$

with $\left(\frac{1}{2}\right)_n = \Gamma\left(\frac{1}{2}+n\right)/\Gamma\left(\frac{1}{2}\right)$.

There are similar expressions for the products $\psi(x)\mu(x')$, $\bar{\psi}(x)\mu(x')$, obtained from the above OPE (5.4.2) by interchanging $\sigma \leftrightarrow \mu$ and $\sqrt{i} \leftrightarrow \sqrt{-i}$. These completely define the fields σ and μ, together with the normalization

$$\langle\sigma(x)\sigma(y)\rangle \sim \mathrm{d}(x,y)^{-\frac{1}{4}}, \quad \langle\mu(x)\mu(y)\rangle \sim \mathrm{d}(x,y)^{-\frac{1}{4}} \quad \text{as } x \to y . \qquad (5.4.7)$$

Note that the fields σ and μ are taken to be $SU(1,1)$ invariant (so that their $SU(1,1)$-dimensions are $h = \bar{h} = 0$). This is in contrast to the order and disorder fields in the boundary Ising field theory on the disk, $\mathcal{A}_{BI}$, about which we perturb as in (5.2.1), where they are usually taken to have the same $SU(1,1)$ dimension as their "scaling" dimension $(1/16, 1/16)$. This contrast comes from the fact that the boundary theory is interpreted as the scaling limit of the lattice Ising model with a boundary, whereas we are interested in the scaling limit of a model on the pseudosphere. The effect of the boundary implies, for instance, that the average of the order field depends on the position, whereas on the pseudosphere, by $SU(1,1)$ invariance, it does not. More precisely, the order field in the boundary theory on flat space is related to that on the pseudosphere by

$$\sigma_{\text{flat}} = (1 - z\bar{z})^{-1/8} \sigma_{\text{pseudosphere}} \tag{5.4.8}$$

and a similar relation holds for the disorder field.

In connection to the last remark, note also that when the parameter r is set to zero in (5.2.3), we obtain a conformal field theory on the disk, and since the operator product expansion is a local entity, it should be independent of the boundary and be the same as on the full plane. The OPE's (5.4.2) are certainly not those of the Ising conformal theory on the plane (those derived in Appendix B.4) when $r = 0$, since we still have covariant derivatives and factors s_n and t_n. However, making the replacement (5.4.8) in (5.4.2) at $r = 0$, we would obtain the usual OPE on the full plane for the fields σ_{flat} and μ_{flat}.

5.5 Two-point functions

5.5.1 Differential equations

All correlation functions of fermion fields in the theory (5.2.3) can be directly constructed by requiring that they be solutions to the equations of motion (5.2.5), and by requiring that these solutions have the singularities specified by the behaviors (5.2.6) at colliding positions of the fields as well as the asymptotic conditions (5.3.8) or (5.3.9) (depending on the regime). Is it possible to similarly construct the correlation functions of order and disorder fields from solutions to differential equations and from appropriate asymptotic behaviors? As in the case of the fields $\mathcal{O}_\alpha$ in the Dirac theory, since order and disorder fields are embedded in a free theory, it turns out that this is possible. The key is to use particular conserved charges in the theory, and relations among them that amount to the equations of motion (5.2.5).

Recall the conserved charges (5.3.3), forming the isometry algebra. Note that the field equations (5.2.5) specify the Casimir of the representation formed by the Fermi fields and their descendants under the action of the isometry algebra,

$$\frac{1}{2}\Big([\mathbf{P},[\bar{\mathbf{P}},\psi]] + [\bar{\mathbf{P}},[\mathbf{P},\psi]]\Big) - [\mathbf{R},[\mathbf{R},\psi]] = \left(r^2 - \frac{1}{4}\right)\psi \ , \tag{5.5.1}$$

with a similar equation for $\bar{\psi}$. This determines correlation functions of Fermi fields. Of course the order and disorder fields do not form such a representation for the isometry algebra.

Since a free theory is certainly integrable, it contains an infinite number of conserved charges. A set of such charges can be constructed from conserved currents obtained by applying an arbitrary number of Lie derivatives on Fermi fields:

$$\bar{\partial}\left(\mathcal{L}'\psi\ \mathcal{L}\psi\right) = \partial\left(\mathcal{L}'\bar{\psi}\ \mathcal{L}\bar{\psi}\right) \ , \tag{5.5.2}$$

where $\mathcal{L}' = \mathcal{P}^{m_1}\bar{\mathcal{P}}^{m_2}\mathcal{R}^{m_3}$, $\mathcal{L} = \mathcal{P}^{n_1}\bar{\mathcal{P}}^{n_2}\mathcal{R}^{n_3}$, m's, n's being some non-negative integers. That the conservation law (5.5.2) holds is a simple consequence of the equations of motion (5.2.5).

Consider the following infinite sum of some of these charges, parametrized by a real variable θ:

$$\mathbf{Q}(\theta) = \sum_{n=0}^{\infty}\frac{\theta^n}{n!}\,\frac{(-1)}{2\pi i}\int_{\mathcal{C}}\left[\psi\,\mathcal{R}^n\psi\,dz - \bar{\psi}\,\mathcal{R}^n\bar{\psi}\,d\bar{z}\right] \tag{5.5.3}$$

which can be written less formally as

$$\mathbf{Q}(\theta) = -\frac{1}{2\pi i}\int_{\mathcal{C}}\left[\psi(z,\bar{z})\,e^{i\frac{\theta}{2}}\psi(e^{i\theta}z,e^{-i\theta}\bar{z})\,dz - \bar{\psi}(z,\bar{z})\,e^{-i\frac{\theta}{2}}\bar{\psi}(e^{i\theta}z,e^{-i\theta}\bar{z})\,d\bar{z}\right] \ . \tag{5.5.4}$$

These are in fact nonlocal conserved charges; for instance, their action on a Fermi field at a given position gives generically Fermi fields at positions separated by a finite distance from the initial position:

$$[\mathbf{Q}(\theta), \psi] = \psi_{-\theta} - \psi_\theta \tag{5.5.5}$$

where ψ_θ is the field ψ rotated by an angle θ around the origin of the disk:

$$\psi_\theta(z, \bar{z}) = e^{-i\frac{\theta}{2}} \psi(e^{-i\theta} z, e^{i\theta} \bar{z}) \ . \tag{5.5.6}$$

These nonlocal charges form, by the adjoint action, a nontrivial module for the isometry algebra with a Casimir given by

$$\frac{1}{2}\Big([\mathbf{P}, [\bar{\mathbf{P}}, \mathbf{Q}(\theta)]] + [\bar{\mathbf{P}}, [\mathbf{P}, \mathbf{Q}(\theta)]]\Big) - [\mathbf{R}, [\mathbf{R}, \mathbf{Q}(\theta)]] = \left(4r^2 - 1\right) \sin^2\left(\frac{\theta}{2}\right) \mathbf{Q}(\theta) \ . \tag{5.5.7}$$

The value $\theta = \pi$ is particularly useful. Define

$$\mathbf{Z}_0 = \frac{1}{2} \mathbf{Q}(\pi) \ . \tag{5.5.8}$$

It generates the following transformation on the Fermi fields:

$$[\mathbf{Z}_0, \psi(z, \bar{z})] = i\psi(-z, -\bar{z}) \ , \qquad [\mathbf{Z}_0, \bar{\psi}(z, \bar{z})] = -i\bar{\psi}(-z, -\bar{z}) \ , \tag{5.5.9}$$

which is the infinitesimal transformation for the group action

$$e^{i\alpha \mathbf{Z}_0} \psi(z, \bar{z}) e^{-i\alpha \mathbf{Z}_0} = \cosh(\alpha)\, \psi(z, \bar{z}) - \sinh(\alpha)\, \psi(-z, -\bar{z}) \ . \tag{5.5.10}$$

It is easy to verify that the equations of motion, the OPE's between Fermi fields and the asymptotic conditions are indeed preserved under this transformation.

The Ward identities coming from the nonlocal charge $\mathbf{Z}_0$ and the cluster property of correlation functions are enough to determine the nonlinear differential equations for order-order and disorder-disorder correlation functions. What is usually the hardest part in finding Ward identities associated to nonlocal charges is to find their action on local fields. In the present case, we consider the set of local fields of the form $\partial^k\bar\partial^l\sigma(z,\bar z)\partial^m\bar\partial^n\sigma(-z,-\bar z)$ as well as similar fields where one or two of the σ's are replaced by disorder fields μ. For convenience, we will use the notation

$$\tilde\sigma(z,\bar z)\equiv\sigma(-z,-\bar z)\ ,\qquad \tilde\mu(z,\bar z)\equiv\mu(-z,-\bar z)\ .$$

Correlation functions containing a Fermi field $\psi(z)$ and fields in the set above at various positions z_i's have square-root branch cuts in the z-plane linking z_i to $-z_i$ for all i. Bilinear combinations of Fermi fields of the type $\psi(z)\psi(-z)$ in such correlation functions are then defined on the plane without cuts. The integrands of the charge $\mathbf{Z}_0$, defined by (5.5.4) with $\theta=\pi$, are such bilinear combinations of Fermi fields. Hence the charge $\mathbf{Z}_0$ has a well-defined action on fields in the set above, that can be calculated by picking up poles at the positions of the fields. The poles can easily be obtained from the full operator product expansion (5.4.2).

In fact, the calculation can be simplified by noticing the following. On any field that contains only holomorphic derivatives or only anti-holomorphic derivatives, for instance $\partial^n\sigma\tilde\sigma$, the action of $\mathbf{Z}_0$ can be calculated using solely the *conformal* OPE, obtained by taking $r=0$ in (5.4.2). For instance, we easily find

$$[\mathbf{Z}_0,\sigma\tilde\mu]=\mu\tilde\sigma\ ,\quad [\mathbf{Z}_0,\mu\tilde\sigma]=-\sigma\tilde\mu\ ,\quad [\mathbf{Z}_0,\sigma\tilde\sigma]=0\ ,\quad [\mathbf{Z}_0,\mu\tilde\mu]=0\ . \tag{5.5.11}$$

Denoting $\partial\tilde\sigma(z,\bar z)\equiv(\partial\sigma/\partial z)(-z,-\bar z)$ and $\partial\tilde\mu(z,\bar z)\equiv(\partial\mu/\partial z)(-z,-\bar z)$, the action on fields with one derivative is given by

$$\begin{aligned}
[\mathbf{Z}_0,\sigma\partial\tilde\mu]&=\partial\mu\tilde\sigma\ ,\qquad &[\mathbf{Z}_0,\mu\partial\tilde\sigma]&=-\partial\sigma\tilde\mu\ ,\\
[\mathbf{Z}_0,d\sigma\tilde\mu]&=\mu\partial\tilde\sigma\ ,\qquad &[\mathbf{Z}_0,\partial\mu\tilde\sigma]&=-\sigma\partial\tilde\mu\ ,\\
[\mathbf{Z}_0,\sigma\partial\tilde\sigma]&=i(\partial\mu\tilde\mu-\mu\partial\tilde\mu)\ ,\qquad &[\mathbf{Z}_0,\mu\partial\tilde\mu]&=i(\partial\sigma\tilde\sigma-\sigma\partial\tilde\sigma)\ ,\\
[\mathbf{Z}_0,\partial\sigma\tilde\sigma]&=i(\mu\partial\tilde\mu-\partial\mu\tilde\mu)\ ,\qquad &[\mathbf{Z}_0,\mu\partial\tilde\mu]&=i(\sigma\partial\tilde\sigma-\partial\sigma\tilde\sigma)\ .
\end{aligned}\tag{5.5.12}$$

On particular linear combinations of fields that contain mixed derivatives, one can use the equations of motion in the form (5.5.7). For instance, consider the linear combination

$$\mathcal{W} = \frac{1}{2}\Big([\mathbf{P},[\bar{\mathbf{P}},\sigma\tilde{\mu}]] + [\bar{\mathbf{P}},[\mathbf{P},\sigma\tilde{\mu}]]\Big) - [\mathbf{R},[\mathbf{R},\sigma\tilde{\mu}]]\ . \tag{5.5.13}$$

Then, the action of $\mathbf{Z}_0$ on $\mathcal{W}$,

$$[\mathbf{Z}_0, \mathcal{W}(z,\bar{z})]\ , \tag{5.5.14}$$

can be expressed using (5.5.7) in terms of the action of $\mathbf{Z}_0$ on the fields $\sigma\tilde{\mu}$ as well as on the fields $[\mathbf{P},\sigma\tilde{\mu}]$, $[\bar{\mathbf{P}},\sigma\tilde{\mu}]$ and $[\mathbf{R},\sigma\tilde{\mu}]$, all of which can be obtained using the conformal OPE:

$$\begin{aligned}
[\mathbf{Z}_0,\mathcal{W}] \quad = \quad & [\mathbf{P},[\mathbf{Z}_0,[\bar{\mathbf{P}},\sigma\tilde{\mu}]]] + [\bar{\mathbf{P}},[\mathbf{Z}_0,[\mathbf{P},\sigma\tilde{\mu}]]] \\
& -\frac{1}{2}\left([\bar{\mathbf{P}},[\mathbf{P},[\mathbf{Z}_0,\sigma\tilde{\mu}]]] + [\mathbf{P},[\bar{\mathbf{P}},[\mathbf{Z}_0,\sigma\tilde{\mu}]]]\right) \\
& +2[\mathbf{R},[\mathbf{Z}_0,[\mathbf{R},\sigma\tilde{\mu}]]] + [\mathbf{R},[\mathbf{R},[\mathbf{Z}_0,\sigma\tilde{\mu}]]] \\
& +(4r^2-1)[\mathbf{Z}_0,\sigma\tilde{\mu}]\ .
\end{aligned} \tag{5.5.15}$$

In order to determine the correlation functions, it is sufficient to consider the following two Ward identities, which only express the fact that $\mathbf{Z}_0$ is conserved:

$$\langle[\mathbf{Z}_0,\mathcal{W}(z,\bar{z})\,(\mu\tilde{\sigma})(z',\bar{z}')]\rangle = 0 \tag{5.5.16}$$

$$\langle[\mathbf{Z}_0,[\mathbf{P},(\sigma\tilde{\sigma})(z,\bar{z})]\,[\mathbf{P},(\mu\tilde{\mu})(z',\bar{z}')]]\rangle = 0\ . \tag{5.5.17}$$

The first Ward identity is in some sense the equations of motion on the fields σ and μ (a similar identity can be used on Fermi fields giving essentially the equation of motion (5.5.1)), and the second is a supplementary condition that relates order-order and disorder-disorder correlation functions. Of course, these Ward identities in fact give linear equations for four-point functions. We can consider a particular case of these equations by taking z and z' very far from the center of the disk but at a finite distance from each other: $z=\rho e^{i\gamma}$, $z'=\rho e^{i\gamma}+\Delta z$, $\rho\to\infty$. Using cluster property and PT-invariance of correlation functions, we obtain bilinear and quadratic differential equations for two-point functions.

More precisely, consider the two-point functions of order and disorder fields

$$G(x,x') = \langle\, \sigma(x)\sigma(x')\,\rangle \ , \qquad \tilde{G}(x,x') = \langle\, \mu(x)\mu(x')\,\rangle \ . \tag{5.5.18}$$

Equation (5.5.16) gives the bilinear differential equation

$$(1-z\bar{z})^2 \left[\partial\bar{\partial}\tilde{G}\,G + \tilde{G}\,\partial\bar{\partial}G - \partial\tilde{G}\,\bar{\partial}G - \bar{\partial}\tilde{G}\,\partial G\right] = \left(r^2 - \frac{1}{4}\right)\tilde{G}\,G \tag{5.5.19}$$

and Equation (5.5.17) gives the quadratic differential equation

$$\begin{aligned} &\partial\tilde{G}\,\partial\tilde{G} - \partial^2\tilde{G}\,\tilde{G} - \bar{z}^4\left(\bar{\partial}\tilde{G}\,\bar{\partial}\tilde{G} - \bar{\partial}^2\tilde{G}\,\tilde{G}\right) + 2\bar{z}^3\bar{\partial}\tilde{G}\,\tilde{G} \\ &= \partial' G\,\partial' G - \partial'^2 G\,G - \bar{z}'^4\left(\bar{\partial}' G\,\bar{\partial}' G - \bar{\partial}'^2 G\,G\right) + 2\bar{z}'^3\bar{\partial}' G\,G \ , \end{aligned} \tag{5.5.20}$$

where $\partial = \partial_z$, $\partial' = \partial_{z'}$, $\bar{\partial} = \partial_{\bar{z}}$ and $\bar{\partial}' = \partial_{\bar{z}'}$. In deriving these equations, it is important to keep track of the phases of the Fermi fields in the integrand of $\mathbf{Z}_0$ in order to get the correct signs.

We are interested in a solution that respects the $SU(1,1)$ symmetry, in which case the two-point correlation functions are simply functions of the projective invariant s (4.1.6). Then, Eqs. (5.5.19) and (5.5.20) above imply the set of equations

$$\begin{aligned} &s(1-s)\left(G'G' - G''G + \tilde{G}'\tilde{G}' - \tilde{G}''\tilde{G}\right) + (2s-1)\left(G'G + \tilde{G}'\tilde{G}\right) = 0 \ , \\ &(s-1)\left(G'G' - G''G - \tilde{G}'\tilde{G}' + \tilde{G}''\tilde{G}\right) - \left(G'G - \tilde{G}'\tilde{G}\right) = 0 \ , \\ &s\left(\tilde{G}''G + \tilde{G}G'' - 2\tilde{G}'G'\right) + \left(\tilde{G}'G + \tilde{G}G'\right) = \frac{r^2 - 1/4}{(1-s)^2}\,\tilde{G}G \ , \end{aligned} \tag{5.5.21}$$

where primes denote derivatives with respect to s and the first two equations of (5.5.21) are consequences of Eq. (5.5.20).

From these equations, we can express the two-point functions in terms of a Painlevé VI transcendent $w(s)$ satisfying (4.1.5) with $\tilde{\lambda} = 1$, $\lambda = 0$ and $\gamma = 0$:

$$\begin{aligned} w'' = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-s}\right)w'^2 - \left(\frac{1}{s} + \frac{1}{s-1} + \frac{1}{w-s}\right)w' \\ + \left(\frac{1}{2} - 2\,r^2\right)\frac{w(w-1)}{s(s-1)(w-s)} \ . \end{aligned} \tag{5.5.22}$$

From the usual parametrization in terms of auxiliary functions $\chi(s)$ and $\varphi(s)$,

$$(2R)^{1/4}\langle\,\sigma(x)\,\sigma(y)\,\rangle \;=\; e^{\chi(s)/2}\cosh(\varphi(s)/2)\ , \tag{5.5.23}$$

$$(2R)^{1/4}\langle\,\mu(x)\,\mu(y)\,\rangle \;=\; e^{\chi(s)/2}\sinh(\varphi(s)/2)\ , \tag{5.5.24}$$

we have

$$\cosh^2\varphi = \frac{1}{w}\ , \tag{5.5.25}$$

$$\chi' = \frac{s(s-1)}{4w(w-s)(w-1)}w'^2 - \frac{s}{2w(w-s)}w' + \left(\frac{1}{4}-r^2\right)\frac{w-1}{(s-1)(w-s)}\ .$$

The function χ is the logarithm of the tau function (4.1.7) associated to the Painlevé equation (5.5.22).

Note that the derivation above is in close connection to the method for deriving differential equations for correlation functions of order and disorder fields in the Ising field theory on flat space developed in [52]. The results above were obtained in [39] by using this method. In the method of [52], one first consider two copies of the Majorana theory. The charge $\mathbf{Z}_0$ above is replaced by a charge that generates a rotation of the Fermi fields among the two copies. Since two copies of the Majorana theory is a Dirac theory (see Section 5.6), the charge $\mathbf{Z}_0$ is just the $U(1)$ charge of the resulting Dirac theory. The product of fields of the type $\sigma\tilde{\sigma}$ above is replaced by a product of fields belonging to different copies, and the cluster property used above is simply replaced by the fact that correlation functions of fields belonging to different copies factorize.

Note also that the method can be applied to the Ising field theory on a surface with boundaries and boundary conditions that break some or all of the spatial symmetries of the bulk. For instance, consider first the infinite flat space, where the two Ward identities (5.5.16) and (5.5.17) lead to the two equations

$$\partial\bar\partial\tilde{G}\,G + \tilde{G}\,\partial\bar\partial G - \partial\tilde{G}\,\bar\partial G - \bar\partial\tilde{G}\,\partial G = m^2\tilde{G}\,G \tag{5.5.26}$$

and

$$\partial\tilde{G}\,\partial\tilde{G} - \partial^2\tilde{G}\,\tilde{G} = \partial' G\,\partial' G - \partial'^2 G\,G\ . \tag{5.5.27}$$

Since these two Ward identities only use the fact that $\mathbf{Z}_0$ is conserved, one would obtain the same equations for a theory on the flat half-plane, where the spatial symmetry perpendicular to the boundary is broken. These equations should fix completely the two-point functions, once the boundary conditions on the line delimiting the half-plane are specified.

5.5.2 Fixing the solutions

As in the case of the scaling fields in the Dirac theory, the appropriate solution to the Painlevé equation (5.5.22) can be fixed, for instance, by the short distance $s \to 0$ behavior

$$w = r^2 s \ln^2 \left(k(r)^2 \, s\right) + O\left(s^2 \ln^4 s\right) \ . \tag{5.5.28}$$

The constant $k(r)$, given by

$$\ln k(r) = \psi(r) + \frac{1}{2\,r} + \gamma - \ln 4 \tag{5.5.29}$$

($\psi(x) = d \ln \Gamma(x)/dx$ and γ is Euler's constant), can be obtained from the vacuum expectation value of the energy field $\langle \varepsilon \rangle$,

$$2R \, \langle \varepsilon \rangle = -2r \left(\psi(r) + \gamma\right) - 1 \ , \tag{5.5.30}$$

by applying conformal perturbation theory. This is done in Section 5.7. I used the condition

$$\frac{d}{dm} \langle \varepsilon \rangle \bigg|_{m=0} = 0 \ , \tag{5.5.31}$$

and the normalization

$$\mathrm{d}(x,y)^2 \, \langle \varepsilon(x) \varepsilon(y) \rangle \to 1 \qquad \text{as} \qquad \mathrm{d}(x,y) \to 0 \ . \tag{5.5.32}$$

The power law in (5.5.28), as well as the behavior

$$\chi(s) = \frac{1}{4} \ln(s) + O(s^{\frac{1}{2}}) \tag{5.5.33}$$

fixing the integration constant for $\chi(s)$ in (5.5.23), (5.5.25), are specified by the leading short distance behaviors (5.4.7).

This solution has the property that at large distances $s \to 1$ it behaves as

$$1 - w = A(r)^2 (1-s)^{1+2r} + O\left((1-s)^{2+2r}, (1-s)^{2+4r}\right) \ , \tag{5.5.34}$$

with the coefficient

$$A(r) = \frac{\Gamma\left(\frac{1}{2}+r\right)}{4^r \sqrt{\pi}\, \Gamma(1+r)} \ . \tag{5.5.35}$$

Furthermore, $\chi(s)$ approaches the constant $4\ln((2R)^{1/8}\langle\sigma\rangle)$, related to the magnetization

$$\langle\sigma\rangle^2 = (R/2)^{-\frac{1}{4}} \prod_{n=1}^{\infty} \left(\frac{1 - \frac{1}{4(mR+n)^2}}{1 - \frac{1}{4n^2}} \right)^n . \tag{5.5.36}$$

obtained in Section 5.6. The leading behavior (5.5.34) are obtained from the form factors of the order and disorder fields (5.6.10) and (5.6.11) in Section 5.6. The behavior (5.5.34) also provides an alternative description of the solution to the Painlevé equation (5.5.22) describing the correlation functions. Together with (5.5.28), it gives a solution to the connection problem relating the singular points $s = 0$ and $s = 1$. This solution cannot be obtained by a simple specialization of the solution showed in Section 4.2 of the previous Chapter: the particular case $\tilde{\lambda} = 1$, $\lambda = 0$ is singular.

5.6 Long distance expansion and form factors

The purpose of this section is to determine the long-distance expansion of the two-point functions $\langle\sigma(x)\sigma(x')\rangle$ and $\langle\mu(x)\mu(x')\rangle$. This fixes the constant A (5.5.35) that specifies the singular behavior of the Painlevé transcendent around $s = 1$ as in (5.5.34) and the vacuum expectation value (5.5.36) that specifies the normalization of the associated tau function describing these two-point functions as explained in the previous section.

The long-distance expansion is obtained by a form factor expansion, as was done in (4.3.11) of (4.8.1):

$$\langle\sigma(x)\sigma(x')\rangle = \sum_{n=0}^{\infty} \sum_{\substack{k_1>\cdots>k_n=0 \\ \sum_i k_i \ \text{odd}}}^{\infty} \langle\text{vac}|\sigma|k_1,\ldots,k_n\rangle\langle k_n,\ldots,k_1|\sigma|\text{vac}\rangle e^{-(\lambda_1+\cdots+\lambda_n)\frac{\mathrm{d}(x,x')}{2R}} \tag{5.6.1}$$

and

$$\langle\mu(x)\mu(x')\rangle = \sum_{n=0}^{\infty} \sum_{\substack{k_1>\cdots>k_n=0 \\ \sum_i k_i \ \text{even}}}^{\infty} \langle\text{vac}|\mu|k_1,\ldots,k_n\rangle\langle k_n,\ldots,k_1|\mu|\text{vac}\rangle e^{-(\lambda_1+\cdots+\lambda_n)\frac{\mathrm{d}(x,x')}{2R}} \tag{5.6.2}$$

where

$$\lambda_j = 1 + 2r + 2k_j .$$

This long-distance expansion gives (recall the notation (5.5.18) and (4.1.6))

$$\begin{aligned} \frac{G(s)}{\langle\sigma\rangle^2} &= 1 + \frac{1+2r}{2^6(1+r)^2} A'(1-s)^{2+2r}\left(1+O(1-s)\right) \\ \frac{\tilde{G}(s)}{\langle\sigma\rangle^2} &= \frac{1}{2}A(1-s)^{1/2+r}\left(1+O(1-s)\right) \end{aligned}$$

where the constants A and A' are given by

$$A = 4^{-r}|\langle\text{vac}|\mu|0\rangle|^2 , \qquad A' = \frac{4^{1-2r}(1+r)^2}{1+2r}|\langle\text{vac}|\sigma|0,1\rangle|^2 . \tag{5.6.3}$$

The description in terms of the differential equation (5.5.22), (5.5.25) imposes that A is the constant appearing in (5.5.34) and further that

$$A' = A^2 . \tag{5.6.4}$$

Along with the constant A, the description in terms of the differential equation then fixes completely the long-distance expansion of the two-point functions, as do the form factor decomposition. Note though that the form factor decomposition gives more information, as it can be used to determine the long-distance expansion of multi-point correlation functions of order and disorder fields as well.

In order to calculate the magnetization (5.5.36), I will use the results of the previous chapter for the one-point function (4.2.8). The strategy is to use the fact that the Dirac theory factorizes in two copies of the Majorana theory, and to show that particular linear combinations of fields $\mathcal{O}_\alpha$ in the Dirac theory factorize into product of order and of disorder fields in the Majorana theory. This factorization will be shown by proving that form factors for these linear combinations in the Dirac theory factorize in an appropriate basis. Then, the vacuum expectation value of these linear combinations give products of vacuum expectation values of order and of disorder fields. In fact, this proof will directly give expressions for the form factors of order and of disorder fields.

5.6.1 Factorization of the Dirac theory

The correlation functions of particular fields $\mathcal{O}_\alpha$ in the Dirac theory are simply related to correlation functions of order and disorder fields in the Ising field theory [148, 113, 132]. Let me explain in more detail what this relation is on flat space.

The tensor product of two independent copies of the Majorana theory can be described by a single copy of the Dirac theory. One can then represent the tensor product of two order fields and of two disorder fields acting non trivially on independent copies of the Majorana theory as a single field in the Dirac theory. Taking the Majorana theory with positive mass to represent the scaling limit of the Ising model in its low-temperature regime, one has the following equivalences [148, 113]:

$$\sigma \otimes \sigma = \mathcal{O}^{(+)}, \quad \mu \otimes \mu = \mathcal{O}^{(-)} \tag{5.6.5}$$

where the fields $\mathcal{O}^{(+)}$ and $\mathcal{O}^{(-)}$ belong to the Dirac theory. They can be expressed in terms of the fields $\mathcal{O}_\alpha$ described above[3]:

$$\mathcal{O}^{(+)} = \frac{1}{\sqrt{2}}\left(\mathcal{O}_{\frac{1}{2}} + \mathcal{O}_{-\frac{1}{2}}\right), \quad \mathcal{O}^{(-)} = \frac{1}{\sqrt{2}}\left(\mathcal{O}_{\frac{1}{2}} - \mathcal{O}_{-\frac{1}{2}}\right). \tag{5.6.6}$$

[3] We also have more generally:

$$\begin{aligned}
\mathcal{O}^{(+)}(x_1)\cdots\mathcal{O}^{(+)}(x_n) &= (\sigma(x_1)\cdots\sigma(x_n))\otimes(\sigma(x_1)\cdots\sigma(x_n)) \\
\mathcal{O}^{(-)}(x_1)\cdots\mathcal{O}^{(-)}(x_n) &= (-1)^{n(n-1)/2}(\mu(x_1)\cdots\mu(x_n))\otimes(\mu(x_1)\cdots\mu(x_n))
\end{aligned}$$

Similarly, the Dirac theory on the pseudosphere is equivalent to a tensor product of two copies of the Majorana theory on the pseudosphere. Consider four real fermion fields $\psi_a, \psi_b, \bar{\psi}_a, \bar{\psi}_b$, defined via

$$\Psi_R = \frac{1}{\sqrt{2}}(\psi_a + i\psi_b), \quad \Psi_L = \frac{1}{\sqrt{2}}(\bar{\psi}_b - i\bar{\psi}_a).$$

It is easy to verify that correlators of these fields factorize; for instance:

$$\langle \psi_a(x_1)\cdots\psi_a(x_n)\,\psi_b(x'_1)\cdots\psi_b(x'_m)\rangle = \langle \psi_a(x_1)\cdots\psi_a(x_n)\rangle\,\langle\psi_b(x'_1)\cdots\psi_b(x'_m)\rangle.$$

This factorization can be expressed by writing the fields $\psi_a, \psi_b, \bar{\psi}_a, \bar{\psi}_b$ as tensor products of fields in two independent copies of the Majorana theory[4]:

$$\psi_a = \psi \otimes \mathbf{1}, \quad \psi_b = \mathbf{1} \otimes \psi, \quad \bar{\psi}_a = \bar{\psi} \otimes \mathbf{1}, \quad \bar{\psi}_b = \mathbf{1} \otimes \bar{\psi}.$$

Here $\psi, \bar{\psi}$ are (real) Majorana fields that satisfy the equations of motion

$$\frac{\partial}{\partial \xi}\bar{\psi} = -\frac{r}{\cos(2\xi_{\rm x})}\psi, \quad \frac{\partial}{\partial \bar{\xi}}\psi = -\frac{r}{\cos(2\xi_{\rm x})}\bar{\psi}$$

with $\xi = \xi_{\rm x} + i\xi_{\rm y}$ and $\bar{\xi} = \xi_{\rm x} - i\xi_{\rm y}$ and have short distance normalization given by

$$\langle \psi(\xi_1,\bar{\xi}_1)\psi(\xi_2,\bar{\xi}_2)\rangle \sim -\frac{1}{2\pi i}\frac{1}{\xi_1 - \xi_2}, \quad \langle \bar{\psi}(\xi_1,\bar{\xi}_1)\bar{\psi}(\xi_2,\bar{\xi}_2)\rangle \sim \frac{1}{2\pi i}\frac{1}{\bar{\xi}_1 - \bar{\xi}_2}.$$

For product of fields, we have

$$\psi_a(x_1)\cdots\psi_a(x_n)\,\psi_b(x'_1)\cdots\psi_b(x'_m) = (\psi(x_1)\cdots\psi(x_n)) \,\otimes\, (\psi(x'_1)\cdots\psi(x'_m))\ ,$$

and in general we must take into account the signs coming from the fact that two Dirac fields anti-commute. For instance, we have

$$\psi_a(x_1)\psi_b(x'_1)\,\psi_a(x_2)\psi_b(x'_2) = -(\psi(x_1)\psi(x_2)) \,\otimes\, (\psi(x'_1)\psi(x'_2))\ .$$

Consequent to this decomposition of the Dirac fermion field, the Hilbert space of the Dirac theory can be written as a tensor product of two copies of the Hilbert

[4]In this section, we use a slightly different normalization of Majorana fermion fields than that used in Section 5.2.

space of the Majorana theory: $\mathcal{H} = \mathcal{H}_M \otimes \mathcal{H}_M$. It is easy to verify that the Hilbert space $\mathcal{H}_M$, within the quantization scheme that we are considering, has a structure similar to that of $\mathcal{H}$. Recall that a basis for $\mathcal{H}_M$ can be taken as the discrete set of states diagonalizing the Hamiltonian of the Majorana theory in this quantization scheme:

$$|k_1, \ldots, k_n\rangle_M,\ k_j \in \mathbb{N},\ n = 0, 1, 2, \ldots,\ k_1 < \cdots < k_n,$$

with vacuum denoted by $|\mathrm{vac}\rangle_M$. These states correspond to eigenvalues $\lambda_1 + \ldots + \lambda_n$. States with different orderings of energy levels are defined by the fact that exchanging the positions of two arguments k_i, k_j brings a factor of (-1). In order to obtain a precise correspondence between the Dirac Hilbert space and a tensor product of two copies of the Majorana Hilbert space, define one-particle states $|k\rangle_a$ and $|k\rangle_b$ in the Dirac theory by

$$|k\rangle_a = \frac{1}{\sqrt{2}}(\,|k\rangle_+ + |k\rangle_-\,), \quad |k\rangle_b = \frac{i}{\sqrt{2}}(\,|k\rangle_+ - |k\rangle_-\,), \tag{5.6.7}$$

and multi-particle states involving states of type a and b by forming exterior products of these one-particle states. Then,

$$|\mathrm{vac}\rangle = |\mathrm{vac}\rangle_M \otimes |\mathrm{vac}\rangle_M$$

and

$$|k_1, \ldots, k_n, k_1', \ldots, k_m'\rangle_{\underbrace{a, a, \ldots,}_{n}\underbrace{b, b, \ldots}_{m}} = |k_1, \ldots, k_n\rangle_M \otimes |k_1', \ldots, k_m'\rangle_M.$$

Here we have fixed some of the phases by requiring the charge conjugation symmetry in the Dirac theory to be implemented by

$$|k\rangle_+ \leftrightarrow |k\rangle_-, \quad \Psi_R^\dagger \leftrightarrow \Psi_R, \quad \Psi_L^\dagger \leftrightarrow -\Psi_L.$$

In what follows, we will omit the subscript M on Majorana states unless required for clarity.

We can expect that on the Poincaré disk, the fields $\mathcal{O}^{(+)}$ and $\mathcal{O}^{(-)}$ given by (5.6.6) factorize as a tensor product of local fields (also called order and disorder fields) belonging to the Majorana theory, the way they do on flat space, like in (5.6.5). Although such a factorization is not *a priori* obvious from the definitions of $\mathcal{O}^{(+)}$ and $\mathcal{O}^{(-)}$, it is a local property and should not be affected by the curvature. The study of the fields $\mathcal{O}_\alpha$ above then gives information about order and disorder fields in the Majorana theory on the pseudosphere. Indeed, the expected correspondence (5.6.5) between fields $\mathcal{O}^{(+)}$ and $\mathcal{O}^{(-)}$ defined in (5.6.6) in the Dirac theory and fields σ and μ in the Majorana theory, and the correspondence described above between the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_M$ of both theories, allow us to write matrix elements in $\mathcal{H}$ of the fields $\mathcal{O}^{(+)}$ and $\mathcal{O}^{(-)}$ in terms of matrix elements in $\mathcal{H}_M$ of the fields σ and μ. Having expressions for form factors of the fields $\mathcal{O}^{(+)}$ and $\mathcal{O}^{(-)}$ in Appendix B.2, this in turn gives us expressions for form factors of the fields σ and μ.

In the next subsections, I obtain expressions for form factors and vacuum expectation values of order and disorder fields by specializing the results of Appendix B.2. From these results, I directly verify the factorization property (5.6.5) of the fields $\mathcal{O}^{(+)}$ and $\mathcal{O}^{(-)}$. More precisely, since the Hilbert space of the Dirac theory on the pseudosphere is a tensor product of two copies of the Hilbert space of the Majorana theory on the pseudosphere, I verify that in the tensor product basis of the Dirac Hilbert space, matrix elements between vacuum and excited states, or form factors, of the fields $\mathcal{O}^{(+)}$ and $\mathcal{O}^{(-)}$ factorize.

5.6.2 Magnetization and form factors

Let me first summarize the results. Consider the Majorana theory with fermion mass m on the pseudosphere with Gaussian curvature $-1/R^2$. Recall that I use the quantization scheme where the Hamiltonian is taken as the generator of a non-compact subgroup of the $SU(1,1)$ isometry group, which gives a set of eigenstates (5.3.17). The Hilbert space of the Dirac theory is just a tensor product of two copies of the Hilbert space of the Majorana theory. Then, I will verify the factorization properties:

$$\langle \mathrm{vac}|\mathcal{O}^{(+)}\left(|k_1,\ldots,k_n\rangle\otimes|k_1',\ldots,k_m'\rangle\right) = \qquad (5.6.8)$$

$$\langle \mathrm{vac}|\sigma|k_1,\ldots,k_n\rangle\langle \mathrm{vac}|\sigma|k_1',\ldots,k_m'\rangle$$

and[5]

$$\langle \mathrm{vac}|\mathcal{O}^{(-)}\left(|k_1,\ldots,k_n\rangle\otimes|k_1',\ldots,k_m'\rangle\right) = \qquad (5.6.9)$$

$$-\langle \mathrm{vac}|\mu|k_1,\ldots,k_n\rangle\langle \mathrm{vac}|\mu|k_1',\ldots,k_m'\rangle.$$

Under the normalization of the states implied by (5.3.15), for instance

$$\langle k_1|k_2\rangle = \delta_{k_1,k_2}\,,$$

the two-particle form factors of order fields are given by

$$\frac{\langle \mathrm{vac}|\sigma|k_1,k_2\rangle}{\langle\sigma\rangle} = (-1)^{\frac{k_1+k_2+1}{2}}\frac{\sqrt{k_2(2r+k_2)}}{\pi(1+2r+k_1+k_2)}\;\times \qquad (5.6.10)$$

$$\sqrt{\frac{\Gamma\left(r+\frac12+\frac{k_1}{2}\right)\Gamma\left(r+\frac{k_2}{2}\right)\Gamma\left(\frac12+\frac{k_1}{2}\right)\Gamma\left(\frac{k_2}{2}\right)}{\Gamma\left(r+1+\frac{k_1}{2}\right)\Gamma\left(r+\frac12+\frac{k_2}{2}\right)\Gamma\left(1+\frac{k_1}{2}\right)\Gamma\left(\frac12+\frac{k_2}{2}\right)}}$$

for k_1 even and k_2 odd, where we use again $r = mR$ as in (4.1.2). For k_1 odd and k_2 even, one can use $\langle \mathrm{vac}|\sigma|k_1,k_2\rangle = -\langle \mathrm{vac}|\sigma|k_2,k_1\rangle$, and in other cases the two-particle form factor is zero. One-particle form factors of disorder fields are given by

$$\frac{\langle \mathrm{vac}|\mu|k\rangle}{\langle\sigma\rangle} = (-1)^{\frac{k}{2}}\sqrt{\frac{\Gamma\left(r+\frac12+\frac{k}{2}\right)\Gamma\left(\frac12+\frac{k}{2}\right)}{\pi\Gamma\left(r+1+\frac{k}{2}\right)\Gamma\left(1+\frac{k}{2}\right)}} \qquad (5.6.11)$$

for k even, zero otherwise. Multiparticle form factors in both cases can be obtained by Wick's theorem on the particle states: contractions between two one-particle states in a multi-particle state is given by the two-particle form factor

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[5] The sign $(-)$ on the right-hand side comes from the identification

$$\mathcal{O}^{(-)}(x)\psi_a(x_1)\cdots\psi_a(x_n)\,\psi_b(x_1')\cdots\psi_b(x_m') = \\ (-1)^n\left(\mu(x)\psi(x_1)\cdots\psi(x_n)\right)\otimes\left(\mu(x)\psi(x_1')\cdots\psi(x_m')\right),$$

which can be obtained, for instance, by analyzing the OPE's in the Dirac and in the Majorana theories, and from the fact that form factors of disorder field μ are zero if they contain an even number of particle.

of the order field, and contraction between a one-particle state and the disorder field is given by the one-particle form factor of the disorder field; there are no other contractions. Above, the fields are at any point of the unique line on the pseudosphere that is both a geodesic and an orbit of the non-compact subgroup generated by the Hamiltonian. On these fields at such points, the Hamiltonian generates geodesic translations in units of $2R$.

The fact that the form factors factorize strongly suggest the validity of the factorization relations (5.6.5). Then, from the vacuum expectation value (4.2.8) of fields $\mathcal{O}_{\frac{1}{2}}$, we can deduce the vacuum expectation value of the order field (5.5.36). From (4.2.1), we have the following normalization of the fields $\mathcal{O}^{(+)}$ and $\mathcal{O}^{(-)}$ in the Dirac theory on the Poincaré disk:

$$\langle\mathcal{O}^{(+)}(x)\mathcal{O}^{(+)}(y)\rangle \sim \mathrm{d}(x,y)^{-\frac{1}{2}}, \quad \langle\mathcal{O}^{(-)}(x)\,\mathcal{O}^{(-)}(y)\rangle \sim -\mathrm{d}(x,y)^{-\frac{1}{2}} \quad \text{as } x\to y\ , \tag{5.6.12}$$

where $\mathrm{d}(x,y)$ is the geodesic distance between the points x and y (B.1.3). This indeed corresponds to the normalization (5.4.7) for the order and disorder fields. Note that in the theory on flat space, form factors were first calculated in [13], and the vacuum expectation value of the spin field was first calculated in [132].

Below, for definiteness, the fields of which I take matrix elements are assumed to be at the center of the Poincaré disk. Using their transformation properties under the $SU(1,1)$ isometry group, they can always be translated to any other points in the Poincaré disk.

5.6.3 Form factors of order field

I now verify the factorization properties of the field $\mathcal{O}^{(+)}$ as defined by (5.6.6) and calculate the multi-particle form factors of the order field σ in the Majorana theory. Using a formula of the form (B.2.10) of Appendix B.2, which essentially states that multi-particle form factors of the fields $\mathcal{O}_\alpha$ can be evaluated in terms of their two-particle form factors through Wick's theorem, the multi-particle form factors of $\mathcal{O}^{(+)}$ in the Dirac theory can be written in the form

$$\frac{1}{\langle \mathcal{O}^{(+)} \rangle} \langle \mathrm{vac} | \mathcal{O}^{(+)} | k_1, \ldots, k_n, k'_1, \ldots, k'_n \rangle_{\underbrace{+,+,\ldots}_{n},\underbrace{-,-,\ldots}_{n}} = (-1)^{n(n-1)/2} \frac{\det(A_+) + \det(A_-)}{2} \tag{5.6.13}$$

where the $n \times n$ matrices A_+ and A_- have matrix elements

$$[A_+]_{ij} = f(k_i, k'_j), \quad [A_-]_{ij} = -f(k'_j, k_i)$$

with $f(k_1, k_2) = f_+(k_1, k_2)$ given by (B.2.12)) in Appendix B.2.5. In Appendix B.5, it is shown that

$$\frac{\det(A_+) + \det(A_-)}{2} = \det\left(\frac{A_+ + A_-}{2} \right). \tag{5.6.14}$$

This equation simply means that we can calculate the form factor in (5.6.13) by using Wick's theorem to pair one-particle states in the asymptotic state, the contractions being given by

$$\overbracket{|k_1\rangle_+ \; |k_2\rangle_-} = \frac{1}{2}(f(k_1, k_2) - f(k_2, k_1)) = \begin{cases} 0 & (k_1 + k_2 \text{ even}) \\ f(k_1, k_2) & (k_1 + k_2 \text{ odd}) \end{cases} \tag{5.6.15}$$

where we used properties (B.2.13) (other contractions being zero). Changing basis to $|k\rangle_a$ and $|k\rangle_b$ through (5.6.7), we can calculate the multi-particle form factors by using Wick's theorem with the contractions

$$\overbracket{|k_1\rangle_a \; |k_2\rangle_a} = \overbracket{|k_1\rangle_b \; |k_2\rangle_b} = \overbracket{|k_1\rangle_+ \; |k_2\rangle_-}$$

and

$$\overbracket{|k_1\rangle_a \; |k_2\rangle_b} = 0.$$

We then obtain

$$\frac{1}{\langle \mathcal{O}^{(+)} \rangle} \langle \mathrm{vac} | \mathcal{O}^{(+)} | k_1, \ldots, k_n, k'_1, \ldots, k'_m \rangle_{\underbrace{a,a,\ldots}_{n},\underbrace{b,b,\ldots}_{m}} = \mathrm{Pf}(\Sigma)\mathrm{Pf}(\Sigma') \tag{5.6.16}$$

for n and m even, the form factor being zero otherwise. Here Pf means the Pfaffian of a matrix. The $n \times n$ matrix Σ and the $m \times m$ matrix Σ' have matrix elements

$$[\Sigma]_{j,l} = \overbracket{|k_j\rangle_+ \quad |k_l\rangle_-} \ , \quad [\Sigma']_{j,l} = \overbracket{|k'_j\rangle_+ \quad |k'_l\rangle_-} \quad .$$

The factorized form of the right-hand side of (5.6.16) strongly suggests that we can identify the field $\mathcal{O}^{(+)}$ in the Dirac theory on the Poincaré disk with a tensor product of order fields σ in two independent copies of the Majorana theory on the Poincaré disk, as in (5.6.5). Comparing with (5.6.8), equation (5.6.16) then leads to the form factors of the order field:

$$\frac{\langle \mathrm{vac}|\sigma|k_1, \ldots, k_n\rangle}{\langle\sigma\rangle} = \mathrm{Pf}(\Sigma) \tag{5.6.17}$$

for n even, zero otherwise. In particular, this gives the two-particle form factor as

$$\frac{\langle \mathrm{vac}|\sigma|k_1, k_2\rangle}{\langle\sigma\rangle} = \begin{cases} 0 \quad (k_1 + k_2 \text{ even}) \\ f(k_1, k_2) \quad (k_1 + k_2 \text{ odd}) \end{cases} \tag{5.6.18}$$

(where I recall that $f(k_1, k_2)$ is given by (B.2.12) in Appendix B.2.5), and says that we can calculate multi-particle form factors of the order field by using Wick's theorem to pair one-particle states in the asymptotic states, the contractions being given by the two-particle form factors. It is possible to verify that the two-particle form factor (5.6.18) can be written as (5.6.10)

5.6.4 Form factors of disorder field

Now we proceed to verify the factorization properties of the field $\mathcal{O}^{(-)}$ and to calculate the multi-particle form factors of the disorder field μ in the Majorana theory. As in the previous subsection, using definitions (5.6.6) and Wick's theorem, the multi-particle form factor of the field $\mathcal{O}^{(-)}$ in the Dirac theory can be written in the form

$$\frac{1}{\langle\mathcal{O}^{(+)}\rangle}\langle \mathrm{vac}|\mathcal{O}^{(-)}|k_1, \ldots, k_n, k'_1, \ldots, k'_n\rangle_{\underbrace{+,+\ldots}_{n},\underbrace{-,-\ldots}_{n}} = \tag{5.6.19}$$

$$(-1)^{n(n-1)/2}\frac{\det(A_+)-\det(A_-)}{2}$$

where, again as in the previous subsection, the $n \times n$ matrices A_+ and A_- have matrix elements

$$[A_+]_{ij} = f(k_i, k'_j), \quad [A_-]_{ij} = -f(k'_j, k_i).$$

In Appendix B.5, it is shown that

$$\frac{\det(A_+)-\det(A_-)}{2} = \mathrm{Res}_w \det(A_+(w)) \tag{5.6.20}$$

where $A_+(w)$ is a matrix with matrix elements depending on an auxiliary (formal) variable w:

$$[A_+(w)]_{ij} = f(k_i, k'_j)\cdot \begin{cases} w^{-1} & (k_i \text{ and } k'_j \text{ even}) \\ 1 & (k_i \text{ or } k'_j \text{ odd}). \end{cases}$$

In equation (5.6.20), the symbol Res_w is just a convenient way of saying that one must keep only the coefficient of the monomial w^{-1} in the determinant $\det(A_+(w))$, that is, one must take the formal residue in the variable w. Equation (5.6.20) means that we can calculate the form factor (5.6.19) by using Wick's theorem with contractions given by

$$\overbracket{|k_1\rangle_+ \; |k_2\rangle_-} = f(k_1, k_2)\cdot \begin{cases} w^{-1} & (k_1 \text{ and } k_2 \text{ even}) \\ 1 & (k_1 \text{ or } k_2 \text{ odd}) \end{cases}$$

(other contractions being zero) and by taking the formal residue in w of the resulting sum of products of contractions. Changing basis to $|k\rangle_a$ and $|k\rangle_b$, we can calculate the multi-particle form factors by using Wick's theorem with the contractions

$$\overbracket{|k_1\rangle_a \; |k_2\rangle_a} = \overbracket{|k_1\rangle_b \; |k_2\rangle_b} = \frac{1}{2}(f(k_1,k_2) - f(k_2,k_1)) = \frac{\langle \mathrm{vac}|\sigma|k_1,k_2\rangle}{\langle\sigma\rangle}$$

and

$$\begin{aligned} \overbracket{|k_1\rangle_a \; |k_2\rangle_b} &= -\frac{i}{2}w^{-1}(f(k_1,k_2) + f(k_2,k_1)) \\ &= -i\, w^{-1}\,(-1)^{\frac{k_1}{2}}\sqrt{f(k_1,k_1)}\,(-1)^{\frac{k_2}{2}}\sqrt{f(k_2,k_2)} \end{aligned}$$

and by taking the residue in w (here we used the first and second equations of (B.2.13)). Then we find

$$\frac{1}{\langle\mathcal{O}^{(+)}\rangle}\langle\mathrm{vac}|\mathcal{O}^{(-)}|k_1,\ldots,k_n,k_1',\ldots,k_m'\rangle_{\underbrace{a,a,\ldots}_{n},\underbrace{b,b,\ldots}_{m}} = \tag{5.6.21}$$
$$-\left(\sum_{j=1}^{n}(-1)^{j-1}\,(-1)^{\frac{k_j}{2}}\sqrt{if(k_j,k_j)}\,\frac{\langle\mathrm{vac}|\sigma|k_1,\ldots,\widehat{k_j},\ldots,k_n\rangle}{\langle\sigma\rangle}\right)\times$$
$$\times\left(\sum_{j=1}^{m}(-1)^{j-1}\,(-1)^{\frac{k_j'}{2}}\sqrt{if(k_j',k_j')}\,\frac{\langle\mathrm{vac}|\sigma|k_1',\ldots,\widehat{k_j'},\ldots,k_m'\rangle}{\langle\sigma\rangle}\right)$$

for n and m odd, the form factor being zero otherwise (where the hat means omission of the argument). Again, as in the case of the field $\mathcal{O}^{(+)}$, the factorized form of the right-hand side of (5.6.21) strongly suggests that we can identify the field $\mathcal{O}^{(-)}$ in the Dirac theory on the Poincaré disk with a tensor product of disorder fields μ in two independent copies of the Majorana theory on the Poincaré disk, as in (5.6.5). Comparing with (5.6.9), equation (5.6.21) then leads to the form factors of the disorder field:

$$\langle\mathrm{vac}|\mu|k_1,\ldots,k_n\rangle = \sum_{j=1}^{n}(-1)^{j-1}\langle\mathrm{vac}|\mu|k_j\rangle\,\frac{\langle\mathrm{vac}|\sigma|k_1,\ldots,\widehat{k_j},\ldots,k_n\rangle}{\langle\sigma\rangle} \tag{5.6.22}$$

for n odd, zero otherwise. Here the one-particle form factor $\langle\mathrm{vac}|\mu|k\rangle$, up to a sign factor independent of k, is given by

$$\langle\mathrm{vac}|\mu|k\rangle = \langle\sigma\rangle\,(-1)^{\frac{k}{2}}\sqrt{if(k,k)} \tag{5.6.23}$$

(where we recall that $f(k,k) = f_+(k,k)$ is given by (B.2.12)). From the last property of (B.2.13), the one-particle form factor $\langle\mathrm{vac}|\mu|k\rangle$ is non-zero only for k even, and it is real since $if(k,k)$ is real and positive. The ambiguous sign factor was chosen to make $\langle\mathrm{vac}|\mu|0\rangle$ positive. This ambiguity is related to the ambiguity in the choice of branch on which to evaluate the correlation function $\langle\psi(x)\mu(y)\rangle$ in the Majorana theory.

It is possible to show that the one-particle form factor (5.6.23) can be expressed as (5.6.11)

5.7 Short distance expansion and conformal perturbation theory

In this section, I calculate the short-distance expansion of the two-point functions $\langle\sigma(x)\sigma(x')\rangle$ and $\langle\mu(x)\mu(x')\rangle$. This fixes the constant k (5.5.29) that specifies the singular behavior of the Painlevé transcendent around $s=0$ as in (5.5.28), which determines the Painlevé transcendent used to describe these two-point functions as explained in the Section 5.5. The form of the short distance asymptotic behavior $s\to 0$ of the appropriate transcendent can be obtained by iteratively solving (4.1.5), (5.5.25) with prescribed leading behavior (5.5.28), (5.5.33). In terms of the combinations $F_-(s)=(2R)^{1/4}s^{1/8}\,G(s)$ and $F_+(s)(2R)^{1/4}s^{1/8}\,\tilde{G}(s)$, it is given by

$$\begin{aligned} F_\pm(s) \;=\; & 1\pm r\,s^{1/2}\Omega+\frac{1}{8}(2r^2-1)s\pm\frac{1}{8}r\,s^{3/2}\left[2+\Omega+2r^2\Omega\right] \\ & +\frac{1}{128}s^2\left[-7+r^2\left(15+8\Omega^2\right)+r^4\left(1+8\Omega-8\Omega^2\right)\right] \\ & \pm\frac{1}{128}r\,s^{5/2}\left[17+7\Omega+3r^2(3+7\Omega)+r^4(-1+5\Omega)\right]+O\left(s^3\,\Omega^2\right)\ , \end{aligned} \tag{5.7.1}$$

where the term Ω contains the logarithmic dependence in s,

$$\Omega=\ln\left(k\,s^{1/2}\right)\ . \tag{5.7.2}$$

The constant k is determined from conformal perturbation theory and from the vacuum expectation value of the energy field $\varepsilon(x)=i(2R)^{-1}(1-z\bar{z})(\psi\bar{\psi})(x)$. The latter can be obtained from the propagator

$$\langle\,\psi(x)\,\bar{\psi}(x')\,\rangle=(1-z\bar{z}')^{-1}\mathcal{G}(s)\ , \tag{5.7.3}$$

where the piece

$$\mathcal{G}(s)=\frac{\Gamma(r)\Gamma(1+r)}{2i\,\Gamma(2r)}(1-s)^rF(r,1+r;1+2r;1-s) \tag{5.7.4}$$

is function of the projective invariant s, Eq. (5.4.4). This propagator is determined by the equations of motion (5.2.5)

$$(1-z\bar{z})\,\partial_z\Big((1-z\bar{z})\partial_{\bar{z}}\langle\psi(x)\bar{\psi}(x')\rangle\Big)=r^2\langle\psi(x)\bar{\psi}(x')\rangle\ , \tag{5.7.5}$$

and the normalization condition

$$\langle\psi(x)\bar{\psi}(x')\rangle \sim \frac{ir}{1-z'\bar{z}'}\ln|z-z'|^2 \qquad \text{as} \qquad |z-z'|\to 0\ , \tag{5.7.6}$$

as well as the condition that it vanishes in the limit of large geodesic distance $s\to 1$. One can calculate the vacuum expectation value of the energy field by point-splitting technique. Due to resonance between the energy field $\varepsilon(x)$ and the identity field multiplied by the mass parameter, $m\,\mathbf{1}$, one needs one more condition to define the energy field. This condition can be taken as (5.5.31) which gives (5.5.30).

Now consider the order-order operator product expansion (OPE) in the massive Majorana theory on the pseudosphere, which has the form

$$\sigma(x)\sigma(x') = (2R)^{-1/4}s^{-1/8}\left(C_{\mathbf{1}}(s,r)\,\mathbf{1} + 2R\,s^{1/2}\,C_\varepsilon(s,r)\,\varepsilon(x') + \ldots\right), \tag{5.7.7}$$

where $C_{\mathbf{1}}(s,r)$ and $C_\varepsilon(s,r)$ are structure functions and the dots represent contibutions coming from descendent fields. In the Majorana theory of mass m on infinite flat space, the main property of such OPE's is that the structure functions involved are analytic in the perturbing parameter m in some region around $m=0$ (in fact, they are entire functions of m). All non-analyticities around $m=0$ of correlation functions come from the vacuum expectation values of the fields appearing in OPE's. In the massive Majorana theory on the pseudosphere, analyticity of structure functions is a trivial statement, since all correlation functions are expected to be analytic in some region around $m=0$ (expected to be finite). This comes from the fact that the negative curvature plays the role of an infrared regulator. A more useful statement is that the flat space limit $R\to\infty$ (that is, the limit where the infrared regulator disappears) should be well defined independently on every term in the expansion in m of the structure functions. This gives, for the structure functions, the form

$$\begin{aligned} C_{\mathbf{1}}(s,r) &= 1 + Cs^{1/2} + O(s) + rs^{1/2}\left(a\ln s + b + O(s^{1/2}\ln s)\right) + O(r^2 s\ln^2 s)\ , \\ C_\varepsilon(s,r) &= \frac{1}{2} + O(s^{1/2}\ln s)\ . \end{aligned} \tag{5.7.8}$$

Logarithmic terms appear in the part proportional to r in $C_{\mathbf{1}}(s,r)$ because of the resonance between ε and $m\,\mathbf{1}$; from (5.7.1) the coefficient a is equal to $-1/2$.

Clearly, the OPE (5.7.7) shows that the constant $-r\ln k$ involved in the short distance expansion (5.7.1) of the order-order correlation function is given by

$$-r\ln k = R\langle\varepsilon\rangle + rb + C \tag{5.7.9}$$

in terms of the vacuum expectation value of the energy field and of the constants b and C appearing in the structure function $C_{\mathbf{1}}(s,r)$. These constants can be obtained from the conformal limit $m\to 0$ [22],

$$\langle\sigma(x)\sigma(x')\rangle_{m=0} = (2R)^{-1/4}\sqrt{s^{1/4}+s^{-1/4}}\ , \tag{5.7.10}$$

which gives $C=0$, and from the known flat space limit $k(r)\to re^{\gamma}/4$ as $r\to\infty$ [132], so that $b=\ln 4$. This gives Eq. (5.5.29).

It is instructive to explicitly evaluate the constant b by conformal perturbation theory, thereby giving a simple derivation of the known flat space limit. This illustrates the use of a negative curvature as an infrared regulator. The form (5.7.8) of the structure functions imply that the constant b can be calculated by perturbation theory of the two point function $\langle\sigma(x)\sigma(x')\rangle$ about the boundary Ising CFT, expanding to first order in m:

$$\begin{aligned}\langle\sigma(x)\,\sigma(x')\rangle &= \langle\sigma(x)\,\sigma(x')\rangle_{m=0} \ + \\ &+\frac{2R^2m}{\pi}\int_{|\zeta|<1}\frac{d\mathrm{x}''d\mathrm{y}''}{(1-\zeta\bar\zeta)^2}\Big(\langle\sigma(x)\sigma(x')\varepsilon(x'')\rangle_{m=0}-(2R)^{-1}\langle\sigma(x)\sigma(x')\rangle_{m=0}\Big)\\ &+O(m^2)\ ,\end{aligned} \tag{5.7.11}$$

where $\zeta=\mathrm{x}''+i\mathrm{y}''$ and $\bar\zeta=\mathrm{x}''-i\mathrm{y}''$. By simple considerations of large geodesic distance asymptotics, the integral above is infrared ($|\zeta|\to 1$) convergent. From the result (5.7.10) and from

$$\begin{aligned}\langle\sigma(x)\sigma(x')\varepsilon(x'')\rangle_{m=0} \ &= \ \frac{(2R)^{-5/4}}{2\sqrt{s^{1/4}+s^{-1/4}}}\Bigg\{s^{1/4}\left(\frac{|\zeta||\zeta-s^{\frac12}|}{|1-s^{\frac12}\zeta|}+\frac{|1-s^{\frac12}\zeta|}{|\zeta||\zeta-s^{\frac12}|}\right)\\ &\qquad + s^{-1/4}\left(\frac{|\zeta-s^{\frac12}|}{|\zeta||1-s^{\frac12}\zeta|}+\frac{|\zeta||1-s^{\frac12}\zeta|}{|\zeta-s^{\frac12}|}\right)\Bigg\}\ ,\end{aligned}$$

where s is the projective distance (5.4.4) between the points x and x', we find

$$\langle\sigma(x)\,\sigma(x')\rangle = \tag{5.7.12}$$
$$(2R)^{\frac{1}{4}}s^{-\frac{1}{8}}\left[1+\frac{1}{2}s^{\frac{1}{2}}+O(s)+rs^{\frac{1}{2}}\left(-\frac{1}{2}\ln s+\ln 4+O\left(s^{\frac{1}{2}}\right)\right)+O(r^2)\right]\,,$$

which, as expected, gives $C=0$, $b=\ln 4$.

5.8 Thermodynamics and Discussion

5.8.1 Relation to the hypothetical statistical model

Consider an Ising-like lattice statistical model embedded into the pseudosphere and recall the picture of the scaling limit described in Chapter 3. According to this picture, the correlation functions of order and disorder variables in the lattice model are evaluated, in the scaling limit, at distances proportional to R, with prescribed proportionality constants. Then the leading asymptotic behaviors of correlation functions in this limit are described by correlation functions in the Ising quantum field theory on the pseudosphere developed in this chapter, with

$$|m| = 1/\xi^{\text{flat}}.$$

The quantity r in the Ising field theory is equal to the ratio R/ξ^{flat} that is kept fixed in the scaling limit.

The order field σ in the Ising field theory can be interpreted as the scaling limit of the lattice spin variable, let's denote it $\hat{\sigma}$, in the microscopic theory. According to this interpretation, the sign of m is positive if the temperature is brought towards the critical temperature from below in the scaling limit, and negative if it is brought from above. Similarly, the lattice disorder variable $\hat{\mu}$ is described in the scaling limit by the disorder field μ in the Ising field theory. The relation between correlation functions in the lattice theory and correlation functions in the Ising field theory is given by, for instance,

$$\langle\hat{\sigma}(x)\hat{\sigma}(y)\rangle \sim \langle\sigma(x)\sigma(y)\rangle \quad \text{in the region } a\ll\xi^{\text{flat}}\,,\ a\ll R\,,\ a\ll \mathrm{d}(x,y)$$

with

$$R/\xi^{\rm flat} = r\,,\ \ {\rm d}(x,y)/\xi^{\rm flat}\ \ \text{fixed},$$

where a is a microscopic distance. The spin and disorder variables on the lattice are normalized in accordance to the conditions (5.4.7). More precisely, the normalization of the spin and disorder variables on the lattice is given by

$$\langle\hat\sigma(x)\hat\sigma(y)\rangle \sim {\rm d}(x,y)^{-\frac14},\quad \langle\hat\mu(x)\hat\mu(y)\rangle \sim {\rm d}(x,y)^{-\frac14}\quad \text{ as } a \ll d(x,y) \ll \xi^{\rm flat}\ .$$

5.8.2 The free energy

An interesting quantity to study is the specific free energy $f(m,R)$ as function of the mass m and curvature radius R, defined through the partition function Z by $f = -\lim_{V\to\infty} \ln Z^{1/V}$ where V is the (two-dimensional) volume of space.

A particularly simple case is the massless one, $m=0$, where the free energy $f(m=0,R)$ can be computed using the defining relation between the trace of the energy-momentum tensor T^μ_μ and the variation of the action S under a scale transformation of the metric $g_{\mu\nu}$, $\sqrt{g}\,T^\mu_\mu = 2g_{\mu\nu}\,\delta S/\delta g_{\mu\nu}$. This gives

$$R\frac{d}{dR}\Big[V(R)\,f(0,R)\Big] = V(R)\langle T^\mu_\mu\rangle\ , \tag{5.8.1}$$

where the volume $V(R)$ must be taken finite (but large) for this equation to make sense and must vary like R^2 under a scale transformation, $R\,dV(R)/dR = 2V(R)$. For the pseudosphere, the trace anomaly is related to the central charge c by $\langle T^\mu_\mu\rangle = c/(12\pi R^2)$, where we have set to zero the constant piece related to the vacuum energy density. With $c=1/2$, this yields

$$f(m=0,R) = \frac{1}{24\pi R^2}\ln\left(\frac{2R}{L}\right)\ , \tag{5.8.2}$$

where L is an integration constant not determined by the quantum field theory but only fixed after specifying the microscopic theory, i.e. the theory with an explicit ultra-violet cutoff. In the scaling limit of the corresponding microscopic theory, that is, setting the temperature equal to the flat-space critical temperature and making R very large in microscopic units, the corresponding free energy per unit volume is expected to have the leading behavior (5.8.2).

Such a geometrical contribution to the specific free energy, and in particular the presence of the non-universal distance L, is expected for theories on a space that is not asymptotically flat. In the case of the pseudosphere, the logarithmic increase of $R^2 f(0,R)$ as R increases is related to the decrease of the "space available" around every site, which decreases the interaction energy and increases the free energy (as opposed to the case of a sphere of radius R, where one observes a decrease of $R^2 f(0,R)$ [143]). In comparison, there is no such contributions to the specific free energy in asymptotically flat spaces without singularities, for instance. There is only a finite and universal contribution to the total, volume-integrated free energy; for a conformal-to-flat metric $g_{\mu\nu} = e^{2\phi}\delta_{\mu,\nu}$, this contribution is given by the well-known formula

$$-\frac{c}{24\pi}\lim_{V\to\infty}\int_V d^2x\,\partial_\mu\phi\,\partial^\mu\phi\;. \tag{5.8.3}$$

The free energy $f(m,R)$ of the Ising field theory at arbitrary mass can be obtained from the vacuum expectation value of the energy field at finite volume V, here denoted by $\langle\varepsilon\rangle_V(x)$, by taking the infinite-volume limit:

$$\frac{d}{dm}f(m,R) = -\frac{1}{2\pi}\lim_{V\to\infty}\frac{1}{V}\int_V d^2x\left[\langle\varepsilon\rangle_V(x) + m\ln\left(\frac{2R}{\epsilon}\right)\right]\;, \tag{5.8.4}$$

where ϵ is another non-universal microscopic distance.

In (5.8.4), it is tempting to take the limit of infinite volume V by simply replacing $\langle\varepsilon\rangle_V(x)$ by its infinite-volume, position-independent expression (5.5.30). However, because on the pseudoshpere the surface enclosing a finite region increases as fast as its volume for large volumes, it is possible that contributions proportional to the surface, arising from integration of $\langle\varepsilon\rangle_V(x)$ at positions x near the boundary (where it is significantly different from (5.5.30)), give in $f(m,R)$ additional finite terms. We have not yet evaluated these contributions, but expect to come back to this problem in a future work.

A similar situation was found in the study of the Ising model on hyperlattices [110]. As the authors did, we focus our attention on a "bulk" free energy, defined by taking for $\langle \varepsilon \rangle_V(x)$ in (5.8.4) the expression (5.5.30), valid at positions x far from the boundary. This gives

$$2\pi R^2 f(m,R) = \ln G(1+r) - \frac{r}{2}\ln 2\pi - \frac{r^2}{2}\ln\left(\frac{2R}{e^{1+\gamma}\epsilon}\right) + \frac{1}{12}\ln\left(\frac{2R}{L}\right) \ , \quad (5.8.5)$$

where $G(z)$ is Barnes' G-function

$$G(z+1) = (2\pi)^{z/2} e^{-[z(z+1)+\gamma z^2]/2} \prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)^n e^{-z+z^2/(2n)}\right] \ , \quad (5.8.6)$$

and γ is Euler's constant. This expression has the small r convergent expansion

$$2\pi R^2 f(m,R) = \frac{1}{12}\ln\left(\frac{2R}{L}\right) - \frac{r}{2} - \frac{r^2}{2}\ln\left(\frac{2R}{\epsilon}\right) + \sum_{n=3}^{\infty}(-1)^{n-1}\zeta(n-1)\frac{r^n}{n} \ , \quad (5.8.7)$$

where $\zeta(n)$ is Riemann's zeta function, as well as the following asymptotic expansion at large r,

$$2\pi R^2 f(m,R) = \frac{r^2}{2}\ln\left(\frac{m\,\epsilon\, e^{\gamma-1/2}}{2}\right) - \frac{1}{12}\ln\left(\frac{m\,L}{2}\right) + \zeta'(-1) + O\left(r^{-2}\right) \ , \quad (5.8.8)$$

with the first term corresponding to the specific free energy of the massive Majorana fermion theory in flat space (the $R \to \infty$ limit). As on the sphere [143], there is no logarithmic term in R in this large r expansion.

From the analytic properties of Barnes' G-function, one can see that the free energy (5.8.5) has logarithmic singularities located at the negative integers $r = -1, -2, \ldots$. In particular, it is regular at $r = 0$, that is, the flat space critical temperature does not correspond to a singularity of the free energy.

It is interesting to note that when we fix $r = 1/\sqrt{6}$, the free energy (5.8.5) does not depend on R anymore, and only the ratio L/ϵ appears. In this case, the logarithmic increase of $R^2 f(m,R)$ as R increases due to effects of the geometry as explained above is exactly cancelled out by the logarithmic decrease due to the increase of the interaction energy as the correlation length grows.

The "bulk" free energy defined above still depends on the asymptotic conditions of the quantum field theory. Specifically, the expression (5.8.5) is valid for "fixed" asymptotic conditions, whilst the replacement $r \mapsto -r$ gives the expression for "free" asymptotic conditions. Both asymptotic conditions, or regimes, are stable in the region $-\frac{1}{2} < r < \frac{1}{2}$ and we intend to discuss the possible transitions between these regimes in a future work. A full treatment of the thermodynamics of the model, in fact, seems to require a better understanding of the nature and importance of the neglected surface terms as well as of the other stable asymptotic conditions that break part or all of the symmetries associated to the isometry of the pseudosphere, as described in Section 2.2.

5.8.3 The magnetization

The expression (5.5.36) for the magnetization $\langle\sigma\rangle$ in the Ising field theory is expected to determine the coefficient of the leading asymptotic behavior of the magnetization in the microscopic theory as the scaling limit is taken. As depicted in Figure 3, it does not vanish at the flat space critical temperature $m = 0$, but rather at a higher temperature, corresponding to the value $m = -\frac{1}{2R}$ of the mass parameter. That is, at $r = -1/2$, there is a change in the power law of the leading asymptotic behavior of the magnetization in the microscopic theory as the scaling limit is taken. From this only, we cannot conclude that there exists an R-dependent temperature at which the magnetization vanishes identically in the microscopic theory for any finite R. However, the vanishing of the magnetization occurs at the value of m below which the ordered regime is unstable and the disordered regime, where the magnetization is zero, is stable. It is plausible that there be a similar point joining an ordered and a disordered regime at finite R in the microscopic theory at a temperature higher than the flat-space critical temperature (higher by an amount that has the power law behavior $\sim R^{-1}$ as the scaling limit is taken). The magnetization would vanish at the turning point between the two regimes, as it has been suggested for the regular lattice theory studied in Ref. [110]. We note, though, that our expression (5.8.5) for the free energy is regular at $r=-1/2$. Of course, as we have pointed out, the expression (5.8.5) probably does not give the full free energy, hence no serious conclusion can be drawn from it yet.

Near the effective "critical" temperature, the magnetization in the Ising field theory vanishes as

$$\langle\sigma\rangle^2 = (2R)^{-1/4}\,\sqrt{\pi}\,\bar{s}_{\mathrm{flat}}^4\left(r+\frac{1}{2}\right)+O\left(\left(r+\frac{1}{2}\right)^2\right)\ , \tag{5.8.9}$$

where $\bar{s}_{\mathrm{flat}} = 2^{1/12}\,e^{-1/8}\,A^{3/2}$ (A being Glaisher's constant). The exponent 1/2 can be explained by recalling that a space of constant negative curvature is effectively infinite-dimensional at large distances due to the fact that the volume grows exponentially [20]. In fact, a theory on the pseudosphere should essentially show, in some sense, a cross-over behavior from a two-dimensional theory to an infinite-dimensional theory. Hence mean-field theory could be used to predict the exponent ruling the vanishing of the magnetization in the Ising field theory, giving 1/2 as above. Assuming that the magnetization in the microscopic theory vanishes similarly at a "critical" temperature, the exponent ruling its vanishing should then be 1/2, which agrees with the results of Ref. [110]. In the flat-space limit $R\to\infty$, the magnetization takes the usual form $\langle\sigma\rangle\to\bar{s}_{\mathrm{flat}}\,m^{1/8}$, and the exponent 1/8 is recovered.

The fact that the effective "critical" temperature is higher than the flat-space critical temperature is expected: the asymptotic conditions have a greater effect on the pseudosphere than they have on flat space, hence the "fixed" asymptotic conditions will render the establishment of disorder more difficult, increasing the effective "critical" temperature. Similar considerations apply in the disordered regime: there the effective temperature at which the average of the disorder variable vanishes is lower than the flat-space critical temperature, because "free" asymptotic conditions make it more difficult to establish order.

5.8.4 Two-point correlation functions and susceptibility

An interesting characteristics of the two-point functions is their exponential decay at large geodesic distances,

$$\frac{\langle\sigma(x)\sigma(x')\rangle}{\langle\sigma\rangle^2} - 1 \sim \frac{\Gamma\left(\frac{1}{2}+r\right)\Gamma\left(\frac{3}{2}+r\right)}{2\pi\,\Gamma^2\left(2+r\right)}\, e^{-2(1+r)\,d(x,x')/R} \quad \text{as} \quad \mathrm{d}(x,x') \to \infty\ , \tag{5.8.10}$$

and

$$\frac{\langle\mu(x)\mu(x')\rangle}{\langle\sigma\rangle^2} \sim \frac{\Gamma\left(\frac{1}{2}+r\right)}{\sqrt{\pi}\,\Gamma\left(1+r\right)}\, e^{-\left(\frac{1}{2}+r\right)d(x,x')/R} \quad \text{as} \quad \mathrm{d}(x,x') \to \infty\ . \tag{5.8.11}$$

As expected, the leading exponential decay is different for order-order and disorder-disorder two-point functions; in the former it comes from two-particle contributions, whereas in the latter it comes from one-particle contributions. However, contrary to the flat-space case, the vanishing of the exponent ruling the leading exponential decay occurs at different values of m in order-order and in disorder-disorder two-point functions. This is simply due to the discreteness of the energy levels, and to Pauli's exclusion principle that forces two particles to be in different energy levels. Hence one cannot define, in this way, a unique correlation length valid for describing the long distance behaviour of both correlation functions. It is natural, however, exponential decay of the disorder-disorder two-point function as the one defining an effective correlation length,

$$\xi = \frac{2R}{1+2r}\ . \tag{5.8.12}$$

We indeed expect this correlation length to diverge at the point $r = -\frac{1}{2}$ in the ordered regime where the magnetization $\langle\sigma\rangle$ vanishes, since at this point, the disorder field acquires a nonzero expectation value and the large distance asymptotic behavior of its two-point function changes. This correlation length is also in accordance with the usual definition, in finite size systems, as the inverse of the gap between the ground state and the first excited state. It diverges as the inverse power of the difference of the temperature to the effective "critical" temperature, as is the case for the Ising model on flat two-dimensional space, but it is defined here only for the behavior from above the point $r = -\frac{1}{2}$, since below this point the system is necessarily in its disordered regime. A corresponding definition of the correlation length in the disordered regime leads to a divergence at the point $r = \frac{1}{2}$ from below. Following considerations similar to those of the previous subsection, we expect to have the same power law behavior of the correlation length in the lattice theory in the vicinity of the critical point. Note that a naive application of general results from mean field theory would predict the power law $\sim (1+2r)^{-1/2}$.

In the ordered regime ($r > -\frac{1}{2}$), and as the point $r = -\frac{1}{2}$ is approached, the two-point function of disorder fields goes at large distances to an almost constant value, before vanishing at larger and larger distances. This almost constant value approaches the value that $\langle\mu\rangle^2$ takes in the disordered regime at $r = -\frac{1}{2}$. More precisely, as $r = -\frac{1}{2}$ is approached in the ordered regime, both order-order and disorder-disorder two-point functions tend to the exact form they have in the disordered regime at $r = -\frac{1}{2}$. This is a consequence of the duality relating the point $r = -\frac{1}{2}$ to the point $r = \frac{1}{2}$, which yields for instance

$$\langle\sigma(x)\sigma(x')\rangle\Big|_{r=\pm\frac{1}{2}} = \langle\mu(x)\mu(x')\rangle\Big|_{r=\mp\frac{1}{2}} . \tag{5.8.13}$$

The relation (5.8.13) can be verified by the short distance expansion (5.7.1) essentially from the property $k(1/2) = k(-1/2)$, for $k(r)$ in (5.5.29). More generally, similar relations should hold for any correlation functions. This implements the fact that as r approaches the value $-1/2$ from above, the "free" asymptotic conditions become unstable and are traded for "fixed" asymptotic conditions. A similar duality has also been observed in the study of the statistical Ising model on a hyperlattice [110].

It is also interesting to consider the general case where an external magnetic field h is added to the Ising field theory (5.2.1), by adding the perturbation $h\int d^2x\; e^{\phi(x)}\,\sigma(x)$. The corresponding susceptibility χ giving the linear response of the magnetization is given by

$$\begin{aligned}\chi &= \frac{1}{2}\int \frac{d^2x}{4R^2}\; e^{\phi(x)}\;\left(\langle\sigma(x)\sigma(0)\rangle - \langle\sigma\rangle^2\right) \\ &= \frac{\pi}{4R}\int_0^\infty ds\;\sinh\left(\frac{s}{R}\right)\;\left(\langle\sigma(x)\sigma(0)\rangle|_{\mathrm{d}(x,0)=s} - \langle\sigma\rangle^2\right)\;.\end{aligned} \tag{5.8.14}$$

Using the asymptotic behavior (5.8.10) for the ordered regime, it is straightforward to see that the integral above is convergent for any $r > -\frac{1}{2}$, with a divergence at $r = -\frac{1}{2}$,

$$\chi \sim \frac{\bar{s}^4_{\text{flat}}}{4\sqrt{\pi}}\,\frac{(2R)^{-1/4}}{1+2r} \qquad \text{as} \qquad r \to -\frac{1}{2}\;. \tag{5.8.15}$$

The susceptibility again shows a divergence at the effective "critical" value $m = -\frac{1}{2R}$, with a mean-field power law behavior. A similar phenomenon was observed for the model studied in [110]. In the disordered regime, using the asymptotic behavior (5.8.11) with $r \mapsto -r$ for the order-order two-point function, one can see that the susceptibility is finite for $r < -\frac{1}{2}$ and diverges with mean-field power law at $r = -\frac{1}{2}$, even though the regime is stable above $-\frac{1}{2}$. Hence in the whole range $-\frac{1}{2} < r < \frac{1}{2}$ in the disordered regime, the response of the magnetization to a magnetic field is not linear at small magnetic field. Note also that the susceptibility diverges at $r = -\frac{1}{2}$ from both directions with the same exponent.

Relating the susceptibility to the expansion of the free energy in powers of the magnetic field in the usual fashion, one could conclude from this analysis that the free energy possesses a singular behavior at small magnetic field in the region $-\frac{1}{2} < r < \frac{1}{2}$ of the disordered regime. However, from considerations similar to those above, it is possible that one needs to take into account surface terms in order to obtain the correct coefficients in the expansion of the free energy in powers of magnetic field. I hope to carry out this analysis in a future work.

5.9 Conclusion

In this chapter, I have first developed the Hilbert space of the Ising quantum field theory and identified its stable $SU(1,1)$-invariant regimes. In the region $-1/2 < r < 1/2$, there are two such regimes, one in which the magnetization is nonzero (ordered regime), the other in which it is zero (disordered regime); in the region $r > 1/2$ only the ordered regime is stable and in the region $r < -1/2$ only the disordered regime is stable. In all regimes, I have calculated the two-point functions of order and of disorder fields and the magnetization. I also calculated the form factors of order and of disorder fields, which can give the long-distance expansion of any multi-point correlation functions. Then, I have calculated a "bulk" part of the free energy, and I have analyzed the singularities of the susceptibility as function of the real parameter r.

The results of the present chapter open the way to a more detailed study of the Ising model on a curved space, and at the same time leave many questions unanswered. First, in order to have better grounds for my statistical interpretation, it would be very interesting to actually construct a lattice statistical model on the pseudosphere. This would tell us about the subtleties involved in taking the scaling limit, and eventually it would be interesting to compare its scaling behavior to some of the results obtained here. Second, many quantities still need to be calculated in the Ising field theory. The free energy very probably possesses terms that come from effects of the "asymptotic regions" of space. More precisely, the finite volume specific free energy has an infinite volume limit which may depend on more details of how the limit is taken than those taken into consideration by the asymptotic conditions, and extra terms may have to be added to the "bulk" free energy calculated in this chapter. A full analysis of the two stable $SU(1,1)$ regimes discussed here must include these terms. Moreover, in the region $-1/2 < r < 1/2$ there are many stable regimes that are not $SU(1,1)$-invariant, which should be taken into consideration. Among these regimes there should be some where "domain walls" are formed, as in the Ising conformal field theory on the disk. It seems essential to understand their thermodynamics, in particular to calculate their free energy, in order to understand fully the Ising field theory in the region $-1/2 < r < 1/2$.

It would be very interesting to understand the phase structure of the model with a magnetic field. The situation with a magnetic field is more involved, though. From the two-point function of order fields one can evaluate the magnetic susceptibility, and more generally from multi-point correlation functions one can obtain an expansion of the free energy (and of other quantities) in powers of the magnetic field. An estimate of the expansion coefficients can in principle be obtained from the form factor of order fields, giving their long-distance behavior, and from their behavior at colliding points. Such expansions give information about the phase space in nonzero magnetic field; for instance, one could estimate from the first few coefficients of the expansion the position of a (possibly complex) "critical" value of the magnetic field. Note however that this program is plagued with the same problem as that affecting the calculation of the free energy without magnetic field; namely, that possible contributions of the "asymptotic regions" of the pseudosphere may be important, and in principle the calculations should be done first in finite volume.

It might be rewarding to slightly change the point of view and to consider the effect of a statistical system on the surface on which it lives. The interaction between microscopic degrees of freedom living on a dynamical surface and subjected to thermodynamical laws would give the surface a nontrivial dynamics which would be interesting to understand. Such a point of view could be useful in the study of quantum gravity.

Finally, let me note that the techniques used to calculate the two-point functions are worth developing by themselves. For instance, I would like to understand if they can be applied to correlation functions of the fields $\mathcal{O}_\alpha$ in the free Dirac theory (on flat space, say). More ambitiously, I would like to know if they can be generalize to theories with a more complicated spectrum than that of free fermions. In particular, I tried in Section 5.5 to put the techniques under a light which shows as much as possible their characteristics that are independent of the free fermion spectrum. I constructed a nonlocal conserved charge, which action on particular local fields I could calculate solely using conformal OPE's and using a Casimir equation for the algebra of spatial symmetries. Ward identities were obtained from conservation of this nonlocal charge, and the cluster property of correlation functions was used finally to obtain nonlinear differential equations for the two-point functions. Nonlocal conserved charges certainly exist in non-free-fermion integrable theories. It would be interesting, for instance, to see how the nonlocal charge that I constructed is deformed under integrable deformation of the critical Ising field theory obtained by adding a magnetic field. An important point would be to understand the structure of the module for the algebra of the spatial symmetries generated by this nonlocal charge.

Appendix A

About the $SU(2)$-Thirring model

A.1 Expansion of the integral $J_n(a,c)$

In this appendix, we give the first few terms in the expansion of $J_n(a,c)$ (2.4.6) around $c=-2$. The coefficients in this expansion involve standard functions of a, which could then easily be used to obtain an expansion of $J_n(\frac{2\omega}{1+\rho}, \frac{-2}{1+\rho})$ in powers of ρ, as is needed in (2.4.29). To simplify the result, we will use the parameter

$$b = c + 2\,.$$

We find the following expansions in b of the functions $A(q, b-2)$, $B(q, b-2)$ (2.4.9) involved in (2.4.10):

$$\begin{aligned}
A(q,b-2) &= \frac{\Gamma(2-b-q)\,\Gamma(b)}{\Gamma(2-b)\Gamma(1+b-q)}\left\{1+b^2\Big(\frac{\pi^2}{6}+\frac{\psi(1-q)+\gamma_E}{q}\Big)+O(b^3)\right\},\\
B(q,b-2) &= \frac{\Gamma(q+b)\Gamma(b)}{\Gamma(q)\Gamma(2b)}\\
&\quad\times\left\{1+\frac{b}{2q}+\frac{b^2}{2q^2}(q^2\psi'(q)-1)+\frac{b^3}{4q}(q\psi''(q)+2\psi'(q))+O(b^4)\right\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
J_n(a,b-2) \;&=\; \frac{\pi^2}{2}\,\frac{4a^2-n^2+2nb}{b^2(1-b)^2}\exp\Big\{-G_n(a)\,b\\
&\quad+\Big(\frac{G_n'''(a)}{12}-2\,\frac{a\,G_n'(a)+G_n(a)}{n^2-4a^2}+\frac{10}{3}\,\zeta(3)\Big)\,b^3+O(b^4)\Big\}\;,
\end{aligned}$$

where

$$G_n(a) = \psi(a+n/2)+\psi(-a+n/2)+2\gamma_E\;,$$

and $G_n'(a) = \frac{d}{da}\,G_n(a)\,,\quad G_n''(a) = \frac{d^2}{da^2}\,G_n(a)\,.$

A.2 Coefficients for the anomalous dimension

In this appendix, we write down explicit expressions for the coefficients u_1, u_2, v_1 and v_2 taking part in the expansion (2.4.19), (2.4.32) of the the function Γ_g.

On the one hand, from the assumption (2.6.6), the coefficients of odd powers of ρ in the exponential factor of $Z_{n,\omega}$ (2.4.34) are completely fixed by the conjectured constant $\mathbf{Z}_n(\omega\beta)$ (2.6.4). This fixes u_1, u_2 uniquely, giving

$$
\begin{aligned}
u_1 &= \left(\omega^2-\frac{n^2}{16}\right)\left(T_n(2\omega)-\frac{3}{2}\right)+\frac{n(n-2)}{16}\,, \\
u_2 &= \frac{1}{12}\left(\omega^2-\frac{n^2}{16}\right)\left(\omega^2+\frac{n^2}{16}-\frac{1}{2}\right)T_n''(2\omega)+\frac{\omega(4\omega^2-1)}{12}\,T_n'(2\omega) \\
&\quad +\frac{1}{4}\left(\omega^2-\frac{1}{12}\right)T_n(2\omega)-\frac{n(n+4)}{768}-\frac{11\omega^2}{48}+\frac{1}{24}\,\bar\tau_0+\frac{\tau_2}{2}\left(\omega^2-\frac{n^2}{16}\right),
\end{aligned}
\tag{A.2.1}
$$

where

$$
\begin{aligned}
T_n(a) &= \psi(a+n/2)+\psi(-a+n/2)+2\gamma_E+2\bar\tau_0\,, \\
T_n'(a) &= \frac{d}{da}\,T_n(a)\,, \qquad T_n''(a)=\frac{d^2}{da^2}\,T_n(a)\,.
\end{aligned}
$$

On the other hand, the expansion in powers of ρ of (2.4.29) uniquely determines the coefficients u_1, v_1 and $v_2-\frac{3}{2}u_2$ in the first equation of (2.4.32). The coefficient u_1 thus obtained is in agreement with (A.2.1), verifying the assumption (2.6.6) to first order. The coefficients v_1 and (using the expressions $u_{1,2}$ from (A.2.1)) v_2 are

$$
\begin{aligned}
v_1 &= \frac{\omega}{2}\left(\omega^2-\frac{n^2}{16}\right)T_n'(2\omega)+\frac{1}{4}\left(\omega^2-\frac{n^2}{16}\right)T_n^2(2\omega) \\
&\quad -\frac{3}{4}\left(\omega^2-\frac{n(5n-4)}{48}\right)T_n(2\omega)+\frac{7}{8}\left(\omega^2-\frac{n(17n-20)}{112}\right)+\frac{u_1}{2}\,, \\
v_2 &= \left(\omega^2-\frac{n^2}{16}\right)\left(\frac{1}{24}-\frac{\omega^2}{4}\right)T_n''(2\omega) \\
&\quad -\left(\left(\omega^2-\frac{n^2}{16}\right)\frac{T_n(2\omega)}{2}+\frac{1}{4}\left(\omega^2+\frac{n(n-4)}{16}-\frac{1}{2}\right)\right)\omega\,T_n'(2\omega) \\
&\quad -\frac{1}{12}\left(\omega^2-\frac{n^2}{16}\right)T_n^3(2\omega)-\frac{1}{8}\left(\omega^2+\frac{n(n-4)}{16}\right)T_n^2(2\omega) \\
&\quad -\frac{1}{8}\left(\omega^2-\frac{n(n-2)}{8}-\frac{1}{2}\right)T_n(2\omega) \\
&\quad -\frac{1}{8}\left(\omega^2-\frac{n^2}{16}\right)\left(2\,\tau_2-14\,\zeta(3)-3\right)-\frac{n(n-8)}{256}+\frac{u_1}{8}+\frac{v_1}{2}+\frac{3u_2}{2}-\frac{\bar\tau_0}{8}\,.
\end{aligned}
$$

A.3 Relations among perturbative coefficients

In this appendix, I derive equations that relate perturbative coefficients of the anomalous dimension Γ_g (2.4.16).

It is convenient to parametrize the perturbative expansion in $g_{||}$ and $g_\perp$ of the anomalous dimension in the following form:

$$\Gamma_g = \Gamma^{(0)}(g_{||}) - \frac{g_\perp^2}{1+\frac{g_{||}}{2}}\left(\frac{n^2}{32} - \frac{1}{2}F'(g_{||}) + V(g_{||},g_\perp^2)\right) \tag{A.3.1}$$

where $\Gamma^{(0)}$ is given in (2.4.20), and where

$$F(g_{||}) = \sum_{n=1}^{\infty} u_n g_{||}^{2n-1}\ , \quad V(g_{||},g_\perp^2) = \sum_{k=0,l=0}^{\infty} v_l^{(k)} g_{||}^l\, g_\perp^{2k}\ .$$

Eq. (2.4.16) (which is equivalent to (2.4.28) with the normalization condition (2.4.8)), Eq. (2.4.15) and the renormalization group flow equations (2.4.21), (2.4.23) give

$$\mathbf{C}_{\mathbf{I}}^{(ren)} = (Mr)^{-4\omega^2-n^2(1+\rho^2)/4}\ \ (g_\perp^2)^{\omega^2-n^2(1-\rho^2)/16}\, e^{-F(g_{||})+S(g_{||},g_\perp^2)}\ , \tag{A.3.2}$$

and

$$Z_{n,\omega} = M^{2d}\left(2^{\rho+1}\sqrt{\rho}\, e^{\tau_0\rho+\tau_2\rho^3+\ldots}\right)^{n^2/2-2d} \quad e^{F(2\rho)-S(2\rho,0)}\ , \tag{A.3.3}$$

generalizing (2.4.33) and (2.4.34). The power series $S(g_{||},g_\perp^2)$ can always be chosen such that $S(g_{||},0)$ does not have odd powers of $g_{||}$, by the redundancy in (A.3.2), and does not have even powers of $g_{||}$ either, by the fact that their coefficients can be set to 0 in (A.3.3) using the transformation (2.4.35). It is further defined by the equation

$$V(g_{||},g_\perp^2) = \frac{1}{2}\frac{\partial}{\partial g_{||}}S(g_{||},g_\perp^2) + g_{||}\frac{\partial}{\partial g_\perp^2}S(g_{||},g_\perp^2)\ . \tag{A.3.4}$$

It is a simple matter to observe that this equation along with the initial condition $S(g_{||},0) = 0$ give the following infinite set of relations for the coefficients $v_l^{(k)}$ defining the power series $V(g_{||},g_\perp^2)$:

$$\sum_{k=0}^{m} \frac{k!}{\left(\frac{1}{2}-m\right)_k}\, v_{2(m-k)}^{(k)} = 0\, , \quad m = 0,1,2,\ldots \tag{A.3.5}$$

The conjecture (2.6.6) and Eq. (A.3.3) completely fix all coefficients u_n in $F(g_{||})$. Then the relations above are nontrivial relations among the perturbative expansion coefficients of Γ_g in (A.3.1).

A.4 Three-particle contribution

In this appendix, I give the formula for the three-particle contribution $F^{(3)}$ (2.7.5) to the fermion two-point function in the $SU(2)$-Thirring model that we used for our numerical calculations. First specialize the expression written in [92] to the case of three-particle form factors of the field $\mathcal{O}^1_{\beta/4}$ for $\beta^2 = 1$:

$$\langle\, \mathrm{vac}\,|\, \mathcal{O}^1_{1/4}(0)\,|\, A_-(\theta_1)\ldots A_+(\theta_k)\ldots A_-(\theta_3)\,\rangle_{in} = -\frac{A_G^{\frac{9}{2}}\,\Gamma^3(\frac{1}{4})}{2^{\frac{15}{4}}\,e^{\frac{3}{8}}\,\pi^{\frac{9}{4}}}\, e^{\frac{i\pi}{8}}\, M^{\frac{1}{4}}$$
$$\times \prod_{m=1}^{3} e^{\frac{\theta_m}{4}} \prod_{m<j} G(\theta_m - \theta_j) \left\{ \int_{C_+} \frac{d\gamma}{2\pi}\, e^{-\frac{\gamma}{2}} \prod_{p=1}^{k} W(\theta_p - \gamma) \prod_{p=k+1}^{3} W(\gamma - \theta_p) \right.$$
$$\left. + \int_{C_-} \frac{d\gamma}{2\pi}\, e^{-\frac{\gamma}{2}} \prod_{p=1}^{k-1} W(\theta_p - \gamma) \prod_{p=k}^{3} W(\gamma - \theta_p) \right\}. \tag{A.4.1}$$

Here the functions G and W are

$$G(\theta) = i\, \frac{2^{-\frac{5}{12}}\, e^{\frac{1}{4}}\, \Gamma(\frac{1}{4})}{\sqrt{\pi}\, A_G^3}\, \sinh(\theta/2)\, \exp\left(\int_0^\infty \frac{dt}{t}\, \frac{\sinh^2 t(1 - i\theta/\pi)\, e^{-t}}{\sinh(2t)\, \cosh(t)} \right) \tag{A.4.2}$$

and

$$W(\theta) = 2\, \frac{\Gamma(\frac{3}{4} - \frac{i\theta}{2\pi})\Gamma(-\frac{1}{4} + \frac{i\theta}{2\pi})}{\Gamma^2(\frac{1}{4})}\,. \tag{A.4.3}$$

The contour C_+ starts from $-\infty$ on the real axis of the complex γ-plane, goes above the poles located at $\gamma = \theta_p + i\pi/2$, $p = 1, \ldots, k$ and below those located at $\gamma = \theta_p - i\pi/2$, $p = k+1, \ldots, 3$, always staying in the strip $-\pi/2 - 0 < \Im m\, \gamma < \pi/2 + 0$, and finally extends to $+\infty$ on the real axis. Similarly, the contour C_- goes above the poles located at $\gamma = \theta_p + i\pi/2$, $p = 1, \ldots, k-1$ and below those at $\gamma = \theta_p - i\pi/2$, $p = k, \ldots, 3$. Notice that the integrals in (A.4.1) can be expressed in terms of the generalized hypergeometric function ${}_3F_2$ at unity.

Using the expressions (A.4.1) and performing one of the rapidity integrals in (2.6.2), one can obtain the following form for the function $F^{(3)}$ in (2.7.5):

$$F^{(3)} = \frac{2\,e^{-\frac{3}{4}} A_G^9}{3\pi\,\Gamma^6(\frac{1}{4})} \int_{-\infty}^{\infty} dx dy\; \frac{e^{-Mr\sqrt{3+2\cosh x+2\cosh y+2\cosh(x-y)}}}{(3+2\cosh x+2\cosh y+2\cosh(x-y))^{\frac{1}{4}}}$$
$$\times\,(\,2\,|R_1(x,y)|^2 + |R_2(x,y)|^2\,)\; |G(x)G(y)G(x-y)|^2\, e^{\frac{x+y}{2}} \left(\frac{e^{-x}+e^{-y}+1}{e^{x}+e^{y}+1}\right)^{\frac{1}{4}} .$$

The functions R_1 and R_2 here are

$$R_2(x,y) = e^{-\frac{x}{2}+\frac{i\pi}{4}}\; R_1(-x,y-x) - e^{-\frac{y}{2}-\frac{i\pi}{4}}\; R_1^*(-y,x-y)$$

and

$$R_1(x,y) = -\frac{\cosh\frac{x}{2}\cosh\frac{y}{2}}{2\sinh x\sinh y}\, U\Big(-\frac{1}{2},-\frac{1}{2}-\frac{ix}{2\pi},-\frac{1}{2}-\frac{iy}{2\pi};-\frac{ix}{2\pi},-\frac{iy}{2\pi}\Big)$$
$$+e^{-\frac{x}{2}}\frac{\cosh\frac{y-x}{2}\cosh\frac{x}{2}}{2\sinh(y-x)\sinh x}\, U\Big(\frac{1}{2},\frac{1}{2}-\frac{i(y-x)}{2\pi},\frac{1}{2}+\frac{ix}{2\pi};1-\frac{i(y-x)}{2\pi},2+\frac{ix}{2\pi}\Big)$$
$$+e^{-\frac{y}{2}}\frac{\cosh\frac{x-y}{2}\cosh\frac{y}{2}}{2\sinh(x-y)\sinh y}\, U\Big(\frac{1}{2},\frac{1}{2}-\frac{i(x-y)}{2\pi},\frac{1}{2}+\frac{iy}{2\pi};1-\frac{i(x-y)}{2\pi},2+\frac{iy}{2\pi}\Big)\,,$$

where $U(a,b,c;d,e)$ is related to the generalized hypergeometric function ${}_3F_2$ by

$$U(a,b,c;d,e) = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(d)\Gamma(e)}\;\; {}_3F_2(a,b,c;d,e;1)\;.$$

Appendix B
About models on the Poincaré disk

B.1 Symmetries of the Poincaré disk

In this thesis, I will mostly use the representation of the pseudosphere on the so-called Poincaré disk. Hence in this section I recall some of the basic results concerning the isometries of the Poincaré disk.

With appropriate complex coordinates $z = \mathrm{x} + i\mathrm{y}$, $\bar{z} = \mathrm{x} - i\mathrm{y}$, the Poincaré disk can be brought to the region $|z| < 1$ in the complex z–plane. For a Gaussian curvature

$$\hat{K} = -\frac{1}{R^2} \tag{B.1.1}$$

the metric is then specified by the length element

$$ds^2 = \frac{dzd\bar{z}(2R)^2}{(1 - z\bar{z})^2}\ . \tag{B.1.2}$$

The geodesic distance $\mathrm{d}(x_1, x_2)$ between points $x_1 = (z_1, \bar{z}_1)$ and $x_2 = (z_2, \bar{z}_2)$ on the Poincaré disk is given by

$$\mathrm{d}(x_1, x_2) = 2R\,\mathrm{arctanh}\left(\frac{|z_1 - z_2|}{|1 - z_1\bar{z}_2|}\right). \tag{B.1.3}$$

The metric of the Poincaré disk is $SU(1,1)$ invariant. The transformation of coordinates corresponding to the group element

$$g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix},\ |a|^2 - |b|^2 = 1$$

is

$$z \mapsto z' = f_g(z) = \frac{az + \bar{b}}{bz + \bar{a}}\ ,\ \bar{z} \mapsto \bar{z}' = \bar{f}_g(\bar{z}) = \frac{\bar{a}\bar{z} + b}{\bar{b}\bar{z} + a}. \tag{B.1.4}$$

A field theory covariantly defined the Poincaré disk possesses an $SU(1,1)$ symmetry group induced by the isometry group of the metric (B.1.2). The local fields can then be classified by their properties under $SU(1,1)$ transformations. A field $\mathcal{O}(x)$ will be said to have $SU(1,1)$-dimension $(h,\bar{h})$ if it transforms under (B.1.4) as

$$\mathcal{O}(x) \mapsto \mathcal{O}'(x') = (\partial f_g)^{-h} (\bar{\partial} \bar{f}_g)^{-\bar{h}} \, \mathcal{O}(x) \tag{B.1.5}$$

where $\partial \equiv \partial/\partial z$ and $\bar{\partial} \equiv \partial/\partial \bar{z}$. One can construct local fields of higher $SU(1,1)$-dimension by using the covariant derivatives

$$\mathcal{D}\,\mathcal{O}(x) = \left(\partial - \frac{2h\bar{z}}{1 - z\bar{z}}\right) \mathcal{O}(x) \ , \qquad \bar{\mathcal{D}}\,\mathcal{O}(x) = \left(\bar{\partial} - \frac{2\bar{h}z}{1 - z\bar{z}}\right) \mathcal{O}(x) \ , \tag{B.1.6}$$

where the holomorphic covariant derivative $\mathcal{D}$ takes a field of $SU(1,1)$-dimension $(h,\bar{h})$ to a field of dimension $(h+1,\bar{h})$, and the anti-holomorphic covariant derivative $\bar{\mathcal{D}}$ to a field of dimension $(h,\bar{h}+1)$.

The functions $(\partial f_g)^{-h}$ and $(\bar{\partial}\bar{f}_g)^{-\bar{h}}$ that appear in (B.1.5) are called automorphic factors, and will sometimes be denoted by

$$H_{h,g}(z) = (\partial f_g)^{-h} = (bz + \bar{a})^{2h} \ , \ \bar{H}_{\bar{h},g}(\bar{z}) = (\bar{\partial}\bar{f}_g)^{-\bar{h}} = (\bar{b}\bar{z} + a)^{2\bar{h}}. \tag{B.1.7}$$

By definition, these functions have the following properties which insure that the transformations (B.1.5) form a representation of the $SU(1,1)$ group on the space of fields:

$$H_{h,g}(f_{g^{-1}}(z)) = H_{-h,g^{-1}}(z) \ , \ \bar{H}_{h,g}(\bar{f}_{g^{-1}}(\bar{z})) = \bar{H}_{-h,g^{-1}}(\bar{z}) \tag{B.1.8}$$

and

$$\begin{aligned} H_{h,g_2^{-1}}(z) H_{h,g_1^{-1}}(f_{g_2^{-1}}(z)) &= H_{h,(g_1 g_2)^{-1}}(z) \\ \bar{H}_{h,g_2^{-1}}(\bar{z}) \bar{H}_{h,g_1^{-1}}(\bar{f}_{g_2^{-1}}(\bar{z})) &= \bar{H}_{h,(g_1 g_2)^{-1}}(\bar{z}) \ . \end{aligned} \tag{B.1.9}$$

The covariant functions of one holomorphic and one anti-holomorphic coordinates z, $\bar{z}$ are powers of $1 - z\bar{z}$, which transforms as

$$1 - f_g(z)\bar{f}_g(\bar{z}) = H_{-\frac{1}{2},g}(z)\bar{H}_{-\frac{1}{2},g}(\bar{z})(1 - z\bar{z}) \ .$$

We note that z^{-1} transforms like an anti-holomorphic coordinate. Then, regarding z^{-1} as an anti-holomorphic coordinate, one can consider powers of z as being covariant functions, with for instance

$$f_g(z) = H_{-\frac{1}{2},g}(z)\bar{H}_{\frac{1}{2},g}(z^{-1})z \; . \tag{B.1.10}$$

Covariant functions of two holomorphic and two anti-holomorphic coordinates z, w, $\bar{z}$, $\bar{w}$ exist for many transformation laws. Consider a function $F(z, w, \bar{z}, \bar{w})$ with the property

$$F\left(f_g(z), f_g(w), \bar{f}_g(\bar{z}), \bar{f}_g(\bar{w})\right) = H_{h_1,g}(z)H_{h_2,g}(w)\bar{H}_{\bar{h}_1,g}(\bar{z})\bar{H}_{\bar{h}_2,g}(\bar{w})F(z, w, \bar{z}, \bar{w}) \; .$$

Fixing the overall power of $1 - z\bar{z}$ to be 0, for instance, then there is a basis of two solutions to this equation:

$$\begin{aligned} &F(z, w, \bar{z}, \bar{w}) = \\ &\quad (1 - w\bar{w})^{h_1+\bar{h}_1-h_2-\bar{h}_2}(z - w)^{\bar{h}_1-h_1+\bar{h}_2-h_2}(1 - \bar{w}z)^{-h_1-\bar{h}_1+h_2-\bar{h}_2}(1 - w\bar{z})^{-2\bar{h}_1} \end{aligned} \tag{B.1.11}$$

and

$$\begin{aligned} &F(z, w, \bar{z}, \bar{w}) = \\ &\quad (1 - w\bar{w})^{h_1+\bar{h}_1-h_2-\bar{h}_2}(\bar{z} - \bar{w})^{h_1-\bar{h}_1+h_2-\bar{h}_2}(1 - w\bar{z})^{-h_1-\bar{h}_1+\bar{h}_2-h_2}(1 - \bar{w}z)^{-2h_1} \; . \end{aligned} \tag{B.1.12}$$

A general solution can be formed by linear combinations of these two solutions with coefficients that are arbitrary functions of the invariant ratio $\left|\frac{z-w}{1-\bar{w}z}\right|$. One can obtain other solutions by allowing factors of powers of $1-z\bar{z}$ and by modifying accordingly both h_1 and $\bar{h}_1$ in the expressions above.

A basis for the isometry algebra can be taken as the generators of the coordinates transformations

$$z \mapsto z + \epsilon\,(1 - z^2) \; , \qquad z \mapsto z + i\,\epsilon\,(1 + z^2) \; , \qquad z \mapsto z + i\,\epsilon\,z \; , \tag{B.1.13}$$

with conjugate transformations for $\bar{z}$ and where ϵ is a real infinitesimal parameter. These give rise to Lie derivatives on the local fields (B.1.5), denoted respectively by $\mathcal{P}_{\rm x}$, $\mathcal{P}_{\rm y}$ and $\mathcal{R}$. Introducing the notation $\mathcal{P} = \frac{1}{2}(\mathcal{P}_{\rm x} - i\,\mathcal{P}_{\rm y})$, $\bar{\mathcal{P}} = \frac{1}{2}(\mathcal{P}_{\rm x} + i\,\mathcal{P}_{\rm y})$, they are given by

$$\begin{aligned}
\mathcal{P}\,\mathcal{O}(x) &= \left(\partial - \bar{z}^2\bar\partial - 2\bar{h}\bar{z}\right)\mathcal{O}(x)\ , \\
\bar{\mathcal{P}}\,\mathcal{O}(x) &= \left(\bar\partial - z^2\partial - 2hz\right)\mathcal{O}(x)\ , \\
\mathcal{R}\,\mathcal{O}(x) &= i\left(z\partial - \bar{z}\bar\partial + h - \bar{h}\right)\mathcal{O}(x)\ .
\end{aligned} \tag{B.1.14}$$

Symmetries lead to conserved currents j^μ, which are divergenceless in correlation functions (excepts at the positions of other local fields): $\langle \partial_\mu j^\mu \cdots \rangle = 0$. The integral over a closed loop of the normal part of a divergenceless current $\oint dx^\mu \epsilon_{\mu,\nu} \langle j^\nu \cdots \rangle$ is independent of the shape of the loop. Hence conserved currents give rise to conserved charges, or integrals of motion, when integrated over closed loops, or, in fact, over lines beginning and ending at a boundary or in an asymptotic region of space where the fields are given boundary or asymptotic conditions respecting the symmetries. Contributions from local fields to the charges $\mathbf{P}$, $\bar{\mathbf{P}}$, $\mathbf{R}$ corresponding to the symmetries associated to the Lie derivatives (B.1.14) are given by:

$$[\mathbf{P}, \mathcal{O}(x)] = i\mathcal{P}\,\mathcal{O}(x)\ , \qquad [\bar{\mathbf{P}}, \mathcal{O}(x)] = -i\bar{\mathcal{P}}\,\mathcal{O}(x)\ , \qquad [\mathbf{R}, \mathcal{O}(x)] = \mathcal{R}\,\mathcal{O}(x)\ . \tag{B.1.15}$$

The commutator notation is used to represent the contribution of the line integral of the current when the line moves infinitesimally through the position x (see (5.3.3) for explicit expressions of these conserved charges in the Majorana theory). Hence in a quantization scheme, conserved charges are generators for the infinitesimal transformations of local fields. The integrals of motion in (B.1.15) satisfy the $su(1,1)$ algebra

$$[\mathbf{P}, \mathbf{R}] = -i\,\mathbf{P}\ , \qquad [\bar{\mathbf{P}}, \mathbf{R}] = i\,\bar{\mathbf{P}}\ , \qquad [\mathbf{P}, \bar{\mathbf{P}}] = -2i\,\mathbf{R}\ . \tag{B.1.16}$$

B.2 Construction of form factors in the Dirac theory

We study the normalized form factors of scaling fields (4.5.16):

$$F_\alpha(\omega_1,\ldots,\omega_n)_{\epsilon_1,\ldots\epsilon_n} = \langle\langle a_{\epsilon_1}(\omega_1)\cdots a_{\epsilon_n}(\omega_n)\rangle\rangle_\alpha,$$

where we use the notation

$$\langle\langle\cdots\rangle\rangle_\alpha = \frac{\mathrm{Tr}\left(e^{-2\pi K+2\pi i\alpha Q}\cdots\right)}{\mathrm{Tr}\left(e^{-2\pi K+2\pi i\alpha Q}\right)}. \tag{B.2.1}$$

We will assume that the spectral parameters ω_j are real, except when explicitly stated.

B.2.1 Construction of "two-particle" form factors

From the cyclic properties of the trace and the anti-commutation relations for the angular quantization modes, we have

$$\langle\langle c_\nu^\dagger c_{\nu'}\rangle\rangle_\alpha = \frac{\delta(\nu-\nu')}{1+e^{2\pi\nu-2\pi i\alpha}}.$$

From the definition (4.5.13), the trace (4.5.16) is then expressed as an integral:

$$\begin{aligned}\langle\langle a_+(\omega_1)a_-(\omega_2)\rangle\rangle_\alpha = {} & \frac{4^{-r}e^{(\omega_2-\omega_1)\frac{\pi}{4}}}{\Gamma(1+r)^2}\times\\ & \times\int_{-\infty}^{\infty}d\nu\,\frac{\Gamma\left(r+\frac12+i\nu\right)\Gamma\left(r+\frac12-i\nu\right)\cosh(\pi\nu)}{1+e^{2\pi\nu-2\pi i\alpha}}\times\\ & \times F\left(r+\frac12-i\nu, r+\frac12-i\frac{\omega_1}{2};1+2r;2+i0\right)\times\\ & \times F\left(r+\frac12-i\nu, r+\frac12-i\frac{\omega_2}{2};1+2r;2-i0\right).\end{aligned}$$

The contour of integration goes between the poles at $\nu = i\alpha-i/2$ and $\nu=i\alpha+i/2$.

The integrand is proportional to $e^{-\pi\nu}(-\nu)^{-1+i(\omega_1-\omega_2)/2}$ when $\nu\to-\infty$, and to $e^{-\pi\nu}(\nu)^{-1-i(\omega_1-\omega_2)/2}$ when $\nu\to\infty$. This can be obtained from the asymptotics

$$\begin{aligned}F(a,b;c;2\pm i0) &= \frac{\Gamma(c)}{\Gamma(b)}(2a)^{b-c}e^{\pm i\pi(a+b-c)}\left(1+O(a^{-1})\right)\\ &\quad+\frac{\Gamma(c)}{\Gamma(c-b)}(2a)^{-b}e^{\pm i\pi b}\left(1+O(a^{-1})\right),\end{aligned} \tag{B.2.2}$$

valid for $|a| \to \infty$, $|\arg(a)| < \pi$. The integral can be regularized by multiplying the integrand by a factor $e^{p\nu}$ with some complex parameter p; the integral is then convergent for $\Re e(p) = \pi$, $\Im m(p) \neq 0$. It can be evaluated by the method of residues, closing the contour for instance in the upper half plane if $\Im m(p) > 0$. The "two-particle" form factor is the analytical continuation to $p = 0$ of the resulting expression. Contributions of poles in the upper half plane give

$$\langle\langle a_+(\omega_1)a_-(\omega_2)\rangle\rangle_\alpha = \frac{4^{-r} i\pi e^{(\omega_2-\omega_1)\frac{\pi}{4}}}{\sin(\pi(r-\alpha))\Gamma(1+r)^2}\left[G_\alpha(\omega_1,\omega_2) - e^{-i\pi(r-\alpha)}G_r(\omega_1,\omega_2)\right], \tag{B.2.3}$$

where $G_\alpha(\omega_1,\omega_2)$ is the analytical continuation to $p=0$ of the following series:

$$\begin{aligned} G_\alpha^{(p)}(\omega_1,\omega_2) = \sum_{n=0}^{\infty}\Bigg[&\sin(\pi\alpha)\frac{\Gamma(1+r+\alpha+n)}{\Gamma(1-r+\alpha+n)}e^{ip(\frac{1}{2}+\alpha+n)}\times \\ &\times\ F\left(r+1+\alpha+n, r+\frac{1}{2}-i\frac{\omega_1}{2}; 2r+1; 2+i0\right)\times \\ &\times\ F\left(r+1+\alpha+n, r+\frac{1}{2}-i\frac{\omega_2}{2}; 2r+1; 2-i0\right)\Bigg] \end{aligned} \tag{B.2.4}$$

and $G_r(\omega_1,\omega_2)$ is the analytical continuation to $p=0$ of the series above with α replaced by r. Note that the function $G_r(\omega_1,\omega_2)$ is shown below to be identically zero.

The series $G_\alpha^{(p)}(\omega_1,\omega_2)$ is convergent in the upper half p-plane $\Im m(p) > 0$ as well as on $\Im m(p) = 0$, $\Re e(p) \neq \{0, \pi, -\pi\}$. The analytical continuation to $p=0$ can be done via a "zeta-regularization". More precisely, we subtract, inside the summation symbol in (B.2.4), the leading large n asymptotics of the summand:

$$\begin{aligned} &\Gamma(2r+1)^2\sin(\pi\alpha)4^{-r}(2n)^{-1}e^{\pi\Delta}e^{ip(\frac{1}{2}+\alpha+n)}\times \\ &\Bigg[\frac{(2n)^{-i\omega}}{\Gamma\left(r+\frac{1}{2}-i\frac{\omega_1}{2}\right)\Gamma\left(r+\frac{1}{2}-i\frac{\omega_2}{2}\right)} + \frac{(2n)^{i\omega}}{\Gamma\left(r+\frac{1}{2}+i\frac{\omega_1}{2}\right)\Gamma\left(r+\frac{1}{2}+i\frac{\omega_2}{2}\right)} + \\ &\frac{e^{i\pi(\alpha-r+n)}(2n)^{-i\Delta}}{\Gamma\left(r+\frac{1}{2}-i\frac{\omega_1}{2}\right)\Gamma\left(r+\frac{1}{2}+i\frac{\omega_2}{2}\right)} + \frac{e^{-i\pi(\alpha-r+n)}(2n)^{i\Delta}}{\Gamma\left(r+\frac{1}{2}+i\frac{\omega_1}{2}\right)\Gamma\left(r+\frac{1}{2}-i\frac{\omega_2}{2}\right)}\Bigg], \end{aligned}$$

where $\omega = (\omega_1+\omega_2)/2$ and $\Delta = (\omega_1-\omega_2)/2$. The resulting series is convergent at $p=0$. We then add to this series at $p=0$ the following quantity:

$$\Gamma(2r+1)^2\sin(\pi\alpha)4^{-r}e^{\pi\Delta}\times$$

$$\left[\frac{1}{\Gamma\left(r+\frac{1}{2}-i\frac{\omega_1}{2}\right)\Gamma\left(r+\frac{1}{2}-i\frac{\omega_2}{2}\right)}\frac{\zeta(1+i\omega)}{2^{1+i\omega}}+\right.$$
$$\frac{1}{\Gamma\left(r+\frac{1}{2}+i\frac{\omega_1}{2}\right)\Gamma\left(r+\frac{1}{2}+i\frac{\omega_2}{2}\right)}\frac{\zeta(1-i\omega)}{2^{1-i\omega}}+$$
$$\frac{e^{i\pi(\alpha-r)}(2^{-i\Delta}-1)}{\Gamma\left(r+\frac{1}{2}-i\frac{\omega_1}{2}\right)\Gamma\left(r+\frac{1}{2}+i\frac{\omega_2}{2}\right)}\frac{\zeta(1+i\Delta)}{2^{1+i\Delta}}+$$
$$\left.\frac{e^{-i\pi(\alpha-r)}(2^{i\Delta}-1)}{\Gamma\left(r+\frac{1}{2}+i\frac{\omega_1}{2}\right)\Gamma\left(r+\frac{1}{2}-i\frac{\omega_2}{2}\right)}\frac{\zeta(1-i\Delta)}{2^{1-i\Delta}}\right],$$

where $\zeta(z)$ is Riemann's zeta function. The result is $G_\alpha(\omega_1,\omega_2)$. Notice that the function $G_\alpha(\omega_1,\omega_2)$ is real for real ω_1,ω_2.

The resulting expression for $G_\alpha(\omega_1,\omega_2)$ defines a function of ω_1 and ω_2 analytical in the region $|\Im m(\omega_1+\omega_2)|<2$, $|\Im m(\omega_1-\omega_2)|<2$. By repeating the procedure above for the full large n asymptotics of the summand of $G_\alpha^{(p)}(\omega_1,\omega_2)$, one can see that the function $G_\alpha(\omega_1,\omega_2)$ thus defined has no singularity in the finite complex ω_1- and ω_2-planes: the function $G_\alpha(\omega_1,\omega_2)$ is an entire function of ω_1 and ω_2.

The function $G_r(\omega_1,\omega_2)$ can now be evaluated in the following way. From (B.2.3), we have

$$\langle\langle a_+(\omega_1)a_-(\omega_2)\rangle\rangle_0 = -\frac{4^{-r}i\pi e^{(\omega_2-\omega_1)\frac{\pi}{4}}}{\sin(\pi r)\Gamma(1+r)^2}e^{-i\pi r}G_r(\omega_1,\omega_2)\ .$$

By covariance under the subgroup $\mathbb{K}$ we find

$$(\omega_1+\omega_2)G_r(\omega_1,\omega_2)=0\ .$$

Since the function $G_r(\omega_1,\omega_2)$ is entire, this implies that it must be identically zero. Notice that a similar calculation leads to $\langle\langle a_-(\omega_1)a_+(\omega_2)\rangle\rangle_0=0$. Since the commutator $\{a_-(\omega_1),a_+(\omega_2)\}$ is a c-number, an immediate consequence is the anti-commutation property (4.5.8).

Hence we finally have

$$F_\alpha(\omega_1,\omega_2)_{+,-}=\frac{4^{-r}i\pi e^{(\omega_2-\omega_1)\frac{\pi}{4}}}{\sin(\pi(r-\alpha))\Gamma(1+r)^2}G_\alpha(\omega_1,\omega_2)\ . \tag{B.2.5}$$

B.2.2 Two-particle form factors in the discrete basis

It is possible to evaluate the function $G_\alpha(\omega_1, \omega_2)$ for some purely imaginary values of ω_1 and ω_2. We will evaluate it for the values $\omega_1 = -i(1+2r+2k_1)$ and $\omega_2 = -i(1+2r+2k_2)$ for integers $k_1 \geq 0$ and $k_2 \geq 0$. This can be done by using the analytical continuation described above. Equivalently, it can be done by simply replacing these values of ω_1 and ω_2 in the expression (B.2.4), evaluating the resulting series at $p=0$ in a region of r where it is convergent and analytically continuing the result in r. With

$$G_{\alpha;\,k_1,k_2} \equiv G_\alpha(-i(1+2r+2k_1), -i(1+2r+2k_2)) \ ,$$

this gives

$$G_{\alpha;\,k_1,k_2} = \sin(\pi\alpha) \sum_{m_1=0}^{k_1} \sum_{m_2=0}^{k_2} \frac{(-k_1)_{m_1}(-k_2)_{m_2} 2^{m_1+m_2}}{(2r+1)_{m_1}(2r+1)_{m_2} m_1! m_2!} H_{\alpha;\,m_1,m_2} \tag{B.2.6}$$

where

$$\begin{aligned} H_{\alpha;\,m_1,m_2} \quad &= \quad \frac{\Gamma(1+r+\alpha+m_1)\Gamma(1+r+\alpha+m_2)}{\Gamma(1-r+\alpha)\Gamma(1+r+\alpha)} \times \\ &\quad \times {}_3F_2(1, 1+r+\alpha+m_1, 1+r+\alpha+m_2; 1+r+\alpha, 1-r+\alpha; 1) \ . \end{aligned}$$

The ${}_3F_2$ hypergeometric function above can be evaluated in closed form, for any given integer m_1 and m_2, in terms of Gamma functions and rational functions of r and α. The two-particle form factors (4.8.2) in the discrete basis are expressed in terms of $G_{\alpha;k_1,k_2}$:

$$\begin{aligned} &f_\alpha(k_1, k_2) = \\ &\quad 2^{2r+1} i (-1)^{k_1} \sqrt{\frac{\Gamma(1+2r+k_1)\Gamma(1+2r+k_2)}{k_1! k_2!}} \frac{G_{\alpha;\,k_1,k_2}}{\Gamma(1+2r)^2 \sin(\pi(r-\alpha))} \ . \end{aligned} \tag{B.2.7}$$

B.2.3 "Multi-particle" form factors

The "multi-particle" form factors of scaling fields can be evaluated by Wick's theorem in terms of the two-particle form factors. We have

$$F_\alpha(\omega_1,\ldots,\omega_n)_{\epsilon_1,\ldots,\epsilon_n} = \sum_{j=2}^{n}(-1)^j F_\alpha(\omega_1,\omega_j)_{\epsilon_1,\epsilon_j} F_\alpha(\omega_2,\ldots,\widehat{\omega_j},\ldots,\omega_n)_{\epsilon_2,\ldots,\widehat{\epsilon_j},\ldots,\epsilon_n} \tag{B.2.8}$$

where the hat over an argument means omission of this argument. Of course this will be non-zero only for $\sum_{j=1}^n \epsilon_j = 0$. Forming the $n\times n$ matrix

$$\mathbf{F}_\alpha(\omega_1,\ldots,\omega_n)_{\epsilon_1,\ldots,\epsilon_n}$$

with matrix elements $[\mathbf{F}_\alpha(\omega_1,\ldots,\omega_n)_{\epsilon_1,\ldots,\epsilon_n}]_{i,j} = F_\alpha(\omega_i,\omega_j)_{\epsilon_i,\epsilon_j}$, the "multi-particle" form factors can be written as Pfaffians[1]:

$$F_\alpha(\omega_1,\ldots,\omega_n)_{\epsilon_1,\ldots,\epsilon_n} = \mathrm{Pf}\left(\mathbf{F}_\alpha(\omega_1,\ldots,\omega_n)_{\epsilon_1,\ldots,\epsilon_n}\right) . \tag{B.2.9}$$

Using (4.5.8), it is always possible to choose the order such that N operators with positive $U(1)$ charge are followed by N operators with negative $U(1)$ charge. Forming the $N\times N$ matrix $\mathbf{G}_\alpha(\omega_1,\ldots,\omega_N;\tilde\omega_1,\ldots\tilde\omega_N)$ with matrix elements $[\mathbf{G}_\alpha(\omega_1,\ldots,\omega_N;\tilde\omega_1,\ldots\tilde\omega_N)]_{i,j} = F_\alpha(\omega_i,\tilde\omega_j)_{+-}$, we have

$$\begin{aligned} F_\alpha(\omega_1,\ldots,\omega_N,\tilde\omega_1,\ldots,\tilde\omega_N)_{\underbrace{+,+,\ldots,}_{N}\underbrace{-,-,\ldots}_{N}} &= \\ &(-1)^{N(N-1)/2}\det(\mathbf{G}_\alpha(\omega_1,\ldots,\omega_N;\tilde\omega_1,\ldots,\tilde\omega_N)) . \end{aligned} \tag{B.2.10}$$

Similar expressions are valid for the form factors $\langle 0|\mathcal{O}_\alpha|k_1,\ldots,k_n\rangle_{\epsilon_1,\ldots,\epsilon_n}$ in the discrete basis.

B.2.4 Flat space limit of form factors

It is a simple matter to verify that the "two-particle" form factors (B.2.5) specialize to the known expression in the flat space limit. Two-particle form factors

[1] Using this, one can immediately write the form factor expansion (4.3.11) as a Fredholm determinant in the case $\alpha = \alpha'$.

of the scaling fields $\mathcal{O}_\alpha$ in the flat space limit, with particle at rapidities β_1 and β_2 and states normalized by ${}_{\epsilon_1}\langle\beta_1|\beta_2\rangle_{\epsilon_2}^{flat} = 2\pi\delta_{\epsilon_1,\epsilon_2}\delta(\beta_1-\beta_2)$, are obtained by

$$\begin{aligned}\langle\text{vac}|\mathcal{O}_\alpha|\beta_1,\beta_2\rangle_{+-}^{flat} = \lim_{r\to\infty} \times \\ \times (2r)^{\alpha^2+1}2\pi\sqrt{\rho(2r\sinh(\beta_1))\rho(2r\sinh(\beta_2))\cosh(\beta_1)\cosh(\beta_2)} \times \\ \times F_\alpha(2r\sinh(\beta_1), 2r\sinh(\beta_2))_{+-}\end{aligned} \tag{B.2.11}$$

and the one-point function is obtained by

$$\langle\mathcal{O}_\alpha\rangle^{flat} = \lim_{r\to\infty}(2r)^{\alpha^2}\langle\mathcal{O}_\alpha\rangle.$$

Using the expression (B.2.5), this gives the known normalized flat space form factors, first obtained in [113]:

$$\frac{\langle\text{vac}|\mathcal{O}_\alpha|\beta_1,\beta_2\rangle_{+-}^{flat}}{\langle\mathcal{O}_\alpha\rangle^{flat}} = \frac{i\sin(\pi\alpha)}{\cosh\left(\frac{\beta_1-\beta_2}{2}\right)}e^{\alpha(\beta_1-\beta_2)}.$$

B.2.5 Properties of two-particle form factors of the scaling fields $\mathcal{O}_{\pm\frac{1}{2}}$

I now analyze the case of the two-particle form factors of the scaling fields $\mathcal{O}_{\pm\frac{1}{2}}$ evaluated in the discrete basis. These form factors are used to construct form factors of order and disorder fields in Section 5.6.

Define the functions $f_\pm(k_1,k_2)$ by

$$f_\pm(k_1,k_2) \equiv \frac{\langle\text{vac}|\mathcal{O}_{\pm\frac{1}{2}}|k_1,k_2\rangle_{+,-}}{\langle\mathcal{O}_{\pm\frac{1}{2}}\rangle}$$

where the field $\mathcal{O}_{\pm\frac{1}{2}}$ are at the center of the Poincaré disk. Formula (B.2.7 gives expressions for these form factors. In Chapter 5, I use for convenience a slightly different phase for the eigenstates, hence here I shall multiplying these expressions by a phase factor $i^{k_1+k_2+2}$. We have then

$$f_\pm(k_1,k_2) = 2^{2r+1}i^{k_2-k_1+1}\sqrt{\frac{\Gamma(1+2r+k_1)\Gamma(1+2r+k_2)}{k_1!k_2!}}\frac{G_{\pm;k_1,k_2}}{\Gamma(1+2r)^2\cos(\pi r)} \tag{B.2.12}$$

where

$$G_{\pm;\,k_1,k_2} = \sum_{m_1=0}^{k_1} \sum_{m_2=0}^{k_2} \frac{(-k_1)_{m_1}(-k_2)_{m_2} 2^{m_1+m_2}}{(2r+1)_{m_1}(2r+1)_{m_2} m_1! m_2!} H_{\pm;\,m_1,m_2}$$

with

$$H_{\pm;\,m_1,m_2} = \frac{\Gamma(1+r\pm 1/2+m_1)\Gamma(1+r\pm 1/2+m_2)}{\Gamma(1-r\pm 1/2)\Gamma(1+r\pm 1/2)} \times \\ \times\, {}_3F_2(1, 1+r\pm 1/2+m_1, 1+r\pm 1/2+m_2; 1+r\pm 1/2, 1-r\pm 1/2; 1)\ .$$

It can be checked that

$$f_+(k_1,k_2) = -f_-(k_2,k_1).$$

As this identity relates $f_-(k_2,k_1)$ to $f_+(k_1,k_2)$, I need only use $f_+(k_1,k_2)$, which I will denote by $f(k_1,k_2)$. It can be verified that this function satisfies the following identities:

$$\begin{aligned} f(k_1,k_2) &= (-1)^{\frac{k_1+k_2}{2}} \sqrt{f(k_1,k_1)}\sqrt{f(k_2,k_2)} \quad (k_1+k_2 \text{ even}) \\ f(k_1,k_2) &= -f(k_2,k_1) \quad (k_1 \text{ or } k_2 \text{ odd}) \\ f(k_1,k_2) &= 0 \quad (k_1 \text{ and } k_2 \text{ odd})\ . \end{aligned} \tag{B.2.13}$$

Of course, the last identity is just a consequence of the first and the second. In the first identity as well as in equation (4.8.2), square roots $\sqrt{z}$ assume their branch delimited by the region $-\pi \leq \arg(z) < \pi$ with $\sqrt{z} \geq 0$ for $\arg(z) = 0$.

B.3 Canonical quantization of the Majorana theory

In this section we briefly sketch the steps for obtaining the mode decomposition (5.3.10). The Hamiltonian (5.3.5) in the isometric system of coordinates (5.3.1) reads

$$\mathbf{H} = \int_{-\pi/4}^{\pi/4} \frac{d\xi_{\mathrm{x}}}{4\pi} (-i\psi^{(\mathrm{iso})}, i\bar{\psi}^{(\mathrm{iso})})\, \mathcal{H} \begin{pmatrix} \psi^{(\mathrm{iso})} \\ \bar{\psi}^{(\mathrm{iso})} \end{pmatrix}\ , \tag{B.3.1}$$

where the Hamiltonian density is

$$\mathcal{H} = \begin{pmatrix} i\frac{d}{d\xi_{\mathrm{x}}} & \frac{2r}{\cos(2\xi_{\mathrm{x}})} \\ \frac{2r}{\cos(2\xi_{\mathrm{x}})} & -i\frac{d}{d\xi_{\mathrm{x}}} \end{pmatrix}\ . \tag{B.3.2}$$

The Hamiltonian is just, in the language of first quantization, the diagonal matrix element of $\mathcal{H}$ in the state represented by the spinor wave function $\Psi = \begin{pmatrix} \psi^{(\mathrm{iso})} \\ \bar{\psi}^{(\mathrm{iso})} \end{pmatrix}$,

$$\mathbf{H} = (\Psi, \mathcal{H}\Psi) \ , \tag{B.3.3}$$

where the inner product between two spinor wave functions Ψ_1 and Ψ_2 is

$$(\Psi_1, \Psi_2) = \int_{-\pi/4}^{\pi/4} \frac{d\xi_{\mathrm{x}}}{4\pi} \Psi_1^\dagger(\xi_{\mathrm{x}}) \Psi_2(\xi_{\mathrm{x}}) \ . \tag{B.3.4}$$

From the condition on the phases of the fermion operators $\psi^{(\mathrm{iso})}(\xi_{\mathrm{x}}, \xi_{\mathrm{y}})^\dagger = i\psi^{(\mathrm{iso})}(\xi_{\mathrm{x}}, \xi_{\mathrm{y}})$ and $\bar{\psi}^{(\mathrm{iso})}(\xi_{\mathrm{x}}, \xi_{\mathrm{y}})^\dagger = -i\bar{\psi}^{(\mathrm{iso})}(\xi_{\mathrm{x}}, \xi_{\mathrm{y}})$, and from the condition that charge conjugation symmetry $i\psi(\xi_{\mathrm{x}}, \xi_{\mathrm{y}}) \leftrightarrow \bar{\psi}(\xi_{\mathrm{x}}, -\xi_{\mathrm{y}})$ be implemented on modes by $A_\omega^\dagger \leftrightarrow A_\omega$, the mode decomposition of the fields has the form

$$\begin{aligned} \psi^{(\mathrm{iso})}(\xi_{\mathrm{x}}, \xi_{\mathrm{y}}) &= \sum_{\omega>0} \left(e^{\omega\xi_{\mathrm{y}}} G_\omega(\xi_{\mathrm{x}}) A_\omega^\dagger - i\, e^{-\omega\xi_{\mathrm{y}}}\, \bar{G}_\omega(\xi_{\mathrm{x}}) A_\omega \right) , \\ \bar{\psi}^{(\mathrm{iso})}(\xi_{\mathrm{x}}, \xi_{\mathrm{y}}) &= \sum_{\omega>0} \left(e^{\omega\xi_{\mathrm{y}}} \bar{G}_\omega(\xi_{\mathrm{x}}) A_\omega^\dagger + i\, e^{-\omega\xi_{\mathrm{y}}}\, G_\omega(\xi_{\mathrm{x}}) A_\omega \right) . \end{aligned} \tag{B.3.5}$$

The spinor wave functions

$$S_\omega(\xi_{\mathrm{x}}) = \begin{pmatrix} G_\omega(\xi_{\mathrm{x}}) \\ \bar{G}_\omega(\xi_{\mathrm{x}}) \end{pmatrix} , \tag{B.3.6}$$

for all real values of ω and with $G_{-\omega}(\xi_{\mathrm{x}}) = -i\bar{G}_\omega(\xi_{\mathrm{x}})$, should form a complete orthogonal set of wave functions diagonalizing the Hamiltonian density (B.3.2),

$$\mathcal{H} S_\omega = \omega S_\omega \ . \tag{B.3.7}$$

A set of independent spinors of the form (B.3.6) diagonalizing the Hamiltonian density with eigenvalue ω is given by

$$s_\omega^+ = \begin{pmatrix} g_\omega^+ \\ \bar{g}_\omega^+ \end{pmatrix} , \qquad s_\omega^- = \begin{pmatrix} ig_\omega^- \\ -i\bar{g}_\omega^- \end{pmatrix} , \tag{B.3.8}$$

where

$$g_\omega^\pm(\xi_{\mathrm{x}}) = e^{-i\omega\xi_{\mathrm{x}} - i\frac{\pi}{4}(\omega - 1 \mp 2r)} (1 + e^{4i\xi_{\mathrm{x}}})^{\pm r} F\left(-\frac{\omega}{2} + \frac{1}{2} \pm r, \pm r; 1 \pm 2r; 1 + e^{4i\xi_{\mathrm{x}}} \right) , \tag{B.3.9}$$

and

$$\bar{g}^{\pm}_{\omega}(\xi_{\rm x}) = -ig^{\pm}_{-\omega}(\xi_{\rm x}) \ . \tag{B.3.10}$$

The branch cut of the hypergeometric function is taken from $-\infty$ to 1, and the hypergeometric function is chosen to be unity in the limit $\xi_{\rm x} \to -\pi/4$.

The (not normalized) spinors (B.3.6) can then be expressed as

$$S_\omega = \begin{cases} s^+_\omega + C_\omega s^-_\omega & (\omega > 0) \\ -s^+_\omega + C_\omega s^-_\omega & (\omega < 0) \end{cases} \ , \tag{B.3.11}$$

for real constants C_ω satisfying $C_{-\omega} = C_\omega$. For a given set of ω and given associated constants C_ω, they will form a complete orthogonal set if the inner product $(S_\omega, S_{\omega'})$ is well defined for all ω and ω' in this set; and if the Hamiltonian is Hermitian, $(S_\omega, \mathcal{H}S_{\omega'}) = (\mathcal{H}S_\omega, S_{\omega'})$. These lead respectively to the conditions

$$\lim_{\epsilon\to 0} \epsilon \left\{ \Re e \left(\bar{G}_\omega G_{\omega'}\right)_{\xi_{\rm x}=\pi/4-\epsilon} + \Re e \left(\bar{G}_\omega G_{\omega'}\right)_{\xi_{\rm x}=-\pi/4+\epsilon} \right\} = 0 \ , \tag{B.3.12}$$

and

$$\lim_{\epsilon\to 0} \left\{ \Im m \left(\bar{G}_\omega G_{\omega'}\right)_{\xi_{\rm x}=\pi/4-\epsilon} - \Im m \left(\bar{G}_\omega G_{\omega'}\right)_{\xi_{\rm x}=-\pi/4+\epsilon} \right\} = 0 \ . \tag{B.3.13}$$

In the case $r > \frac{1}{2}$, condition (B.3.12) is satisfied only with $C_\omega = 0$ and by taking the hypergeometric function in (B.3.9) to have trivial monodromy: $\omega = \pm(1 + 2r + 2n)$, $n = 0, 1, 2, \ldots$. The function G_ω then vanishes at the boundaries $\xi_{\rm x} = \pm\pi/4$. With this wave decomposition, the Fermi fields vanish as $e^{-m{\rm d}}$ as the geodesic distance d to the center of the disk goes to infinity.

In the case $0 < r < \frac{1}{2}$, condition (B.3.12) is always satisfied. Condition (B.3.13) is then satisfied for many sets $\{\omega; C_\omega\}$. They correspond to many sets of stable asymptotic conditions on the fields, hence to many stable regimes of the quantum field theory with different thermodynamic properties. In this paper we concentrate our attention on the regimes which preserve the $SU(1,1)$ symmetry; we intend to analyse other regimes in a future paper. The charges (5.3.3) must then be well defined (and the Hilbert space must form a lowest weight module for the $su(1,1)$ algebra that they generate), which imposes again that the hypergeometric function in (B.3.9) has trivial monodromy, but not that the function G_ω be vanishing at the boundaries $\xi_{\rm x} = \pm\pi/4$. Hence there are two possible sets: $\omega = \pm(1+2r+2n)$, $n = 0,1,2,\ldots$ with $C_\omega = 0$; and $\omega = \pm(1-2r+2n)$, $n=0,1,2,\ldots$ with $C_\omega \to \infty$. In the first set, the Fermi fields vanish as $e^{-m{\rm d}}$ as the geodesic distance d to the center of the disk goes to infinity, whereas in the second set, they diverge as $e^{m{\rm d}}(1+O(e^{-{\rm d}/R}))$. These correspond respectively to the "fixed" an "free" asymptotic conditions on the order field σ.

The decomposition (5.3.10) follows from these considerations, with the identification $A^\dagger_\omega \mapsto e^{i\frac{\pi}{2}n}A^\dagger_n$ and $A_\omega \mapsto e^{-i\frac{\pi}{2}n}A_n$; this choice of phases insures hemiticity of the conserved charges (5.3.3) on the Hilbert space.

B.4 OPE $\psi\sigma$ in CFT

We start from the following form for the OPE's

$$\begin{aligned} \langle\psi(z)\sigma(0)\cdots\rangle &= \sum_{n\geq 0} z^{n-1/2}\sqrt{\frac{i}{2}}c_n\langle\partial^n\mu(0)\cdots\rangle \\ \langle\psi(z)\mu(0)\cdots\rangle &= \sum_{n\geq 0} z^{n-1/2}\sqrt{\frac{-i}{2}}c_n\langle\partial^n\sigma(0)\cdots\rangle \end{aligned} \tag{B.4.1}$$

where c_n are to be determined, and use

$$\langle\psi(z_1)\psi(z_2)\cdots\rangle \sim \frac{1}{z_1-z_2}\langle\cdots\rangle\ .$$

Consider the function

$$f_{m,n}(z_1,z_2) = z_1^{-m-1/2}z_2^{-n-1/2}\langle\psi(z_1)\psi(z_2)\partial\sigma(0)\cdots\rangle$$

for $m,n\in\mathbb{Z}$, where $\cdots$ represents fields at other points ζ_i with $|\zeta_i|>|z_1|$ and $|\zeta_i|>|z_2|$ and where the correlation function is evaluated in the NS sector. This function is single-valued in a region around $z_1=0$ and $z_2=0$. Then the statement of associativity of the OPE is the statement:

$$\mathrm{Res}_{z_1=0}\mathrm{Res}_{z_2=0}f_{m,n}(z_1,z_2) \;-\; \mathrm{Res}_{z_2=0}\mathrm{Res}_{z_1=0}f_{m,n}(z_1,z_2) = \qquad \mathrm{Res}_{z_2=0}\mathrm{Res}_{z_1=z_2}f_{m,n}(z_1,z_2)\;. \tag{B.4.2}$$

On the right hand side, we have

$$\mathrm{Res}_{z_2=0}\mathrm{Res}_{z_1=z_2}f_{m,n}(z_1,z_2) = \delta_{m+n,0}\langle\partial\sigma(0)\cdots\rangle$$

In an expansion where $|z_2|<|z_1|$, we have

$$\begin{aligned}
\langle\psi(z_1)\psi(z_2)\partial\sigma(0)\cdots\rangle = \sum_{m\geq 0}\sum_{n\geq 0}\sum_{j=0}^{n}\frac{1}{2}c_m c_n \begin{pmatrix} n \\ j \end{pmatrix} \frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma\left(m-j+\frac{1}{2}\right)}(-1)^j \;\times \\
\times\,\Bigg[\left(\frac{1}{2}-n\right) z_1^{m-j-1/2} z_2^{n-3/2}\langle\partial^{m+n-j}\sigma(0)\cdots\rangle \;+ \\
+\;\left(\frac{1}{2}+j-m\right) z_1^{m-j-3/2} z_2^{n-1/2}\langle\partial^{m+n-j}\sigma(0)\cdots\rangle \;+ \\
+\; z_1^{-m-1/2} z_2^{-n-1/2}\langle\partial^{m+n-j-1}\sigma(0)\cdots\rangle\Bigg]\;.
\end{aligned}$$

This gives

$$\begin{aligned}
&\mathrm{Res}_{z_1=0}\mathrm{Res}_{z_2=0}f_{m,n}(z_1,z_2) = \\
&\frac{1}{2}\sum_{j\in\mathbb{Z}}(-1)^j\Bigg[c_{n+1}c_{m+j}\begin{pmatrix} n+1 \\ j \end{pmatrix}\left(m+\frac{1}{2}\right)_j\left(-\frac{1}{2}-n\right) \;+ \\
&\quad +\; c_n c_{m+j+1}\begin{pmatrix} n \\ j \end{pmatrix}\left(m+\frac{3}{2}\right)_j\left(-\frac{1}{2}-m\right) \;+ \\
&\quad +\; c_n c_{m+j}\begin{pmatrix} n \\ j \end{pmatrix}\left(m+\frac{1}{2}\right)_j\Bigg]\;\langle\partial^{m+n+1}\sigma\cdots\rangle
\end{aligned} \tag{B.4.3}$$

which is valid for all $m,n\in\mathbb{Z}$ if we assume $c_n=0$ for $n<0$ (the sum over j is the sum of a finite number of terms because of the binomial expansion coefficients). Also, we have

$$\mathrm{Res}_{z_2=0}\mathrm{Res}_{z_1=0}f_{m,n}(z_1,z_2) = -\mathrm{Res}_{z_1=0}\mathrm{Res}_{z_2=0}f_{n,m}(z_1,z_2)\;.$$

Now let's take $m = -n-1$ for $n \geq 1$. Only the first term of (B.4.2) contributes, and only one value of j in the sum over j in (B.4.4) contributes. This gives

$$c_{n+1} = \frac{c_n}{\frac{1}{2}+n}$$

for $n \geq 1$, which is solved to

$$c_n = \frac{2}{\left(\frac{1}{2}\right)_n} \quad (n \geq 1)$$

using the fact that $c_1 = 4$ (which can be found by considering the case $m = -1, n = 0$). The formula is not valid for $n = 0$, in which case $c_0 = 1$.

B.5 Proof of formulas (5.6.14) and (5.6.20)

We see that $[A_+]_{ij} = [A_-]_{ij}$ when k_i or k'_j is odd, and that $[A_+]_{ij} = -[A_-]_{ij}$ when k_i and k'_j are even. Also, when k_i and k'_j are even, the matrix elements factorize. Arrange the order of the k_i's and k'_j's so that all even ones are at the beginning: k_i is even if and only if $i < I$ and k'_j is even if and only if $j < J$. Then the matrices A_+ and A_- have the following form:

$$A_+ = M + S, \quad A_- = M - S \tag{B.5.1}$$

where

$$[S]_{ij} = s_i s'_j, \quad s_i = 0 \text{ if } i \geq I, \quad s'_j = 0 \text{ if } j \geq J, \quad [M]_{ij} = 0 \text{ if } i < I \text{ and } j < J.$$

Using the technique of minors to calculate determinants, the determinant of A_+, for instance, can always be written as a sum $\sum_i a_i$. In this sum, each term a_i can be factorized as $a_i = b_i \det(B_i)$, where $\det(B_i)$ is the determinant of a submatrix B_i of A_+ that has the same horizontal dimension as that of S, and that contains a certain number (if any) of full lines of S. When written in such a way, in each term $b_i \det(B_i)$, the only factor where matrix elements $[S]_{jk}$ of S enter is in the determinant $\det(B_i)$. A similar expression can be written for $\det(A_-)$, with the sub-matrix S replaced by the sub-matrix $-S$. If more than one line of S is contained in B_i, then $\det(B_i) = 0$ because the elements of S factorize. If only one line of S is contained in B_i, then the same term will appear in both the expressions for $\det(A_+)$ and for $\det(A_-)$ but with opposite signs. If no line of S is contained in B_i, then the same term will appear in both the expressions for $\det(A_+)$ and for $\det(A_-)$ (with the same sign).

From these properties, in the sum of the expressions for the determinants of A_+ and A_-, the only terms remaining are those containing no elements of S as factors. Hence, $\det(A_+) + \det(A_-) = 2\det(M)$. This proves Equation (5.6.14). On the other hand, in the difference of the expressions for the determinants of A_+ and A_-, the only terms remaining are the terms which are linear in elements of S. This prescription can be implemented by multiplying the elements of the sub-matrix S of A_+ by the inverse of a formal variable, w^{-1}, thus forming a matrix which we denote $A_+(w)$, and by taking the formal residue of the determinant of $A_+(w)$. This proves Equation (5.6.20).

Appendix C
Vertex Operator Algebras

C.1 Introduction

The theory of vertex operator algebras (see the books [54] and [87] for pedagogical developments from different perspectives and for historical accounts and many references on the original works) is a mathematical theory that was developed initially in the context of infinite-dimensional Lie algebras. In this context, it gave rise to the construction of modules for affine Lie algebras and for central extensions of algebras of formal differential operators (like the Virasoro algebra and its generalizations). The theory also provided the "moonshine module," a module for the Monster group, the largest finite simple sporadic group: the Monster was realized as the automorphism group of an "algebra of vertex operators."

The theory of vertex operator algebras gives a mathematically rigorous framework for two-dimensional conformal field theory. A vertex operator algebra is the axiomatization of the operator formalism of radial quantization (or any quantization on a simple closed line) for the sector containing the holomorphic currents in models of conformal field theory on a Riemann surface of genus 0. Holomorphicity and the field-state correspondence of conformal field theory are at the basis of vertex operator algebra theory, and locality and the associativity of the operator product algebra give its most important axioms.

In our work with my collaborators J. Lepowsky and A. Milas, announced in [40] and including all proofs in [41], we were interested in the algebra of holomorphic currents in general, and more precisely in models of conformal field theory with $W_{1+\infty}$ symmetry currents.

In the next subsections, I will explain the general principles of vertex operator algebras from the properties of local holomorphic currents in conformal field theory, and I will describe the main results and ideas of our work [41].

C.2 Vertex operator algebras

A vertex operator algebra is a vector space V endowed with a countably infinite set of maps $V \to \mathrm{End}(V) \; : \; v \mapsto v_n, \; n \in \mathbb{Z}$ for each element v of V. The space V is the space of local holomorphic currents, and the infinite set of maps associated to a given element $v \in V$ are the radial modes of the current associated to v. The fact that these modes are endomorphisms of the vector space of local currents is possible thanks to the field-state correspondence. The action of the endomorphism v_n on an element $w \in V$ can be seen as a product labelled by the integer n between the elements v and w. From this point of view, a vertex operator algebra comprises an infinity of algebra products among vectors in a vector space.

The infinite set of maps v_n can be put into a generating function, called a *vertex operator*, with a formal variable x. More precisely, a vertex operator $Y(v,x) \in \mathrm{End}(V)[[x,x^{-1}]]$ is a formal series in the formal variable x of the following form:

$$Y(v,x) = \sum_{n\in\mathbf{Z}} v_n x^{-n-1} \; .$$

The vertex operator $Y(v,x)$ is the radially-evolved operator in radial quantization representing the holomorphic field associated to the vector v, and the variable x plays the role of the holomorphic variable. Mathematically, the x^{-n-1}'s are just "placeholders," endowed with a product operation according to which the powers are simply added.

In a vertex operator algebra, there are two distinguished vectors. One is simply the vacuum vector of radial quantization, or identity field, usually denoted by $\mathbf{1}$

(we will also use below the notation $|\text{vac}\rangle$ for the vacuum vector). The other is the conformal vector, usually denoted by ω, which is such that its modes ω_n generate a representation of the Virasoro algebra: $\omega_{n+1} = L(n)$ with

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12}\delta_{m+n,0}\, c\ . \tag{C.2.1}$$

The vertex operator associated to it, $Y(\omega, x)$, is just the holomorphic part of the stress-energy tensor in conformal field theory. In particular, the vector space V is graded by the eigenvalues of $L(0)$, called *weights*. Weights are integers on V, since it is the space of holomorphic currents.

The endomorphisms v_n's associated to an element v can be fully determined by the operator product expansions in a given model of conformal field theory. Indeed, the operator product expansion of the product between a field associated to v and a field associated to w is simply, if we put the field associated to w at the origin for convenience (that is, the infinite past of radial quantization) and use the field-state correspondence,

$$Y(v,x)w = \sum_{n\in\mathbf{Z}} v_n w\ x^{-n-1}\ .$$

The operator product expansion is constrained by the locality of vertex operators and by the associativity of the algebra that it forms. Mutual locality between two holomorphic fields $\mathcal{O}_1(z_1)$ and $\mathcal{O}_2(z_2)$ says that correlation functions comprising these two fields $\langle\cdots\mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\cdots\rangle$ are analytical functions of z_1 with possible singularities at z_2 that can only be poles (of any finite order); then, bringing z_1 around z_2 gives the same function.

In radial quantization, correlation functions are vacuum expectation values of radially ordered products of operators. Then, a vacuum expectation value of the form $\langle\text{vac}|\cdots Y(v_1,x_1)Y(v_2,x_2)\cdots|\text{vac}\rangle$, when the formal variables x_1 and x_2 are specialized to complex numbers z_1 and z_2, respectively, in the region $|z_1| > |z_2|$, is an analytical function of z_1 with possible poles at z_2. The same function in the complementary region $|z_1| < |z_2|$ is obtained with a different radial ordering: $\langle \text{vac}| \cdots Y(v_2, x_2) Y(v_1, x_1) \cdots |\text{vac}\rangle$. On the other hand, the vacuum expectation value $\langle \text{vac}| \cdots Y(v_1, x_1) Y(v_2, x_2) \cdots |\text{vac}\rangle$ is a formal series in x_1 and x_2. Since it can be specialized to a formal series in complex numbers z_1, z_2 in the region $|z_1| > |z_2|$, it is a Taylor expansion in this region. Similarly, $\langle \text{vac}| \cdots Y(v_2, x_2) Y(v_1, x_1) \cdots |\text{vac}\rangle$ becomes an expansion in the complementary region $|z_1| < |z_2|$. Hence, generically, the vacuum expectation values $\langle \text{vac}| \cdots Y(v_1, x_1) Y(v_2, x_2) \cdots |\text{vac}\rangle$ and $\langle \text{vac}| \cdots Y(v_2, x_2) Y(v_1, x_1) \cdots |\text{vac}\rangle$ are *different* formal series, the difference coming from the different expansions of poles of the form $(z_1 - z_2)^{-m}$ for $m > 0$. Since these poles have a maximum order m, that depends on the vectors v_1 and v_2, then there exists a finite integer k such that we have, as a formal series relation,

$$(x_1 - x_2)^k \, [Y(v_1, x_1), Y(v_2, x_2)] = 0 \; . \tag{C.2.2}$$

This relation is called *weak commutativity* of vertex operators.

Now consider associativity of the operator product expansion. This says that we can evaluate correlation functions involving the product of three fields $\mathcal{O}_1(z_0 + z_2)\mathcal{O}_2(z_2)\mathcal{O}_3(0)$ by using the operator product expansion first of $\mathcal{O}_2(z_2)\mathcal{O}_3(0)$ then of $\mathcal{O}_1(z_0 + z_2)$ multiplied with the resulting fields at 0, or using the operator product expansion first of $\mathcal{O}_1(z_0 + z_2)\mathcal{O}_2(z_2)$ then of the resulting fields at z_2 multiplied with $\mathcal{O}_3(0)$. These two ways of evaluating the correlation functions correspond to expansions of the same function in different (overlapping) regions. In the first case, it is an expansion in the variables $z_0 + z_2$ and z_2 in the region $|z_0 + z_2| > |z_2|$, whereas in the second case, it is an expansion in z_0 and z_2 in the region $|z_2| > |z_0|$. In the first case, one can also rewrite the expansion as an expansion in z_0 and z_2 in the region $|z_2| < |z_0|$. Hence it is more natural to see both ways of calculating as two expansions in different regions of the same function in the variables z_0 and z_2. The resulting analytical function of z_0 and z_2 has possible poles at $z_0 + z_2 = 0$, corresponding to the singularities in the operator product expansion of $\mathcal{O}_1(z_0+z_2)\mathcal{O}_3(0)$.

In terms of vacuum expectation values of radially ordered product of operators, the first way is represented by $Y(v_1, x_0+x_2)Y(v_2, x_2)v_3$, whilst the second way is represented by $Y(Y(v_1, x_0)v_2, x_2)v_3$, with the specialization $x_0 \mapsto z_0$ and $x_2 \mapsto z_2$ in vacuum expectation values. The first expression is easy to understand, and the binomial expression x_0+x_2 should be expanded in nonnegative powers of x_2. The second expression contains an *iterate* of vertex operators. The expression $Y(v_1, x_0)v_2$ inside the vertex operator corresponds to the operator product expansion of $\mathcal{O}_1(z_0+z_2)\mathcal{O}_2(z_2)$, where vectors correspond to fields at z_2. From each vector in the formal series $Y(v_1, x_0)v_2$ we then take the associated vertex operator at x_2, that is, $Y(Y(v_1, x_0)v_2, x_2)$, and calculate the operator product expansion with the field at 0 corresponding to v_3. The difference between the two formal series $Y(v_1, x_0+x_2)Y(v_2, x_2)v_3$ and $Y(Y(v_1, x_0)v_2, x_2)v_3$ comes from different expansions of the poles $(z_0+z_2)^{-m}$ for $m>0$ in the corresponding analytical function. Since these poles have a maximum order m, there exists an integer number l, depending on v_1 and v_3, such that

$$(x_0+x_2)^l\, Y(v_1, x_0+x_2)Y(v_2, x_2)v_3 = (x_0+x_2)^l Y(Y(v_1, x_0)v_2, x_2)v_3\ . \quad \text{(C.2.3)}$$

This relation is called *weak associativity* of vertex operators.

It can be shown that weak commutativity (C.2.2) and weak associativity (C.2.3), along with natural quasi-finiteness requirements on the vector space V, are equivalent to the following single formal equation, called the *Jacobi identity* (because it plays the same role in vertex operator algebras as the Jacobi identity in Lie algebras):

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y(v, x_2)Y(v, x_1) \\ & = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(Y(u, x_0)v, x_2)\ . \end{aligned} \quad \text{(C.2.4)}$$

In this formal equation, the formal delta function $\delta(x)$ is defined as

$$\delta(x) = \sum_{n \in \mathbb{Z}} x^n \ . \tag{C.2.5}$$

It has the obvious property that

$$\delta(x_1/x_2) f(x_1) = \delta(x_1/x_2) f(x_2) \tag{C.2.6}$$

for any formal series $f(x)$. Also, in order to interpret the Jacobi identity correctly, one must use the "binomial expansion" convention: any binomial expression of the type $x_1 - x_2$ raised to a negative power $(x_1 - x_2)^{-n}$ has to be expanded in nonnegative powers of the second variable (x_2 in this example). These rules and other results (see for instance [87]) are at the basis of formal calculus, the "calculus" of doubly-infinite formal series, which is the language of vertex operator algebras.

It is a subtle matter to verify that each term of the Jacobi identity is a well-defined formal series in x_0, x_1 and x_2. But once this is understood, many identities can be obtained from the Jacobi identity, and many results can be found simply and rigorously, without having to appeal to the analytical properties of correlation functions, which are embedded in this algebraic framework. For instance, one can easily prove the equivalence between weak commutativity and weak associativity, under simple natural axioms (see [87]).

The Jacobi identity and quasi-finiteness requirements on V form the axioms defining a vertex operator algebra. More precisely, these axioms are as follows:

A **vertex operator algebra** $(V, Y, \mathbf{1}, \omega)$, or V for short, is a $\mathbb{Z}$–graded vector space

$$V = \coprod_{n \in \mathbb{Z}} V_{(n)}; \ \text{for } v \in V_{(n)}, \ \text{wt } v = n,$$

such that

$$V_{(n)} = 0 \quad \text{for } \ n \text{ sufficiently negative,}$$

$$\dim V_{(n)} < \infty \quad \text{for } \ n \in \mathbb{Z},$$

equipped with a linear map $Y(\cdot, x)$:

$$\begin{aligned} Y(\cdot, x) : V &\to (\mathrm{End}\ V)[[x, x^{-1}]] \\ v &\mapsto Y(v, x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1}\,, \quad v_n \in \mathrm{End}\ V, \end{aligned}$$

where $Y(v, x)$ is called the *vertex operator* associated with v, and two particular vectors, $\mathbf{1}$, $\omega \in V$, called respectively the *vacuum vector* and the *conformal vector*, with the following properties:

truncation condition: For every $v, w \in V$

$$v_n w = 0$$

for $n \in \mathbf{Z}$ sufficiently large;

vacuum property:

$$Y(\mathbf{1}, x) = 1_V \qquad (1_V \text{ is the identity on } V);$$

creation property:

$$Y(v, x)\mathbf{1} \in V[[x]] \quad \text{ and } \quad \lim_{x\to 0} Y(v, x)\mathbf{1} = v\ ;$$

Virasoro algebra conditions: Let

$$L(n) = \omega_{n+1} \ \text{ for } n \in \mathbb{Z}, \ \text{ i.e., } \ Y(\omega, x) = \sum_{n\in\mathbb{Z}} L(n) x^{-n-2}\ .$$

Then

$$[L(m), L(n)] = (m-n)L(m+n) + c_V \frac{m^3 - m}{12}\, \delta_{n+m,0}\, 1_V$$

for $m, n \in \mathbb{Z}$, where $c_V \in \mathbb{C}$ is the central charge (also called "rank" of V),

$$L(0)v = (\mathrm{wt}\ v)v$$

for every homogeneous element v, and we have the $L(-1)$*–derivative property:*

$$Y(L(-1)u, x) = \frac{d}{dx} Y(u, x)\ ;$$

Jacobi identity:

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u,x_1)Y(v,x_2)-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y(v,x_2)Y(u,x_1)$$
$$=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(Y(u,x_0)v,x_2)\ .$$

C.3 Modules and twisted modules

A module for a vertex operator algebra is a vector space on which elements of a vertex operator algebra act in such a way that appropriately adjusted weak commutativity and weak associativity relations are satisfied. That is, fields represented by vectors in V should still be local and their operator product expansions should still be associative, but the vector space on which they act is not V. Vertex operators acting on a module W are denoted by $Y_W(v,x)$ for $v\in V$, and they satisfy the Jacobi identity

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_W(u,x_1)Y_W(v,x_2)-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y_W(v,x_2)Y_W(v,x_1)$$
$$=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y_W(Y(u,x_0)v,x_2)\ . \tag{C.3.1}$$

In general, in a model of conformal field theory, one is given a set of local holomorphic fields with integer weights, the local currents from the symmetries of the model, forming a vertex operator algebra, and a set of primary fields, each generating a module for the Virasoro algebra and for the algebras associated to other symmetries. Each primary field generate also a corresponding module for the vertex operator algebra of local currents. For instance, in minimal models, the various modules composing a given model correspond to the various primary fields (which have rational weights), and the holomorphic fields forming a vertex operator algebra are the identity field and its descendants, the stress-energy tensor and its derivatives. This vertex operator algebra is one of the simplest and is called the Virasoro vertex operator algebra. The restrictions on the allowed modules forming a model of conformal field theory is set by the requirements of modular invariance (which is essentially invariance under the choice of a "time" direction on the torus) and unitarity (although some applications do not require unitarity).

The primary fields also correspond to vertex operators: the so-called intertwining operators, which map different modules among themselves. The theory of intertwining vertex operators is involved and very interesting, but out of the scope of this introduction.

The notion which is at the heart of my work with my collaborators is that of *twisted* modules. A twisted module for a vertex operator algebra can be defined when there exists an automorphism of finite period, say p, of the vertex operator algebra. That is, there must exist a linear automorphism ν of the vector space V preserving the identity vector $\mathbf{1}$ and the conformal vector ω such that

$$\nu Y(u,x)\nu^{-1} = Y(\nu u, x)$$

and $\nu^p = 1$. In conformal field theory language, the twisting procedure corresponds to imposing quasi-periodic conditions, twisted by the automorphism, around the origin of radial quantization. For a twisted module M, the twisted vertex operators acting on it are denoted by $Y_M(v,x)$ for $v \in V$. The twisting procedure then essentially requires

$$Y_M(\nu u, x_1)\big|_{x_1^{1/p} = \omega_p x_2^{1/p}} = Y_M(u, x_2)$$

where ω_p is a primitive p-th root of unity. Twisted vertex operators satisfy yet another Jacobi identity:

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_M(u,x_1)Y_M(v,x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y_M(v,x_2)Y_M(u,x_1) \\ = \frac{1}{p}x_2^{-1}\sum_{r=0}^{p-1}\delta\left(\omega_p^r\left(\frac{x_1 - x_0}{x_2}\right)^{1/p}\right) Y_M(Y(\nu^r u, x_0)v, x_2)\ . \qquad \text{(C.3.2)} \end{aligned}$$

A twisted module for a vertex operator algebra is not a true module, as is apparent from the fact that the form of the Jacobi identity above is different from that of the Jacobi identity (C.2.4). But it is a module for the subalgebra of the vertex operator algebra that is invariant under the automorphism ν, that is, the subalgebra $\{u \in V \mid \nu u = u\}$. For instance, since the conformal vector ω is invariant, its modes still satisfy the Virasoro algebra on M. In particular, the eigenvalue of the mode $L(0)$ on the highest weight vector of M is related to the effect of the twisted quasi-periodic conditions on the Casimir energy.

C.4 Relations to infinite-dimensional Lie algebras of differential operators

In our work with my collaborators J. Lepowsky and A. Milas [40, 41], we were interested in studying representations of a central extension of a certain infinite-dimensional Lie algebra of formal differential operators on the circle. This Lie algebra, which we denote $\hat{\mathcal{D}}^+$, is a natural subalgebra of the so-called $\mathcal{W}_{1+\infty}$ algebra. It is a simple extension of the Virasoro algebra and contains it as a subalgebra. More precisely, it is the central extension of the algebra of formal differential operators spanned by $t^n \left(t\,\frac{d}{dt}\right)^r$, where $n \in \mathbb{Z}$, $r \in \mathbb{N}$ (the nonnegative integers). In the same way as the Virasoro algebra is the algebra of the modes of a holomorphic current, the stress-energy tensor, the algebra $\hat{\mathcal{D}}^+$ is also spanned by the modes of holomorphic currents, hence it can be embedded into a vertex operator algebra. If this vertex operator algebra possesses an automorphism of finite order, then we can form a twisted module for it, call it M. In the module M that we construct, the current algebra generating $\hat{\mathcal{D}}^+$ is invariant under this automorphism. Then M is a module for $\hat{\mathcal{D}}^+$. From the general study of vertex operator algebras and their twisted modules, we derived formulas for the eigenvalue of the elements of the "Cartan subalgebra" (a natural infinite-dimensional abelian subalgebra) of $\hat{\mathcal{D}}^+$ on the highest weight vector of M. This is the generalization of the Casimir energy due to twisted quasi-periodic conditions.

Another, related, purpose of our work was to understand the effects of a transformation to cylindrical coordinates on this "generalized" Casimir energy. A transformation to cylindrical coordinates, from the point of view of vertex operator algebras, is a particular isomorphism between two vertex operator algebras. In other words, on a vector space V, one can define two vertex operator maps, say $Y(\cdot,x)$ and $Y[\cdot,y]$, that generate different algebra products among the elements of V, in such a way that there is an automorphism of the vector space V under which the algebra products generated by $Y(\cdot,x)$ are mapped to those generated by $Y[\cdot,y]$. For instance, it was shown by Zhu [145], [146] that under a transformation to cylindrical coordinates, the algebra products are generated by the vertex operator map $Y[\cdot,y]$ given by the following formula:

$$Y[u,y] = Y(e^{yL(0)}u, e^y - 1) \ .$$

As is well known, a transformation to cylindrical coordinates of the stress-energy tensor produces a term proportional to the identity operator, giving a contribution to the Casimir energy, that can be computed using the Schwarzian derivative. A heuristic way of obtaining the same term is by the so-called zeta regularization. One represents the stress-energy tensor by a normal-ordered product of free fields $h(x)$, the modes of which generate an Heisenberg algebra:

$$h(x) = \sum_{n\in\mathbb{Z}} h(n)\, x^{-n}$$

with

$$[h(m), h(n)] = m\, \delta_{m+n,0} \ .$$

The stress-energy tensor, generating function of Virasoro modes $L(n)$, can be written

$$T(x) = \frac{1}{2} {:}h(x)h(x){:} = \sum_{n\in\mathbb{Z}} L(n)x^{-n} \tag{C.4.1}$$

with the usual normal ordering that brings the $h(n)$'s with positive n to the right. Note that $T(x)$ is not exactly the vertex operator $Y(\omega,x) = \sum_{n\in\mathbb{Z}} L(n)\, x^{-n-2}$, but rather an "homogeneous" version of it, where the powers of x are shifted by

2. From the expression for $T(x)$, if one "erases" the normal-ordering operation one obtains an ill-defined operator $\bar{T}(x)$, that can be rewritten as

$$\bar{T}(x) = \frac{1}{2} {:}h(x)h(x){:} + \frac{1}{2} \sum_{n \geq 1} n \ .$$

By the zeta regularization one can then replace $\sum_{n\geq 1} n$ by $\zeta(-1) = -\frac{1}{12}$, finding that $\bar{T}(x)$ is the energy momentum tensor on the cylinder. The new modes $\bar{L}(n)$, defined by

$$\bar{T}(x) = \sum_{n \in \mathbb{Z}} \bar{L}(n) x^{-n} \ ,$$

form a basis for the Virasoro algebra with bracket relations where the central term is slightly simpler than that of (C.2.1), being a pure monomial:

$$[\bar{L}(m), \bar{L}(n)] = (m-n)\bar{L}(m+n) + \frac{m^3}{12}\delta_{m+n,0} \ .$$

As was found by Bloch [16], a construction of the algebra $\hat{\mathcal{D}}^+$ in terms of holomorphic currents similar to (C.4.1), and an "erasing" of the normal-ordering operation followed by a zeta-regularization, give a basis for $\hat{\mathcal{D}}^+$ with bracket relations where the central term is also very simple. From general results in the theory of vertex operator algebras, we explained the relation between this heuristic zeta-regularization and the well-defined transformation to cylindrical coordinates, and the fact that this transformation to cylindrical coordinates simplifies the central term in bracket relations. This vertex-operator explanation was announced by J. Lepowsky in [85] and in [86], and in our work [41] we give the full proofs and further clarifications. In fact, the transformation to cylindrical coordinates gave us a simple way of deriving the "generalized" Casimir energy in the twisted setting as well.

References

[1] C. Acerbi, G. Mussardo and A. Valleriani, Form-factors and correlation functions of the stress - energy tensor in massive deformation of the minimal models $E^{(1)}_{(N)} \times E^{(1)}_{(N)}/E^{(2)}_{(N)}$. *Int. J. Mod. Phys.* **A11** (1996) 5327-5364.

[2] D. B. Ali and J. A. Gracey, Four loop wave function renormalization in the non-abelian Thirring model. *Nucl. Phys.* **B605** (2001) 337-364.

[3] N. Andrei and J. H. Lowenstein, Diagonalization of the Chiral-Invariant Gross-Neveu Hamiltonian. *Phys. Rev. Lett.* **43** (1979) 1698-1701.

[4] N. Andrei and J. H. Lowenstein, A direct calculation of the S-matrix of the chiral invariant Gross-Neveu model. *Phys. Lett.* **B91** (1980) 401-405.

[5] I. Ya. Arefyeva and V. E. Korepin, Scattering in a two dimensional model having the Lagrangian $1/\gamma((1/2)(\partial_\mu u)^2/ + m^2(\cos u - 1))$ *Pis'ma Zh. Eksp. Teor. Fiz.* **20** (1974) 680-684.

[6] O. Babelon and D. Bernard, From form factors to correlation functions: the Ising model. *Phys. Lett.* **B288** (1992) 113-120.

[7] H. Babujian, A. Fring, M. Karowski and A. Zapletal, Exact form-factors in integrable quantum field theories: the sine-Gordon model. *Nucl. Phys.* **B538** (1999) 535-586.

[8] J. Balog and M. Niedermaier, Off-shell dynamics of the O(3) nonlinear sigma model beyond Monte Carlo and perturbation theory. *Nucl. Phys.* **B500** (1997) 421-461.

[9] A. A. Belavin, Exact solution of the two-dimensional model with asymptotic freedom. *Phys. Lett.* **B87** (1979) 117-121.

[10] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Infinite conformal symmetries in two-dimensional quantum field theory. *Nucl. Phys.* **B241** (1984) 333-380.

[11] J. F. Bennett and J. A. Gracey, Three-loop renormalization of the $SU(N_c)$ non-abelian Thirring model. *Nucl. Phys.* **B563** (1999) 390-436.

[12] D. Bernard and A. Leclair, Differential equations for sine-Gordon correlation functions at the free fermion point. *Nucl. Phys.* **B426** (1994) 534-558. Errata, *Nucl. Phys.* **B498** (1997) 619-622.

[13] B. Berg, M. Karowski and P. Weisz, Construction of Green's functions from an exact S-matrix. *Phys. Rev.* **D19** (1979) 2477-2479.

[14] I. P. Bindloss and S. A. Kivelson, Spectral functions of a Luttinger liquid coupled to phonons and ARPES measurements in the cuprate superconductors. Preprint `cond-mat/0402457`.

[15] N. B. Birrel and P. C. W. Davies, *Quantum Fields in Curved Space*, Cambridge University Press, Cambridge (1982).

[16] S. Bloch, Zeta values and differential operators on the circle, *J. Algebra* **182** (1996), 476–500.

[17] C. Bourbonnais and D. Jerome, The normal phase of quasi-one-dimensional organic superconductors. In *Advances in Synthetic Metals, Twenty years of Progress in Science and Technology*, ed. by P. Bernier , S. Lefrant and E. Bidan, Elsevier, New York (1999).

[18] V. Brazhnikov and S. Lukyanov, Angular quantization and form-factors in massive integrable models. *Nucl. Phys.* **B512** (1998) 616-636, preprint `hep-th/9707091`.

[19] P. Breitenlohner, D. Z. Freedman, Positive energy in anti-de Sitter backgrounds and gauged extended supergravity. *Phys. Lett.* **B115** (1982) 197-201.

[20] C. Callan and F. Wilczek, Infrared behavior at negative curvature. *Nucl. Phys.* **B340** (1990) 366-386.

[21] J. L. Cardy, Conformal invariance and statistical mechanics. In *Fields, Strings and Critical Phenomena*, E. Brèzin and J. Zinn-Justin, eds., *Les Houches 1988, Session XLIX*, North-Holland, Amsterdam (1990) 169-246.

[22] J. Cardy, Conformal invariance and surface critical behaviour. *Nucl. Phys.* **B240** (1984) 514-532.

[23] J. L. Cardy, Operator content of two-dimensional conformally invariant theories. *Nucl. Phys.* **B270** (1986) 186-204.

[24] J. Cardy, *Scaling and renormalization in statistical physics*, Cambridge University Press (1996).

[25] J. Cardy and G. Mussardo, S-matrix of the Yang-Lee edge singularity in two dimensions. *Phys. Lett.* **B225** (1989) 275-278

[26] J. Cardy and G. Mussardo, Form factors of descendent operators in perturbed conformal field theories. *Nucl. Phys.* **B340** (1990) 387-402.

[27] S. Coleman, The quantum sine-Gordon equation as the massive Thirring model. *Phys. Rev.* **D11** (1975) 2088-2097.

[28] S. Coleman and J. Mandula, All possible symmetries of the S matrix. *Phys. Rev.* **159** (1967) 1251-1256.

[29] D. Controzzi, F.H.L. Essler and A.M. Tsvelik, Optical conductivity of one-dimensional Mott insulators. *Phys. Rev. Lett* **86** (2001) 680-683, preprint `cond-mat/0005349`.

[30] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, On the Lambert W function. *Adv. Comput. Math.* **5**, no.4 (1996) 329-359; D. J. Jeffrey, D. E. G. Hare and R. M. Corless, Unwinding the branches of the Lambert W function. *Math. Sci.* **21** (1996) 1-7.

[31] D. B. Creamer, H. B. Thacker and D. Wilkinson, Gelfand-Levitan method for operator fields. *Phys. Rev.* **D21** (1980) 1523-1528.

[32] R. F. Dashen, B. Hasslacher and A. Neveu, Particle spectrum in model field theories from semiclassical functional integral techniques. *Phys. Rev.* **D11** (1975) 34243450.

[33] G. Delfino, Off critical correlations in the Ashkin-Teller model. *Phys. Lett.* **B450** (1999) 196-201.

[34] P. Di Francesco, P. Mathieu and D. Senechal, *Conformal field theory*, Springer, New York (1997).

[35] P. Dorey, Exact S-matrices. Preprint `hep-th/9810026`.

[36] V. Dotsenko, M. Picco and P. Pujoi, Renormalization group calculation of correlation functions for the 2D random bound Ising and Potts models. *Nucl. Phys.* **B455** (1995) 701-723.

[37] B. Doyon, Two-point functions of scaling fields in the Dirac theory on the Poincaré disk. *Nucl. Phys.* **B675** (2003) 607-630, preprint `hep-th/0304190`.

[38] B. Doyon, Form factor of Ising spin and disorder fields on the Poincaré disk. *J. Phys.* **A37** (2003) 359-370, special edition, proceedings of the workshop *Recent Advances in the Theory of Integrable Systems* (Annecy, 2003).

[39] B. Doyon and P. Fonseca, Ising field theory on the pseudosphere. preprint `hep-th/0404136`.

[40] B. Doyon, J. Lepowsky and A. Milas, Twisted modules for vertex operator algebras and Bernoulli polynomials. *I.M.R.N.* **44** (2003) 2391-2408, preprint `math.QA/0303193`.

[41] B. Doyon, J. Lepowsky and A. Milas, Twisted vertex operators and Bernoulli polynomials. Submitted on 11 November 2003 to *Communications in Mathematical Physics*, 56 pages, preprint `math.QA/0311151`.

[42] B. Doyon and S. Lukyanov, Fermion Schwinger's functions for the SU(2)-Thirring model. *Nucl. Phys.* **B644** (2002) 451-475, preprint `hep-th/0203135`.

[43] R. J. Eden, P. V. Landshoff, D. I. Olive and J. C. Polkinghorne, *The Analytic S-Matrix*, Cambridge University Press, Cambridge (1966).

[44] V. J. Emery, A. Luther and I. Peschel, Solution of the one-dimensional electron gas on a lattice. *Phys. Rev.* **B13** (1976) 1272-1276.

[45] F. H. L. Essler and A. M. Tsevelik, Weakly coupled one-dimensional Mott insulator. *Phys. Rev.* **B65** (2002) 115117-115130, preprint `cond-mat/0108382`.

[46] L. D. Faddeev, Quantization of solitons. Proceedings, *18th International Conference on High Energy Physics, Tbilisi, USSR, 1976*, Joint Inst. Nucl. Research (1977).

[47] L. D. Faddeev, Quantum completely integrable models of field theory. *Sov. Sci. Rev. Math. Phys.* **1C** (1980) 107-155.

[48] V. Fateev, D. Fradkin, S. Lukyanov, A. B. Zamolodchikov and Al. B. Zamolodchikov, Expectation values of descendant fields in the sine-Gordon model. *Nucl. Phys.* **B540** (1999) 587-609.

[49] S. Ferrara, A. F. Gatto and R. Grillo, Conformal algebra in two space-time dimensions and the Thirring model. *Nuovo Cimento* **12A** (1972) 959-968.

[50] V. M. Filev, Spectrum of two-dimensional relativistic model. *Teor. Mat. Fiz.* **33** (1977) 119-124.

[51] P. Fonseca, A. Zamolodchikov, Ising Field Theory in a Magnetic Field: Analytic Properties of the Free Energy. *J. Stat. Phys.* **110** (2003) 527-590, preprint `hep-th/0112167`.

[52] P. Fonseca, A. Zamolodchikov, Ward identities and integrable differential equations in the Ising field theory. preprint `hep-th/0309228`.

[53] E. Fradkin and L. P. Kadanoff, Disorder variables and para-fermions in two-dimensional statistical mechanics. *Nucl. Phys.* **B170** (1980) 1-15.

[54] I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Appl. Math., Vol. 134, Academic Press, Boston (1988).

[55] D. Friedan, Z. Qiu, S. Shenker, Conformal invariance, unitarity, and critical exponents in two dimensions. *Phys. Rev. Lett.* **52** (1984) 1575-1578.

[56] D. Friedan and S. Shenker, Details of the non-unitarity proof for highest weight representations of the Virasoro algebra. *Commun.Math.Phys.* **107** (1986) 535-547.

[57] I. M. Gelfand and D. B. Fuks, Cohomologies of the Lie algebra of vector fields on the circle. *Funkcional. Anal.* **2** (1968) 92-93.

[58] A. O. Gogolin, A. A. Nersesyan and A. M. Tsvelik, *Bosonization and Strongly Correlated Systems*, Cambridge University Press (1999).

[59] G. Gruner, *Density Waves in Solids*, Addison-Wesley (1994).

[60] R. Guida and N. Magnoli, All order IR finite expansion for short distance behavior of massless theories perturbed by a relevant operator. *Nucl. Phys.* **B471** (1996) 361-388.

[61] R. Guida and N. Magnoli, Tricritical Ising model near criticality. *Int. J. Mod. Phys.* **A13** (1998) 1145-1158.

[62] J. Hubbard, Electron correlations in narrow energy bands. *Proc. Roy. Soc.* **A276** (1963) 237-257.

[63] Y. Hwu et al., Photoemission near the Fermi energy in one dimension. *Phys. Rev.* **B46** (1992) 13624-13626; F. Zwick et al., Absence of quasiparticles in the photoemission spectra of quasi-one dimensional Berchgaard salts. *Phys. Rev. Lett.* **79** (1997) 3982-3985.

[64] C. Itzykson, J.–M. Drouffe, *Statistical field theory*, Cambridge University Press (1989).

[65] C. Itzykson, H. Saleur and J.B. Zuber, eds, *Conformal Invariance and Applications to Statistical Mechanics*. Singapore, World Scientific (1988).

[66] M. Jimbo, Monodromy problem and the boundary condition for some Painlevé equations. *Publ. RIMS, Kyoto University*, **18** (1982) 1137-1161.

[67] M. Jimbo and T. Miwa, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II. *Physica* **2D** (1981) 407-448.

[68] M. Jimbo, T. Miwa and K. Ueno, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. General theory and τ-function. *Physica* **2D** (1981) 306-352.

[69] L. P. Kadanof and H. Ceva, Determination of an operator algebra for the two-dimensional Ising model. *Phys. Rev.* **B3** (1971) 3918-3939.

[70] M. Karowski, On the bound state problem in 1+1 dimensional field theories. *Nucl. Phys.* **B153** (1979) 244-252.

[71] M. Karowski and H.-J. Thun, Complete S-matrix of the $O(2N)$ Gross-Neveu model. *Nucl. Phys.* **B190** [FS3] (1981) 61-92.

[72] M. Karowski and P. Weisz, Exact form factors in (1+1)-dimensional field theoretic models with soliton behaviour. *Nucl. Phys.* **B139** (1978) 455-476.

[73] B. Kaufman, Crystal Statistics. II. Partition function evaluated by spinor analysis. *Phys. Rev.* **76** (1949) 1232-1243.

[74] S. Khoroshkin, A. Leclair and S. Pakuliak, Angular quantization of the sine-Gordon model at the free fermion point. *Adv. Theor. Math. Phys.* **3** (1999) 1227-1287, preprint `hep-th/9904082`.

[75] C. Kim et al., Observation of Spin-Charge Separation in One-Dimensional $SrCuO_2$. *Phys. Rev. Lett.* **77** (1996) 4054-4057; Separation of spin and charge excitations in one-dimensional $SrCuO_2$. *Phys. Rev.* **B56** (1998) 15589-15595; K. Kobayashi et al., Single-Particle Excitations in One-Dimensional Mott-Hubbard Insulator NaV_2O_5 *Phys. Rev. Lett.* **80** (1998) 3121-3124.

[76] A. N. Kirillov and F. A. Smirnov, A representation of the current algebra connected with the $SU(2)$-invariant Thirring model. *Phys. Lett.* **B198** (1987) 506-510.

[77] A. N. Kirillov and F. A. Smirnov, *Int. J. Mod. Phys.* **A3** (1988) 731.

[78] V. E. Korepin, Direct calculation of the S-matrix in the massive Thirring model. *Theor. Math. Phys.* **41** (1979) 953-967 [*Teor. Mat. Fiz.* **41** (1979) 169-189].

[79] V. E. Korepin, N. M. Bogoliubov and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions*, Cambridge University Press, 1993.

[80] V. E. Korepin and F. H. L. Essler, Scattering matrix and excitation spectrum of the Hubbard model. *Phys. Rev. Lett.* **72** (1994) 908-911; $SU(2) \otimes SU(2)$ invariant scattering matrix of the Hubbard model. *Nucl. Phys.* **B426** (1994) 505-533.

[81] V. E. Korepin and L. D. Faddeev, Quantization of solitons. *Theor. Math. Phys.* **25** (1975) 1039-1049.

[82] A. Koubek, The space of local operators in perturbed conformal field theories. *Nucl. Phys.* **B435** (1995) 703-734; A method to determine the operator content of perturbed conformal field theories. *Phys. Lett.* **B346** (1995) 275-283.

[83] P. P. Kulish, Factorization of the classical and quantum S matrix and conservation laws. *Theor. Math. Phys.* **26** (1976) 132 [*Teor. Mat. Fiz.* **26** (1976) 198-205].

[84] A. LeClair, Spectrum generating affine Lie algebras in massive field theory. *Nucl. Phys.* **B415** (1994) 734-780, preprint `hep-th/9305110`.

[85] J. Lepowsky, Vertex operator algebras and the zeta function. In: Jing, N., Misra, K.C.: *Recent Developments in Quantum Affine Algebras and Related Topics*, Contemporary Math., Vol. 248, Amer. Math. Soc., Providence (1999) 327-340.

[86] J. Lepowsky, Application of a 'Jacobi Identity' for vertex operator algebras to zeta values and differential operators. *Lett. Math. Phys.* **53** (2000) 87-103.

[87] Lepowsky, J., Li, H.: *Introduction to Vertex Operator Algebras and Their Representations*, Progress in Mathematics, Birkhäuser, Boston (2003).

[88] E. H. Lieb and F. Y. Wu, Absence of Mott Transition in an Exact Solution of the Short-Range, One-Band Model in One Dimension. *Phys. Rev. Lett.* **20** (1968) 1445-1448.

[89] S. Lukyanov, Free field representation for massive integrable models. *Commun. Math. Phys.* **167** (1995) 183-226, preprint `hep-th/9307196`.

[90] S. Lukyanov, Form-factors of exponential fields in the sine-Gordon model. *Mod. Phys. Lett.* **A12** (1997) 2543-2550.

[91] S. Lukuyanov and A. B. Zamolodchikov, Exact expectation values of local fields in quantum sine-Gordon model. *Nucl. Phys.* **B493** (1997) 571-587 `hep-th/9611238`.

[92] S. Lukyanov and A. Zamolodchikov, Form factors of soliton-creating operators in the sine-Gordon model. *Nucl. Phys.* **B607** [FS] (2001) 437-455.

[93] A. Luther, Eigenvalue spectrum of interacting massive fermions in one dimension. *Phys. Rev.* **B14** (1976) 2153-2159.

[94] A. Luther, Quantum solitons in one-dimensional conductors. *Phys. Rev.* **B15** (1977) 403-412.

[95] A. Luther, I. Peschel, Calculation of critical exponents in two dimensions from quantum field theory in one dimension, *Phys. Rev.* **B 12** (1975) 3908-3917.

[96] S. Mandelstam, Soliton operators for the quantized sine-Gordon equation. *Phys.Rev.* **D11** (1975) 3026-3030.

[97] S. D. Mathur, Quantum Kac-Moody symmetry in integrable field theories. *Nucl. Phys.* **B369** (1992) 433-460.

[98] B. M. McCoy and T. T. Wu, *The Two-Dimensional Ising Model*, Harvard University Press, Cambridge, Massachusetts (1973).

[99] E. Melzer, On the scaling limit of the 1-d Hubbard model at half filling. *Nucl. Phys.* **B443** [FS] (1995) 553-564.

[100] G. Mussardo, Off-critical statistical models: Factorized scattering theories and bootstrap program. *Phys. Rep.* **218** (1992) 215-379.

[101] E. Ogievetsky, N. Reshetikhin and P. Wiegmann, The principal chiral field in two dimensions on classical Lie algebras: The Bethe-ansatz solution and factorized theory of scattering. *Nucl. Phys.* **B280** [FS 18] (1987) 45-96.

[102] L. Onsager, Crystal Statistics. I. A two-dimensional model with an order-disorder transition. *Phys. Rev.* **65** (1944) 117-149.

[103] J. Palmer, Determinants of Cauchy-Riemann operators as τ-functions. *Acta Applicandae Mathematicae* **18** (1990) 199-223.

[104] J. Palmer, M. Beatty and C. A. Tracy, Tau functions for the Dirac operator on the Poincaré disk. *Commun. Math. Phys.* **165** (1994) 97-174, preprint `hep-th/9309017`.

[105] S. Parke, Absence of particle production and factorization of the S-matrix in 1+1 dimensional models. *Nucl. Phys.* **B174** (1980) 166-182.

[106] D. Pines and P. Nozières, *The Theory of Quantum Liquids. Normal Fermi Liquids*, W.A. Benjamin, New York (1966)

[107] A. M. Polyakov, Hidden symmetry of the two-dimensional chiral fields. *Phys. Lett.* **B72** (1977) 224-226

[108] A. M. Polyakov, *Zh. Eksp. Teor. Fiz.* **57** (1969) 271.

[109] A. M. Polyakov, Conformal symmetry of critical fluctuations. *Pis'ma Zh. Eksp. Teor. Fiz.* **12** (1970) 538-541 [*JETP Lett.* **12** (1970) 381-383]; Non-Hamiltonian approach to conformal quantum field thoery. *Zh. Eksp. Teor. Fiz.* **66** (1974) 23-42 [*Sov. Phys. JETP* **39** (1974) 10-18.]

[110] R. Rietman, B. Nienhuis, J. Oitmaa, The Ising model on hyperlattices. *J. Phys.* **A25** (1992) 6577-6592.

[111] M. Sato, M. Jimbo and T. Miwa, *Lecture notes in physics* vol. 126, Springer (1980) 429-491.

[112] M. Sato, T. Miwa and M. Jimbo, Holonomic quantum fields, *Publ. RIMS, Kyoto University.* **15** (1979) III: 577-629, IV: 871-972.

[113] B. Schroer and T. T. Truong, The order/disorder quantum field operators associated with the two-dimensional Ising model in the continuum limit. *Nucl. Phys.* **B144** (1978) 80-122.

[114] B. Schroer, T. T. Truong and P. H. Weisz, Towards an explicit construction of the sine-Gordon theory. *Phys. Lett.* **B63** (1976) 422-424.

[115] R. Shankar and E. Witten, S matrix of the supersymmetric nonlinear *sigma* model. *Phys. Rev.* **D17** (1978) 2134-2143.

[116] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, QCD and resonance physics. Theoretical foundations. *Nucl. Phys.* **B147** (1979) 385-447.

[117] E. K. Sklyanin, L. D. Faddeev and L. A. Takhtajan, The quantum inverse problem method. *Theor. Math. Phys.* **40** (1980) 688-706 [*Teor. Mat. Fiz.* **40** (1979) 194-220].

[118] F. A. Smirnov, *Form factors in completely integrable models of quantum field theory*, World Scientific, Singapore (1992).

[119] F. A. Smirnov, The quantum Gelfand-Levitan-Marchenko equations and form factors in the sine-Gordon model. *J. Phys.* **A17** (1984) L873-L878.

[120] F. A. Smirnov, A general formula for soliton form factors in the quantum sine-Gordon model. *J. Phys.* **A19** (1986) L575-L578.

[121] P. M. Stevenson, Optimized perturbation theory. *Phys. Rev.* **D23** (1981) 2916-2943.

[122] H. M. Srivastava and J. Choi *Series Associated with the Zeta and Related Functions.* Kluwer Academic Publishers, Dordrecht, Boston, London (2001).

[123] S. N. Vergeles and V. M. Gryanik, Two-dimensional quantum field theories having exact solutions. (In Russian). *Yad. Fiz.* **23** (1976) 1324-1334.

[124] M. A. Virasoro, Subsidiary conditions and ghosts in dual-resonance models. *Phys. Rev.* **D1** (1970) 2933-2936.

[125] J. Voit, One-dimensional Fermi Liquids. *Rep. Prog. Phys.* **57** (1994) 977-1116.

[126] R. M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*, The University of Chicago Press, Chicago, 1994.

[127] S. Weinberg, *The Quantum Theory of Fields*, Cambridge University Press, Cambridge (1995).

[128] K. G. Wilson, Non-lagrangian models of current algebras. *Phys. Rev.* **179** (1969) 1499-1512.

[129] F. Woynarovich, Excitations with complex wave numbers in a Hubbard chain. 1. States with one pair of complex wave numbers. *J. Phys.* **C15** (1982) 85-96; 2. States with several pairs of complex wave numbers. *J. Phys.* **C15** (1982) 97-109.

[130] F. Woynarovich and P. Forgacs, Scaling limit of the one-dimensional attractive Hubbard model: The half-filled band case. *Nucl. Phys.* **B498** (1997) 565-603.

[131] T. T. Wu, B. M. McCoy, *The two-dimensional Ising model*, Harvard University Press (1973).

[132] T. T. Wu, B. M. McCoy, C. A. Tracy and E. Barouch, Spin-spin correlation functions for the two-dimensional Ising model: exact theory in the scaling region. *Phys. Rev.* **B13** (1976) 316-374.

[133] V. P. Yurov and Al. B. Zamolodchikov, Correlation functions of integrable 2D models of the relativistic field theory; Ising model. *Int. Journ. Mod. Phys.* **A6** (1991) 3419-3440.

[134] A. B. Zamolodchikov, Integrable field theory from conformal field theory. *Advanced studies in pure math.* **19** (1989) 641-674.

[135] A. B. Zamolodchikov, Exact two-particle S-matrix of quantum sine-Gordon solitons. *Pisma Zh.Eksp.Teor.Fiz.* **25** (1977) 499-502; *Comm. Math. Phys.* **55** (1977) 183-186.

[136] A. B. Zamolodchikov, Integrals of motion and S-matrix of the (scaled) $T = T_c$ Ising model with a magnetic field. *Int. J. Mod. Phys.* **A4** (1989) 4235-4252.

[137] A. B. Zamolodchikov, Quantum sine-Gordon model. The soliton form factors. Preprint ITEP 45-1977 (1977) 12pp.

[138] A. B. Zamolodchikov and Al. B. Zamolodchikov, Exact S matrix of Gross-Neveu "elementary" fermions. *Phys. Lett.* **B72**, (1978) 481-483.

[139] A. B. Zamolodchikov and Al. B. Zamolodchikov, Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models, *Ann. Phys.* **120** (1979) 253-291.

[140] Al. B. Zamolodchikov, unpublished.

[141] Al. B. Zamolodchikov, unpublished.

[142] Al. B. Zamolodchikov, Mass scale in the sine-Gordon model and its reduction. *Int. J. Mod. Phys.* **A10** (1995) 1125-1150.

[143] Al. B. Zamolodchikov, Scaling Lee-Yang model on a sphere I. Partition function. *JHEP* 0207:029 (2002), preprint `hep-th/0109078`.

[144] Al.B. Zamolodchikov, Two-point correlation function in scaling Lee-Yang model. *Nucl. Phys.* **B348** (1991) 619-641.

[145] Y. Zhu, Vertex operators, Elliptic functions and modular forms, Ph.D. thesis, Yale University, 1990.

[146] Y. Zhu, Modular invariance of characters of vertex operator algebras, *J. Amer. Math. Soc.* **9** (1996) 237-307.

[147] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Clarendon Press, Oxford (1996) (third edition).

[148] J. B. Zuber and C. Itzykson, Quantum field theory and the two-dimensional Ising model. *Phys. Rev.* **D15** (1977) 2875-2884.

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2003	B. Doyon, Two-point functions of scaling fields in the Dirac theory on the Poincaré disk. *Nucl. Phys.* **B675** (2003) 607-630, preprint `hep-th/0304190`.

2003 B. Doyon, Form factor of Ising spin and disorder fields on the Poincaré disk. *J. Phys.* **A37** (2003) 359-370, special edition, proceedings of the workshop *Recent Advances in the Theory of Integrable Systems* (Annecy, 2003).

2003 B. Doyon, J. Lepowsky and A. Milas, Twisted vertex operators and Bernoulli polynomials. Submitted on 11 November 2003 to *Communications in Mathematical Physics*, 56 pages, preprint `math.QA/0311151`.

2004 B. Doyon and P. Fonseca, Ising field theory on a pseudosphere. Submitted on 8 June 2004 to *Journal of Statistical Mechanics: Theory and Experiment*, 39 pages, preprint `hep-th/0404136`.