Conformal restriction, conformal geometry and vertex algebras

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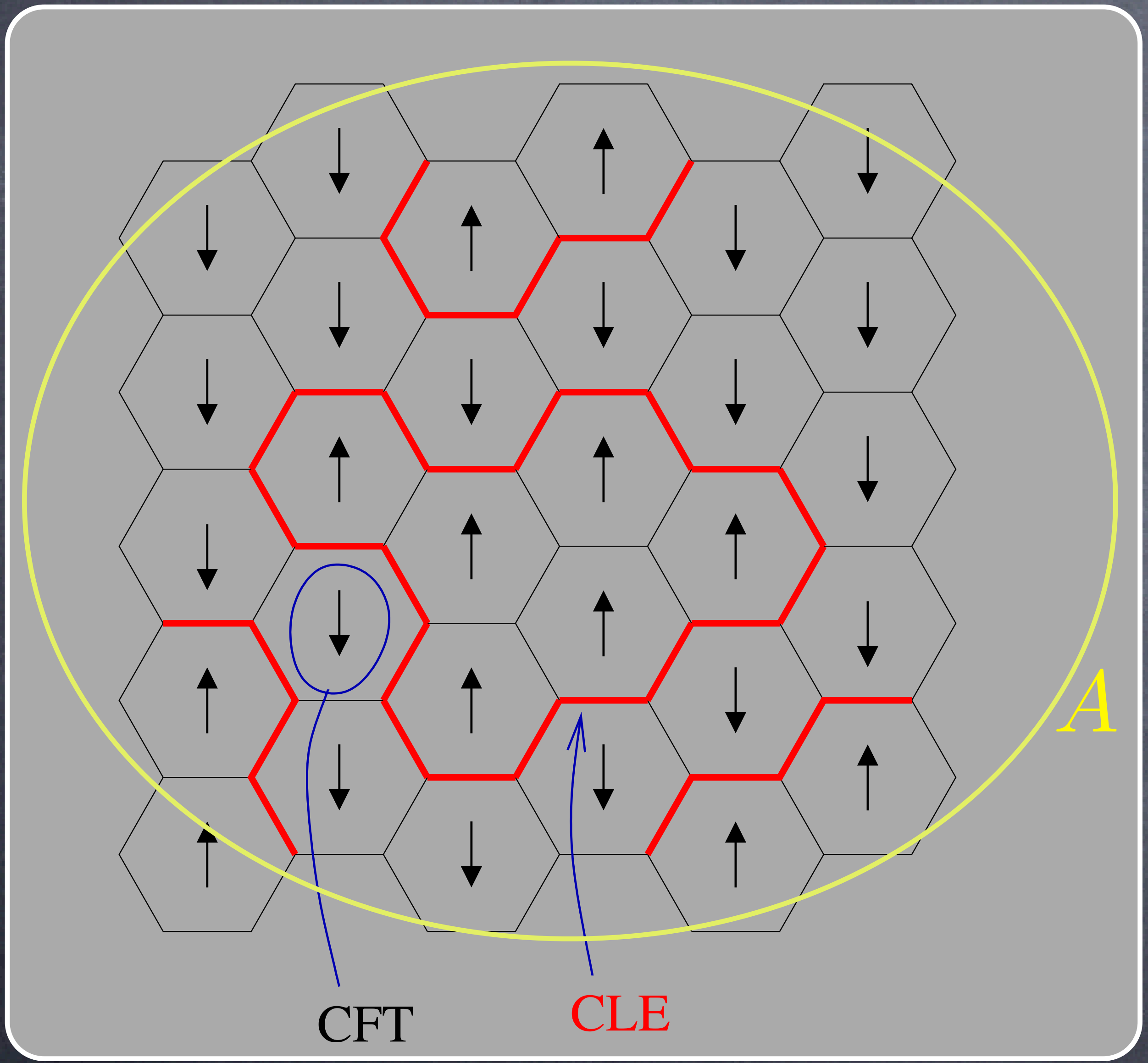
Scaling limit of statistical models: CFT



Example: O(n) model

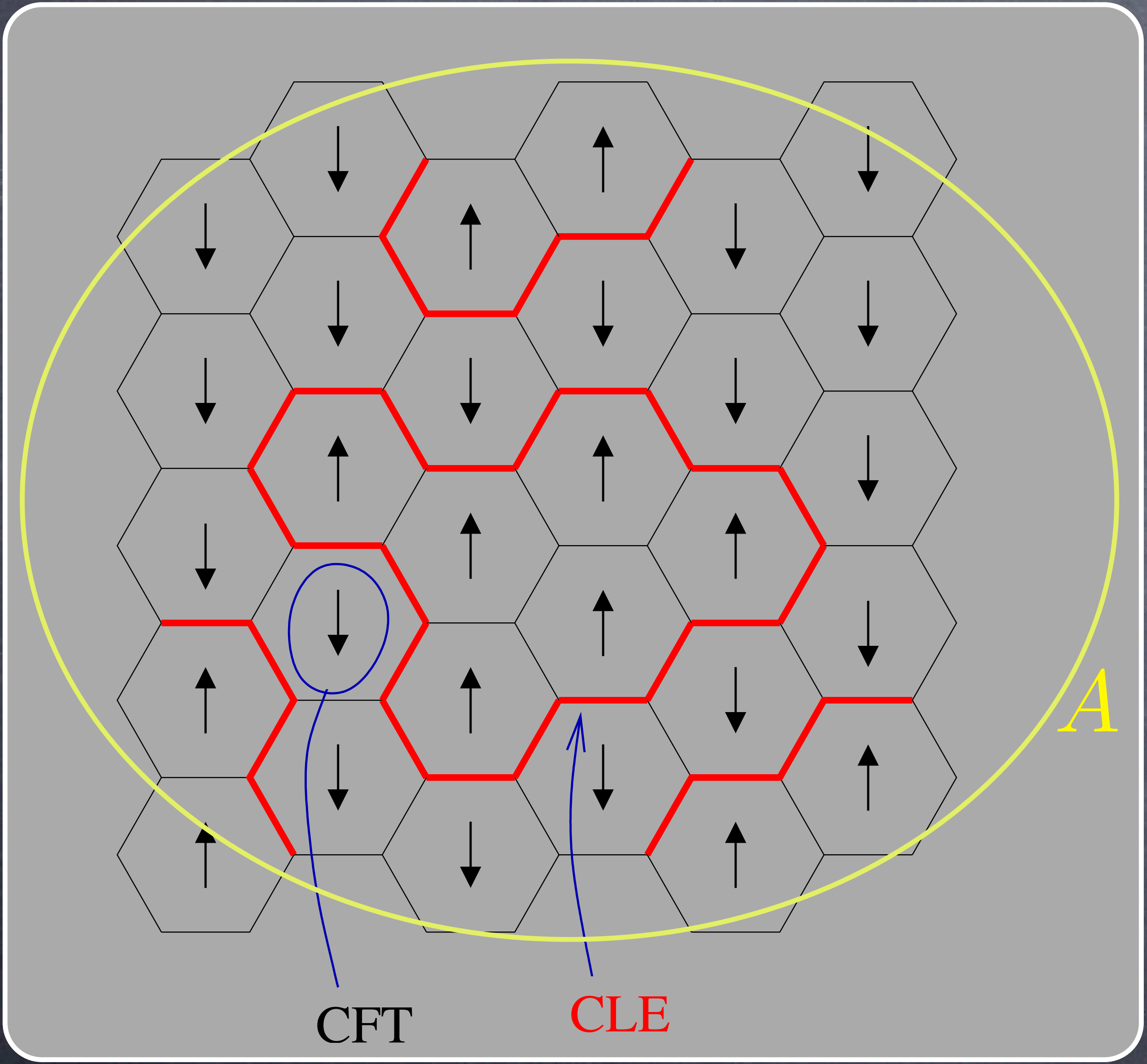
$$\mu = x_c(n)^{\text{length}} n^{\#\text{loops}}$$

Scaling limit:
CFT correlation functions

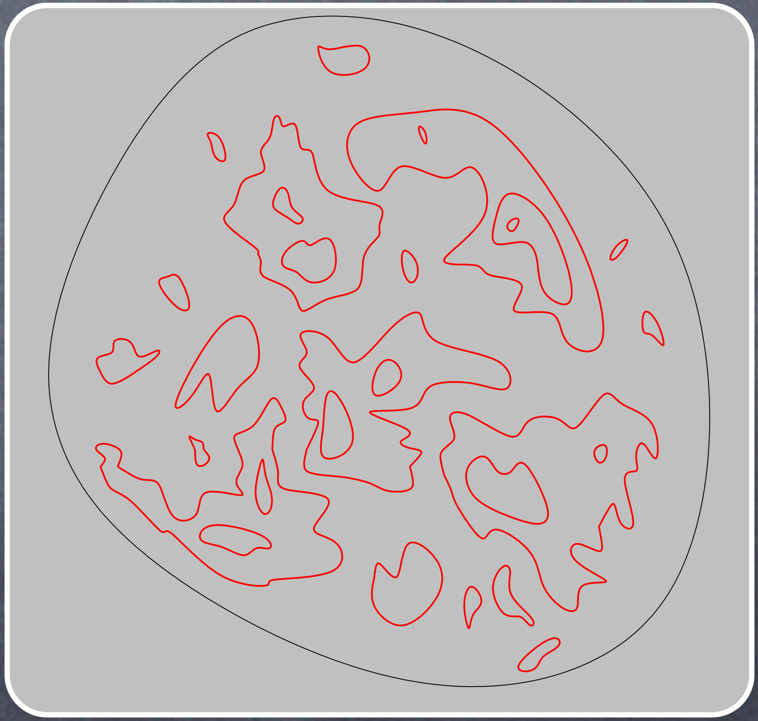
$$\lim_{r\to\infty} r^{kd}\mathbf{E}_{rA}(\sigma_{z_1 r}\cdots\sigma_{z_k r}) = \langle\sigma(z_1)\cdots\sigma(z_k)\rangle_A$$

existence and conformal property proof for Ising case:
Chelkak, Hongler, Izyurov 2013

Scaling limit of statistical models: CLE



Scaling limit:
conformal loop ensembles



CLE: Lawler, Sheffield, Werner 2006-
Scaling limit proof for Ising case: Smirnov, ... 2006-

CFT ⟷ CLE

CFT: central charge $c = c(n) \in [0,1]$

CLE: Hausdorff dimension of loops $\delta = \delta(n) \in [4/3, 7/4]$

$$c = \frac{(7-4\delta)(3\delta-4)}{\delta-1}$$

Bauer, Bernard, Beffara, Cardy, Neinhuis, Schramm, Rhodes, Lawler, Sheffield, Werner

CFT: powerful algebraic structures giving critical exponents and correlation functions (e.g. null-vector equations from Virasoro algebra representations)

construct CFT algebraic structures from CLE concepts?

The Virasoro vertex operator algebra

Borcherds, Frenkel, Huang, Kac, Lepowsky, Meurman, …
Belavin, Polyakov, Zamolodvhikov, …
1984-

Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

Identity module: a particular Verma module V

$$L_m \mathbf{1} = 0 \quad (m \geq -1)$$

$$V = \text{span}\big(L_{\mathbf{m}_j}\mathbf{1} : j \geq 0,\ m_k \leq -2\ (k = 1, \ldots, j)\big)$$

$$L_{\mathbf{m}_j} = L_{m_j} \cdots L_{m_1}$$

Vertex operator map (defines infinitely-many products)

$$Y(\cdot, x) : V \mapsto \text{End}(V)[[x, x^{-1}]]$$

Recursive definition of the vertex operator map

$$Y(\mathbf{1}, x) = 1$$

$$Y(L_m \mathbf{1}, x) = \frac{1}{(-2-m)!} \frac{d^{-2-m}}{dx^{-2-m}} \sum_{n \in \mathbb{Z}} L_n x^{-n-2} \qquad (m \leq -2)$$

$$Y(L_{\mathbf{m}_j} \mathbf{1}, x) = \, : Y(L_{m_j} \mathbf{1}, x) Y(L_{\mathbf{m}_{j-1}} \mathbf{1}, x) :$$

involving the normal-ordered product

$$: Y(L_m \mathbf{1}, x) A : = Y^+(L_m \mathbf{1}, x) A + A Y^-(L_m \mathbf{1}, x)$$

$Y^+(L_m \mathbf{1}, x)$ → nonnegative powers

$Y^-(L_m \mathbf{1}, x)$ → negative powers

Satisfies the main relations of vertex operator algebras

Commutativity

$\forall v, w : \exists k :$

$$(x-y)^k Y(v,x)Y(w,y) = (x-y)^k Y(w,y)Y(v,x)$$

Associativity

$\forall u, w : \exists l \mid \forall v :$

$$(x+y)^l Y(u,x+y)Y(v,y)w = (x+y)^l Y(Y(u,x)v,y)w$$

A relation with CFT correlation functions: inner product

$$(\cdot,\cdot): \quad (\mathbf{1},\mathbf{1}) = 1, \quad L_n^\dagger = L_{-n}$$

$$\langle v_1(x_1)\cdots v_n(x_n)\rangle_{\mathbb{C}} = \iota_{|x_1|>\cdots>|x_n|}(\mathbf{1}, Y(v_1,x_1)\cdots Y(v_n,x_n)\mathbf{1})$$

convergent series in this region: change to complex variables

- The stress-energy tensor:

$$T(x) \ \leftrightarrow \ Y(L_{-2}\mathbf{1}, x)$$

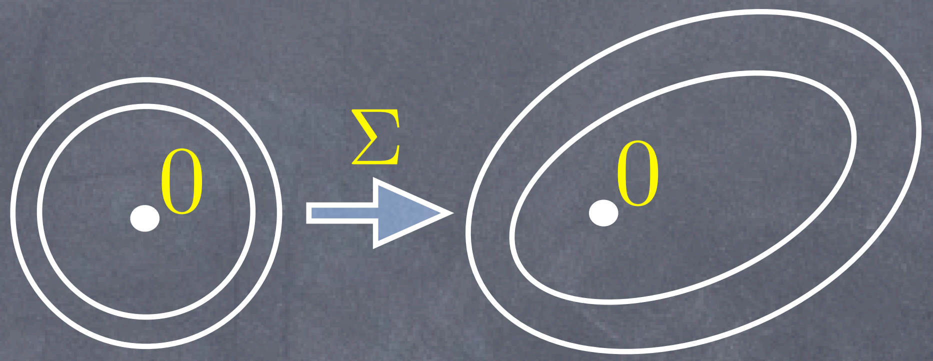
- The operator product expansion:

$$\text{OPE of } u(x+y)v(y) \ \leftrightarrow \ Y(Y(u,x)v,y)$$

Vertex operators and conformal maps

D. 2010-

$\Sigma \in$ { conformal maps on annulus that keep 0 / infinity in inner / outer holes and preserve orientation }



Derivative wrt small conformal deformations (Lie derivatives):

$$\Delta_{m,w} f(\Sigma) := \left(\frac{\partial}{\partial \eta} f(g_\eta \circ \Sigma)\Big|_{\eta=0} \right)_{\text{hol.}} \quad , \; g_\eta(z) = z - \eta(z-w)^{m+1}$$

Assume existence of a particular function satisfying

$$\Delta_{-2,w} \log Z(\Sigma) = (\partial g(w))^2 (\Delta_{-2,g(w)} \log Z)(g \circ \Sigma) + \frac{c}{12}\{g,w\}$$

Schwartzian derivative

Virasoro representation:

$$L_\ell \mapsto Z^{-1}\Delta_{\ell,0}Z \quad (\ell \leq -2), \quad \Delta_{\ell,0} \quad (\ell \geq -1)$$

Vertex operator algebra representation:

$$Y(L_{\mathbf{m}_j}\mathbf{1}, x) \mapsto Z^{-1}\Delta_{m_j,x}\cdots\Delta_{m_1,x}Z$$

Highest weight vector, logarighmic vector, ...:

$$f(\Sigma) = r_\Sigma^\nu$$

$r_\Sigma =$ Conformal radius wrt 0 of $\Sigma(\partial\mathbb{D})$

$$f(\Sigma) = r_\Sigma^\nu \log r_\Sigma$$

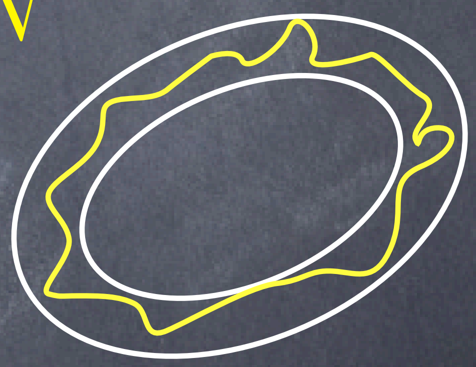
Conformal Ward identities, Cardy's boundary condition, any domain...

$$\langle T(w)\phi(z_1)\cdots\phi(z_n)\rangle = Z^{-1}\Delta_{-2,w}Z\langle\phi(z_1)\cdots\phi(z_n)\rangle$$

CLE

(here «dilute» regime)

- Random variables X with concept of support: where we need the loop information to know the variable
- Indicator variable $I(N)$ that a loop lies and winds in an annular region (tubular neighborhood) N



- Expectation value functionals $\mathbf{E}[\cdot]_A$ on domain A
- Conformal invariance
- A certain conformal restriction

conformal restriction systems

D. 2012

Supported unital algebra

$$X \in \mathfrak{X},\ \mathrm{Supp}(X) = \text{set of closed subsets of } \hat{\mathbb{C}}$$

$$\mathfrak{X}^{(A)} : \text{subalgebra supported in } A$$

Set of linear functionals

$$\mathbf{E}[\cdot]_A : \mathfrak{X}^{(A)} \to \mathbb{C}, \quad \mathbf{E}[\mathbf{1}] = 1$$

Linear representation of groupoid of conformal maps

$$\mathfrak{X}^{(A)} \to \mathfrak{X}^{(g(A))}$$

$$X \to g \cdot X \qquad\qquad g \cdot (g' \cdot X) = (g \circ g') \cdot X$$

Tubular-neighborhood variables

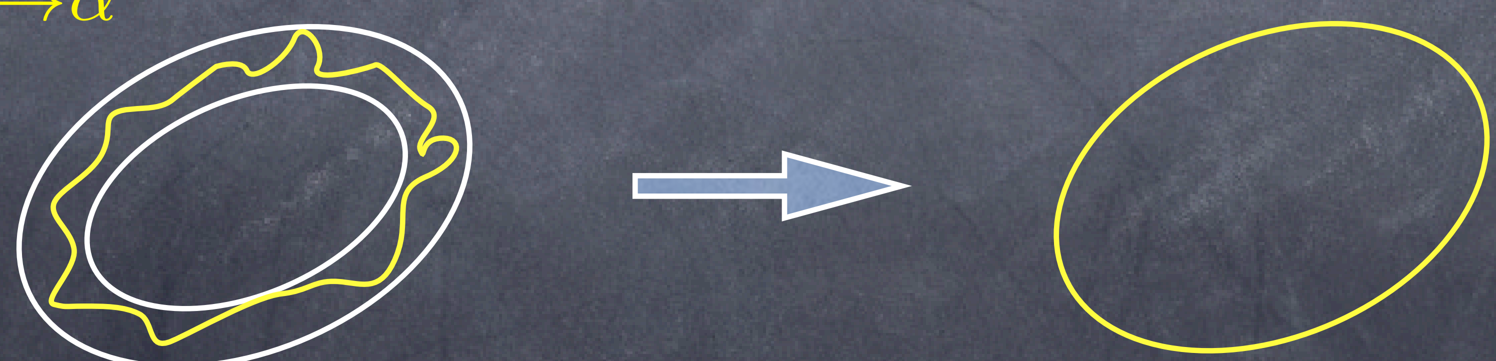
$$N = \text{tubular neighborhood} : I(N) \in \mathfrak{X}$$

$$N \in \text{Supp}(I(N)), \quad g \cdot I(N) = I(g(N)), \quad \mathbf{E}[I(N)]_A \neq 0,\ A \supset N$$

$$E(N) := \frac{I(N)}{\mathbf{E}[I(N)]_{\hat{\mathbb{C}}}}$$

«Weak-local» limits: the «configuration-separating» variables

$$E(\alpha) := \lim_{N \to \alpha} E(N) \qquad \alpha = \text{Jordan curve}$$



weak-local: inside expectations, multiplied by variables supported elsewhere

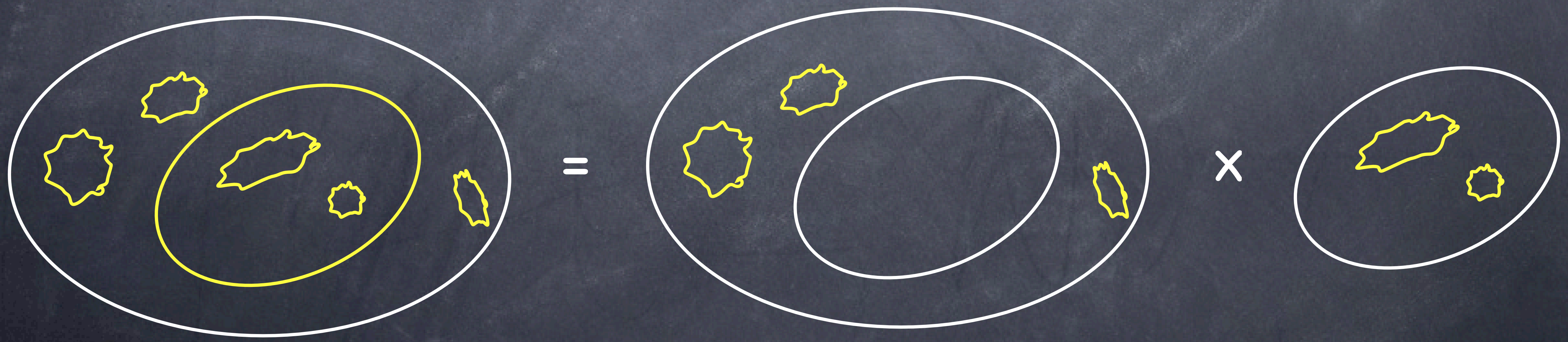
Conformal invariance

$$\mathbf{E}[g\cdot X]_{g(A)} = \mathbf{E}[X]_A \quad \forall \quad X\in\mathfrak{X}^{(A)},\ g \text{ conformal on } A$$

Restriction: conditional = on new domain

$$\frac{\mathbf{E}[E(\alpha)X]_A}{\mathbf{E}[E(\alpha)]_A} = \mathbf{E}[X]_{A\setminus\alpha}$$

X : product of factors supported on components of $A\setminus\alpha$



Smoothness

$$g \mapsto \mathbf{E}[g \cdot X]_{g \cdot A}, \quad g \mapsto \mathbf{E}[X]_{g \cdot A} \quad \text{are smooth at } g = \mathrm{id}$$

where

g conformal on a support of X and boundary components of A

$g \cdot A$ domain bounded by conformally mapped boundary components

Local covering

$$\lim_{\epsilon\to 0} \nabla\mathbf{E}[X]_{A\setminus K_\epsilon} = \nabla\mathbf{E}[X]_A$$

∇ → derivative under small conformal deformations, of any order

if

$$\lim_{\epsilon\to 0} K_\epsilon = \{z\} \quad \text{(in the sense of smallest covering disks)}$$

X supported in $A \setminus \{z\}$

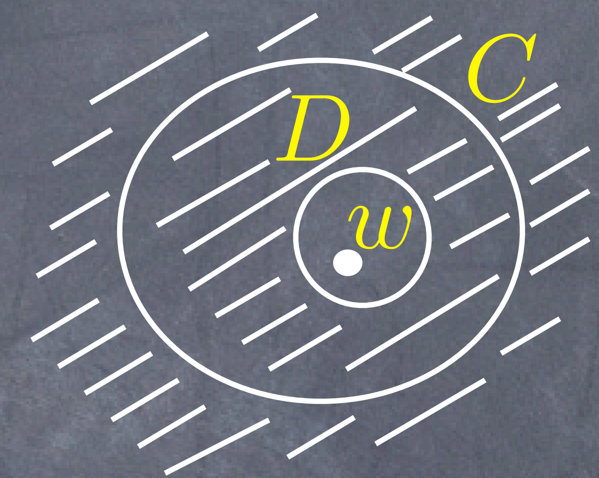
Idea: in CLE there are always infinitely many loops surrounding a point, so taking away a small domain, in the limit, will not change anything.

Theorem 1: relative partition function

D. 2009-2012

Relative partition function (simply connected Jordan domains)

$$Z(\partial C, \partial D) := \frac{1}{\mathbf{E}[E(\partial C)]_{\hat{\mathbb{C}}\setminus\overline{D}}}$$



Theorem:

There exists a complex number $c \in \mathbb{C}$ such that

$$\Delta_{-2,w} \log Z(\partial C, \partial D) = \frac{c}{12}\{s, w\}, \quad s : C \to \mathbb{D} \text{ conformally}$$

$$\Sigma \mapsto Z(\Sigma(\partial C), \Sigma(\partial D))$$

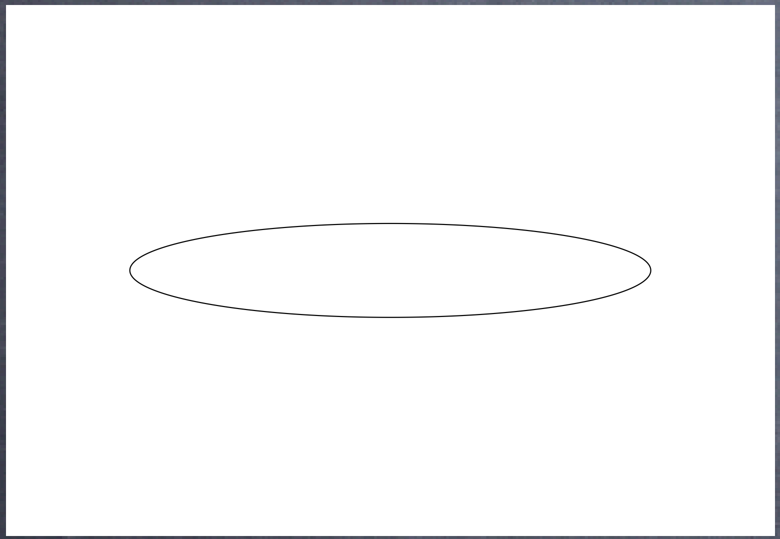
$$w \in D, \quad \overline{D} \subset C$$

Theorem 2: (partial) VOA structure

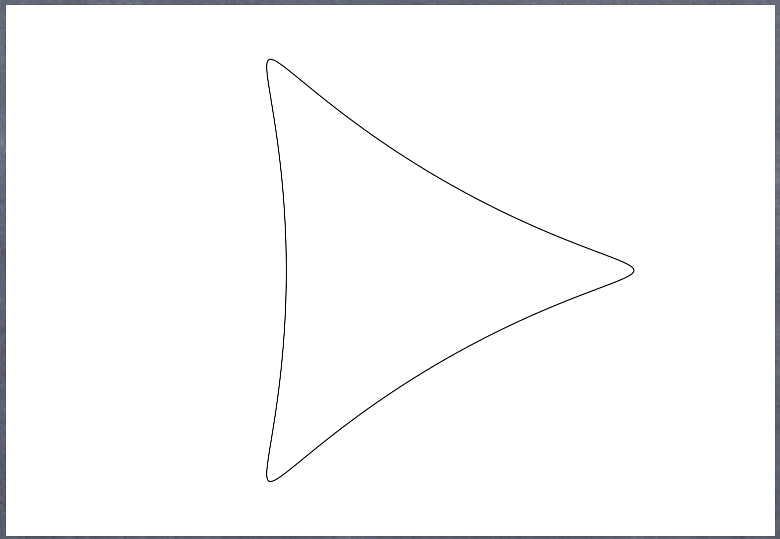
D. 2009-2012

hypotrochoids (ellipses and generalization): $C_k(w,\epsilon,\theta)$

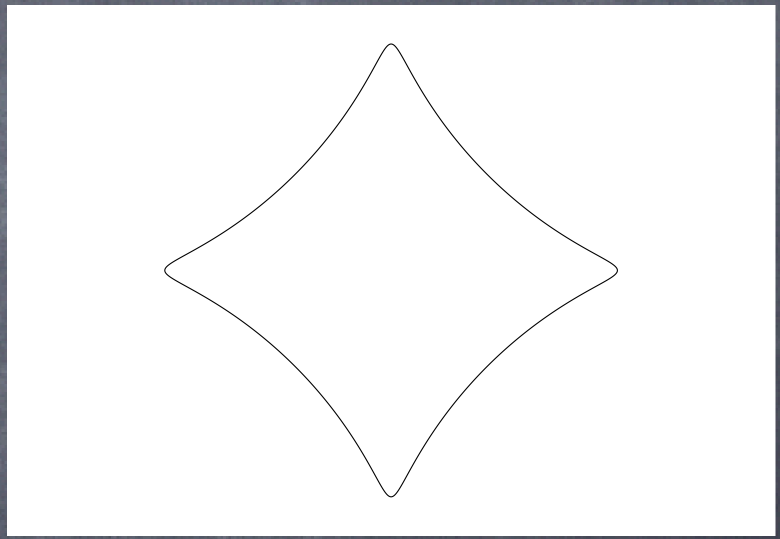
extent ϵ angle θ position w



$k=2$



$k=3$



$k=4$

$$C_k(w,\epsilon,\theta) := \left\{ w + \epsilon e^{i\theta}(b e^{i\alpha} + b^{1-k} e^{(1-k)i\alpha}) : \alpha \in [0,2\pi) \right\}$$

$$k = 2,3,4,\ldots, \quad w \in \mathbb{C}, \quad b > (k-1)^{1/k}$$

Asymptotic expansion in powers of $u = \epsilon e^{i\theta}$ and $\bar{u}$ exists

$$E(C_k(w, \epsilon, \theta)) \sim$$

$$\mathbf{1} + o((u\bar{u})^0) + \sum_{m=1}^{\infty} \frac{u^{km}}{m!} \left(T_{k,m}(w) + o((u\bar{u})^0)\right) + c.c.$$

Equivalently, we define the Fourier-transform variables

$$T_{k,m}(w) := \lim_{\epsilon \to 0} \frac{m!}{2\pi\epsilon^{km}} \int_0^{2\pi} d\theta \, e^{-kmi\theta} E(C_k(w, \epsilon, \theta))$$

Some Virasoro descendents

$$v_{k,1} = L_{-k}\mathbf{1}$$

$$v_{k,2} = (L_{-k}^2 + (k-1)L_{-2k})\mathbf{1}$$

$$v_{k,3} = (L_{-k}^3 + 3(k-1)L_{-2k}L_{-k} + 2(k-1)(2k-1)L_{-3k})\mathbf{1}$$

General recursive definition

$$v_{k,m} := \sum_{\lambda \in \Phi(m)} B_\lambda (k-1)^{m-|\lambda|} L_{-k\lambda_{|\lambda|}} \cdots L_{-k\lambda_1}\mathbf{1}$$

$$\Phi(m) = \{(\lambda_1, \ldots, \lambda_j) : j \geq 1, \sum_i \lambda_i = \lambda, \lambda_i \geq 1\} \qquad |\lambda| = j$$

$$B_{(\lambda_1,\ldots,\lambda_j)} = \delta_{\lambda_j,1} B_{(\lambda_1,\ldots,\lambda_{j-1})} + \sum_i (\lambda_i - 1) B_{(\lambda_1,\ldots,\lambda_i - 1,\ldots,\lambda_j)}$$

Theorem:

The variables $T_{k,m}$ have the vertex operator algebra substructure of the descendents $v_{k,m}$, with the identification $T_{k,m} \equiv v_{k,m}$

Laurent expansions of expectation values are reproduced by products of vertex operators,

$$\Big[\mathbf{E}[T_{k_1,m_1}(w_1)\cdots T_{k_p,m_p}(w_p)X]_A\Big]_{\text{expansion } |w_1|>\cdots|w_p|}$$

$$= Y(v_{k_1,m_1}, w_1)\cdots Y(v_{k_p,m_p}, w_p)\mathbf{E}[X]_A$$

$$\Sigma \mapsto \mathbf{E}(\Sigma \cdot X)_{\Sigma\cdot A}$$

with the domain-dependent representation

$$L_\ell \mapsto Z(\partial A, v)^{-1}\Delta_{\ell,0}Z(\partial A, v) \quad (\ell \leq -2), \quad \Delta_{\ell,0} \quad (\ell \geq -1)$$

v : Jordan curve in A surrounding 0

Conclusion, outlook

- Part of the Virasoro vertex operator algebra recovered as local random variables. This includes the stress-energy tensor, the Virasoro algebra itself, the associated conformal Ward identities. Any central charge (but no yet identified the central element with the CLE «c»).

- Full VOA? Geometric interpretation of variables?

- Other symmetries? What replaces conformal transformations?

- Proving the axioms of conformal restriction systems from conformal loop ensembles?