Random loops and conformal field theory

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Comm. Math. Phys 268 (2006) 687-716, with V. Riva and J. Cardy
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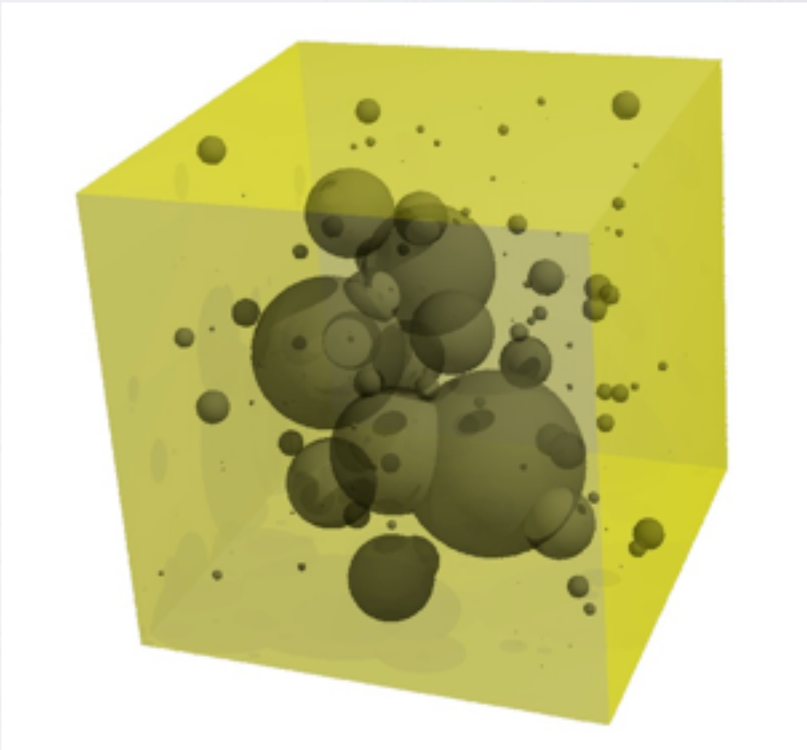
Criticality and nucleation

Thermodynamic criticality may be seen as occurring when two phases start to coexist simultaneously.

Criticality displays (at least!) three fundamental properties:

Sensitivity: divergence of susceptibilities, critical exponents.
Scale invariance: statistical self-similarity, scaling relations.
Universality: independence from microscopic detail.

A simple picture of criticality is that of nucleation: at criticality it costs no free energy to create a phase boundary, so by entropic considerations phase boundaries (bubbles) are created at all scales and both phases coexist.



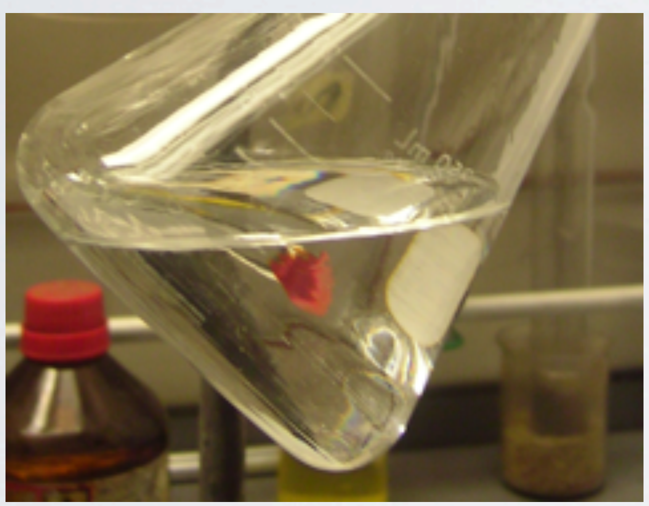
Criticality and nucleation

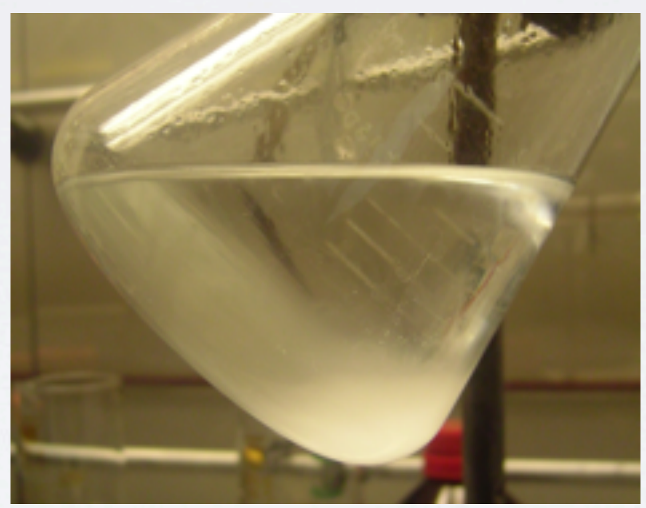
Sensitivity:

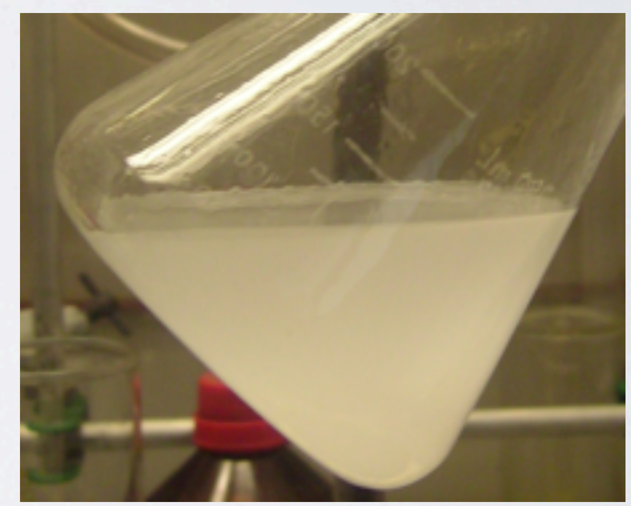
- phase boundaries are easily destroyed by a small external field;
- fluctuations are efficiently transferred from small to large scales.

Scale invariance:

- bubbles occur at all scales creating a scale invariant distribution;
- this may manifest itself through *critical opalescence:*







Universality:

- the bubbles can be seen as large-scale *emergent random objects*, that keep only partial information from microscopic fluctuations.

Criticality and nucleation

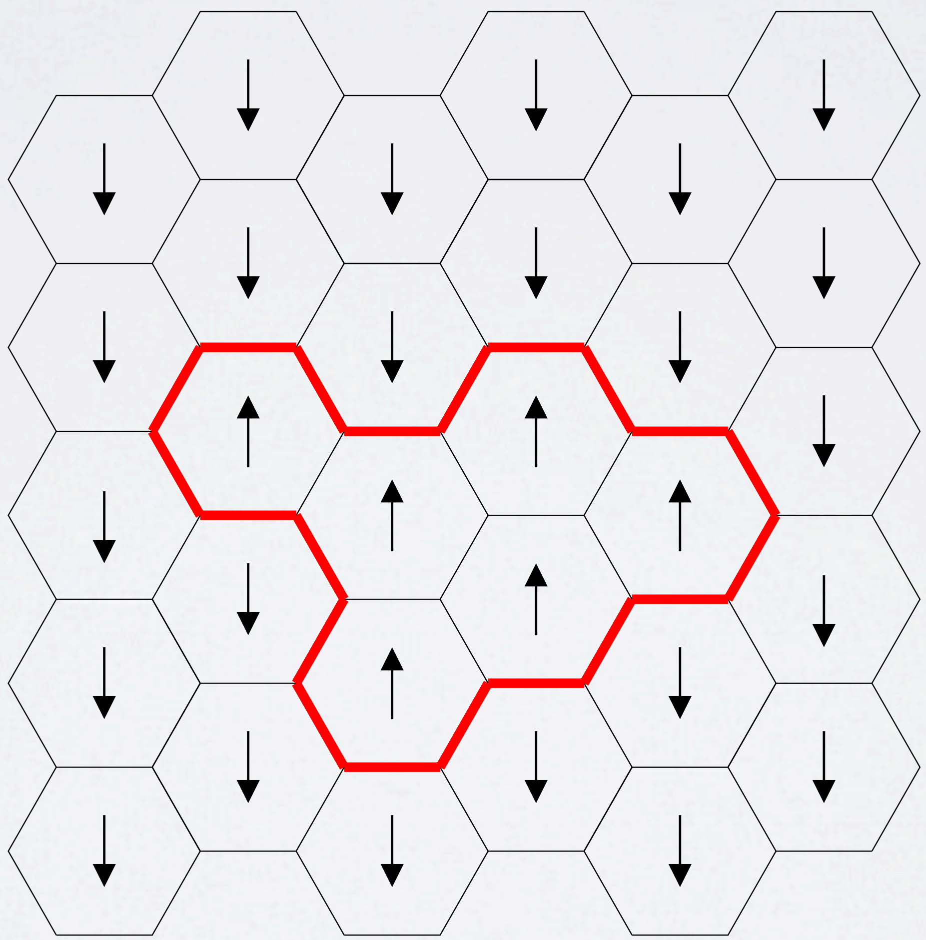
A proposed concept: attempting to connect these properties together in a quantitative way, there should be a universal quantity (to be defined!) describing how much of the fluctuations of small bubbles are carried to large scale fluctuations:

$$c$$

This and the bubble picture can be made somewhat more precise in *two-dimensional models.*

Two dimensions: spins and random loops

Consider a homogeneous, isotropic model of local spins (up or down) where the two phases are spin-up and spin-down; for instance the well-known Ising model at its critical temperature.



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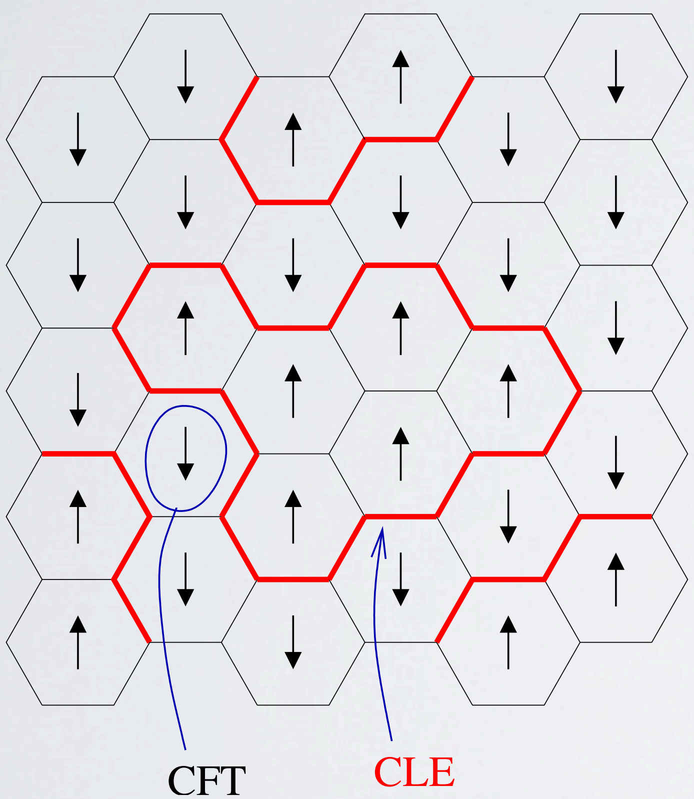

In the standard critical scaling limit, one observes *correlations* between local spins (or other local observables) at large separations:

$$\langle \sigma(s\,w_1)\cdots\sigma(s\,w_n)\rangle \sim s^{-\eta}F(w_1,\ldots,w_n)$$

The exponent η and the function $F(w_1,\ldots,w_n)$ are *universal quantities*, and are described by ***Conformal field theory***.

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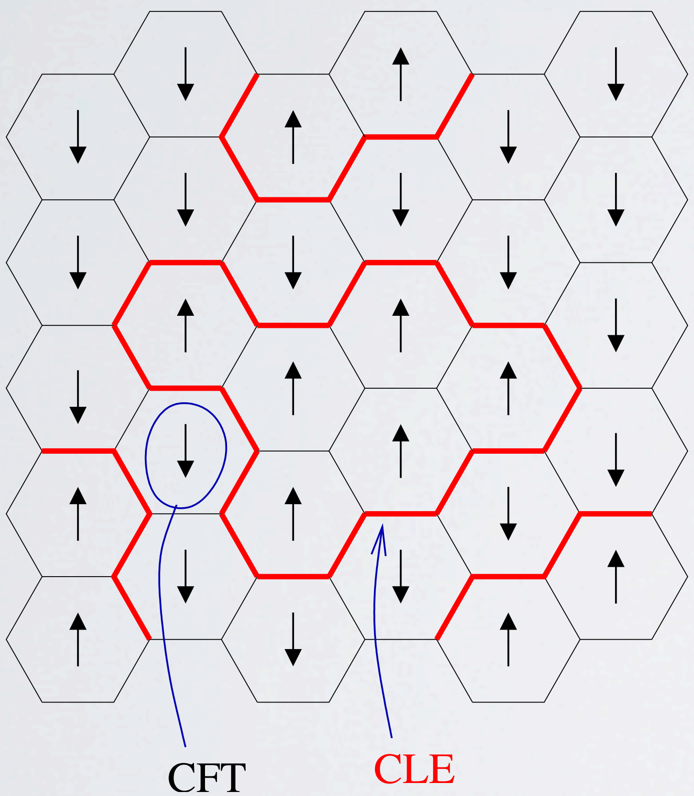


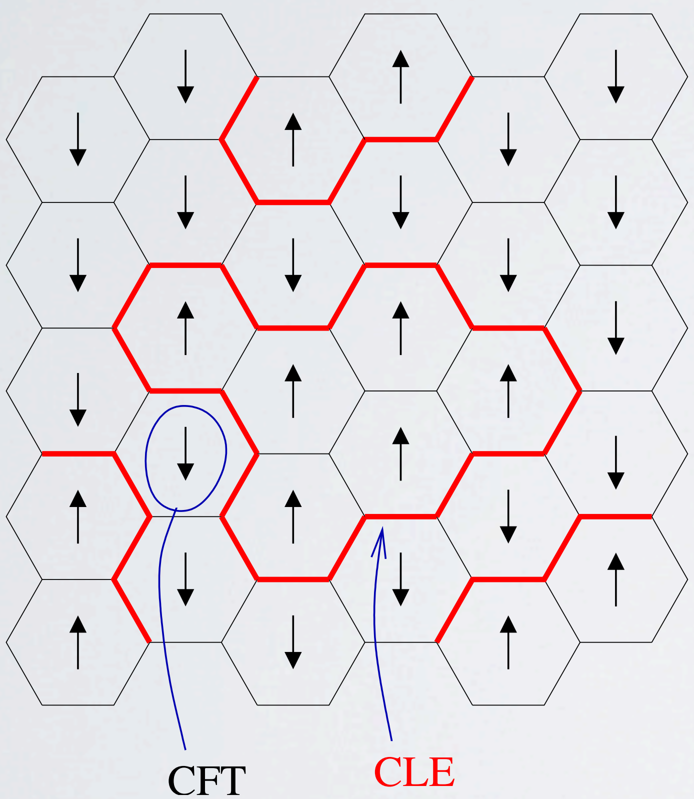
But there is another way of taking the scaling limit: observing the cluster boundaries (the bubbles) at large scales:

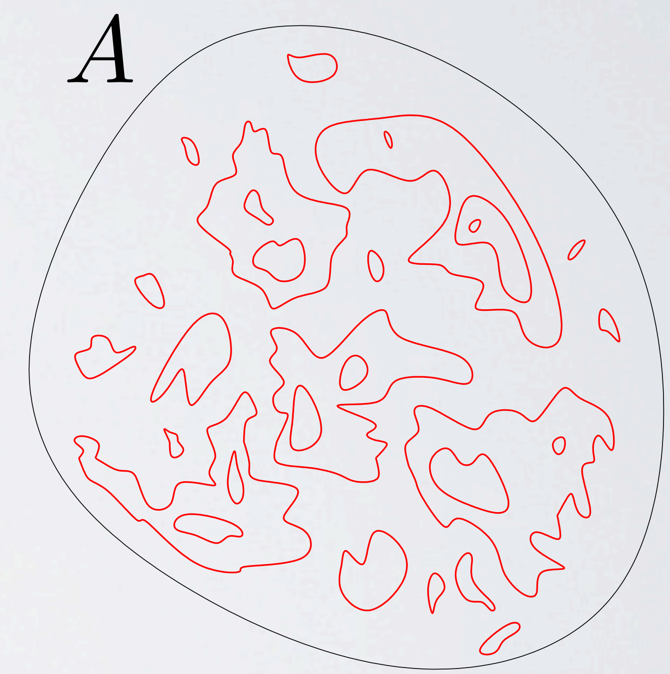
The *measure* μ_A on the resulting set of nested disjoint loops is *universal*, and is described by ***Conformal Loop Ensembles***.

SLE and CLE: Schramm (1999), Lawler, Sheffield, Werner (2006-); Scaling limit proof (Ising): Smirnov et al. (2006-)

Two dimensions: spins and random loops

There are many models besides the Ising models where explicit domain boundaries may be define naturally, e.g. the so-called O(n) models.





$$\mu_A = \text{scaling limit of } x^{\text{total length}} n^{\#\text{loops}}$$

Case n=1 (Ising): proved by Chelkak and Smirnov (2008)

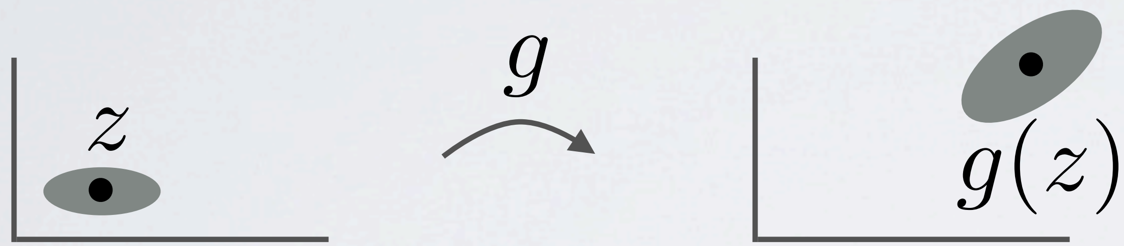
$$x = \frac{1}{\sqrt{2+\sqrt{2-n}}}$$

Nienhuis (1982)

Conformal loop ensembles

S. Sheffield and W. Werner (2006-; Ann. Math. 2012)

Conformal invariance: consequence of scale invariance + homogeneity + isotropy + locality: a conformal transformation is locally a homotety + translation + rotation.



$$\mu_A = \mu_{g(A)} \circ g$$

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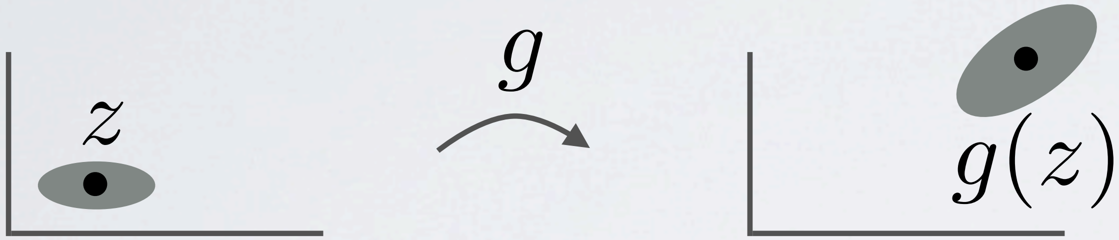

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Restriction / a cluster boundary is a domain boundary: spins don't know if a wall of e.g. plus-spins is a domain boundary, a restriction condition, or cluster boundary.

Inside loops: 

Outside loops: 

Conformal loop ensembles

S. Sheffield and W. Werner (2006-; Ann. Math. 2012)

By the Riemann Mapping Theorem, any simply connected hyperbolic domain can be conformally mapped to the unit disk: *the above form a strong set of conditions*.

As a consequence, there is a *unique one-parameter family of solutions to the above conditions* (on such domains): CLE_c

$$c \in (0, 1]$$

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The parameter c is related in a precisely known fashion to:

- the ***Poisson density*** in the stochastic construction of CLE;
- the ***fractal dimension*** d of any loop;
- the ***Boltzman weight*** n for the presence of a loop.

$$c = \frac{(7-4d)(3d-4)}{d-1}$$

$$c = \frac{(2-3y)(4y-1)}{1-y}, \quad \cos(2\pi y) = -\frac{n}{2}$$

Bauer, Bernard, Cardy; Beffara, Sheffield, Werner (2002-)

Measuring the shape of small loops

BD (2011,2012,2013)

Can we relate c to the *fluctuations* of loops? For this purpose, let us try to measure the shape of small loops. Consider the *indicator variable* $\mathtt{I}(N)$ that there be *at least one loop winding in the annular domain* N.

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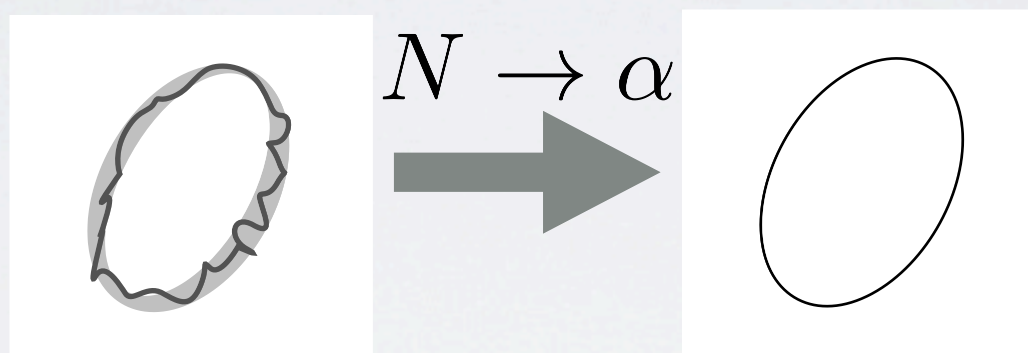

Taking the normalized limit where the annular domain becomes a curve α we obtain a weight for loops that are «near» that shape:

$$\mathtt{E}(\alpha) := \lim_{N \to \alpha} \frac{\mathtt{I}(N)}{\mathbb{E}\big[\mathtt{I}(N)\big]_{\hat{\mathbb{C}}}}$$

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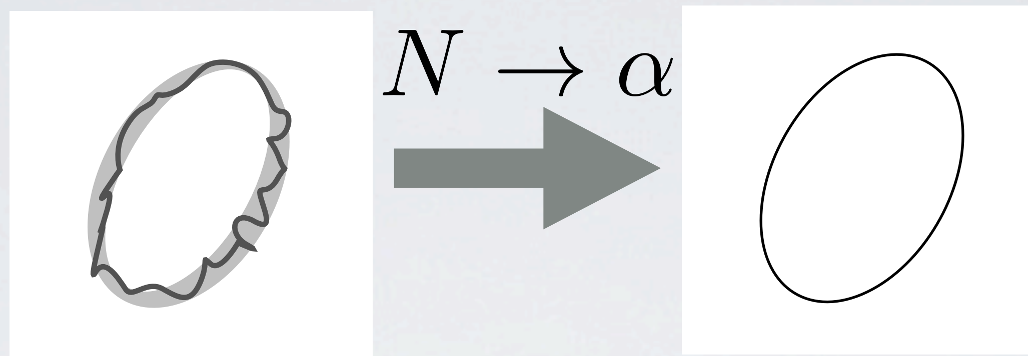

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The normalization factor $\mathbb{E}\big[\mathtt{I}(N)\big]_{\hat{\mathbb{C}}}$ is *zero* in the limit $N \to \alpha$, and the way it tends to zero depends on how *wiggly* the loops tend to be (cf. fractal dimension). This is a nontrivial ***renormalization***.

Measuring the shape of small loops

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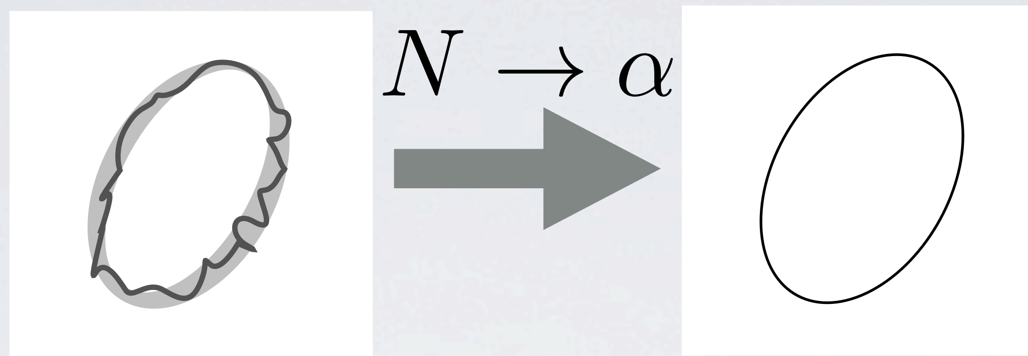

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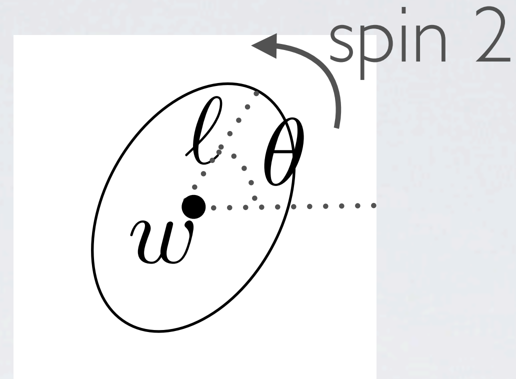
We may think of the variable $\mathtt{E}(\alpha)$ as a Wilson loop (or more precisely, the dual to a Wilson loop).

Ellipses

BD (2011,2012,2013)



Consider the curve $\alpha = \alpha(w, \theta, \ell, e)$ to be an ellipse as above of eccentricity e, and define a *further renormalized* variable corresponding to a rotating infinitesimal ellipse of spin 2:

$$\mathcal{T}(w) := \lim_{\ell \to 0} \frac{1}{2\pi\epsilon^2} \int_0^{2\pi} d\theta \, e^{-2i\theta} \, \mathtt{E}(\alpha(w, \theta, \ell, e)) \qquad \left(\epsilon := \frac{\ell e}{2} \right)$$

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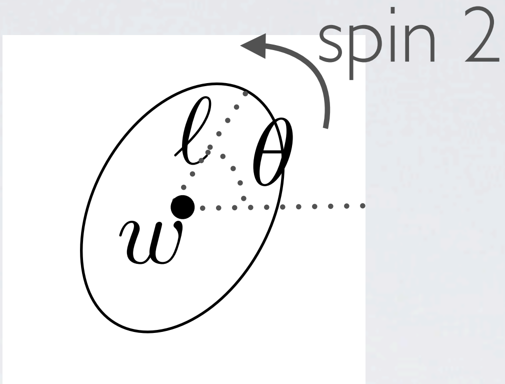

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Then, the following expectation on the Riemann sphere is:

$$\mathbb{E}\big[\mathcal{T}(w_1)\mathcal{T}(w_2)\big]_{\hat{\mathbb{C}}} = \frac{c/2}{(w_1 - w_2)^4}$$

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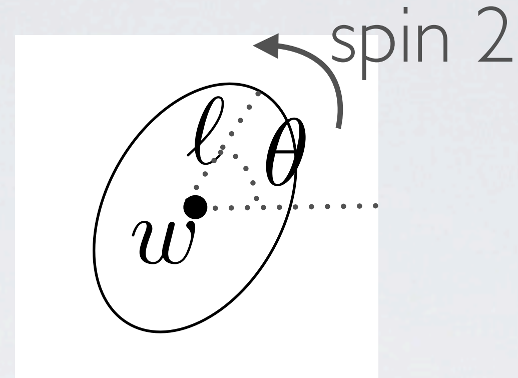

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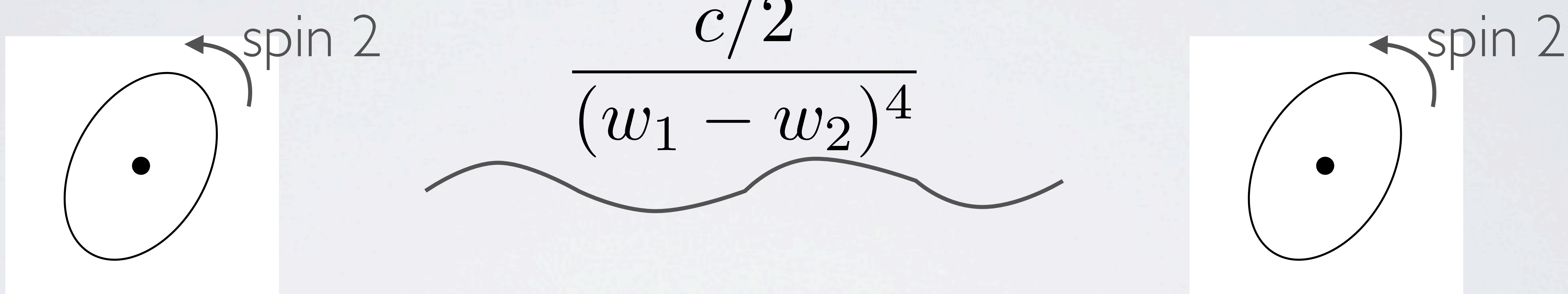
Note: CFT two-point function of holomorphic stress-energy tensor

Ellipses

That is, the number c measures the strength of the correlation between the «events» that separated infinitesimal loops are elliptic and rotating with spin 2 (*shape correlations*).



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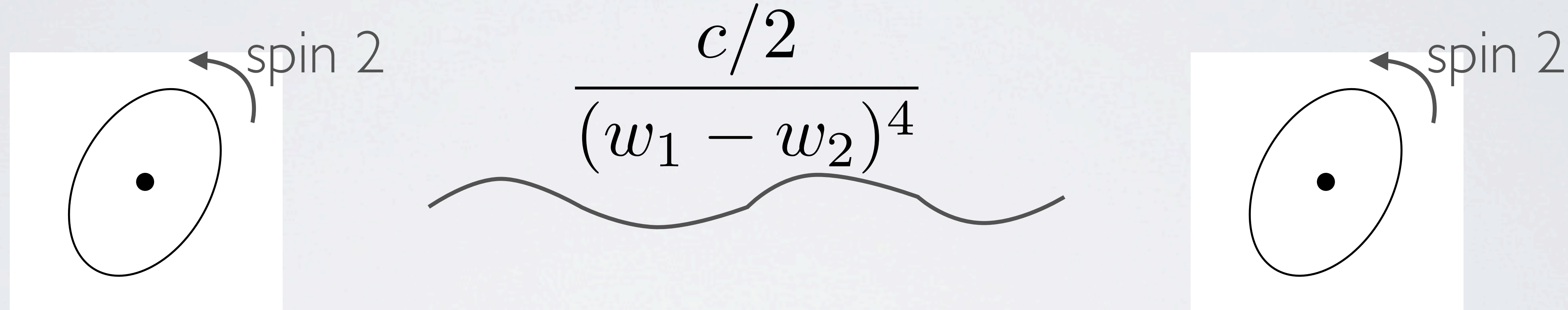
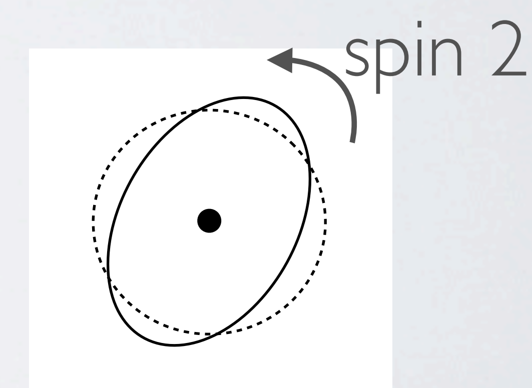


A rotating ellipse of spin 2 can be interpreted as a two-crest wave propagating at the spin-2 speed on a circle: we find large-distance correlations between such **wave-like fluctuations** at small scales.



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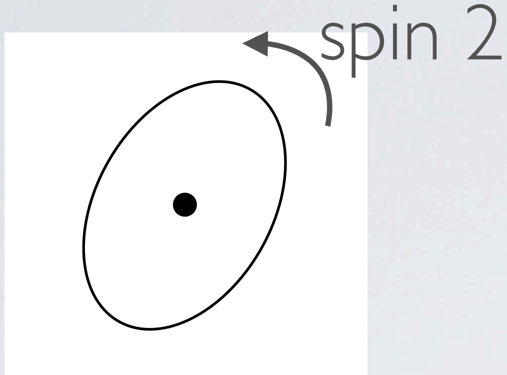

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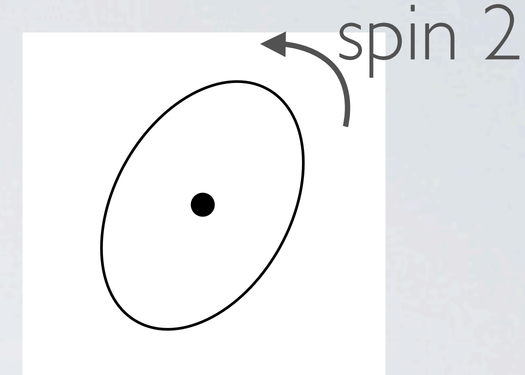
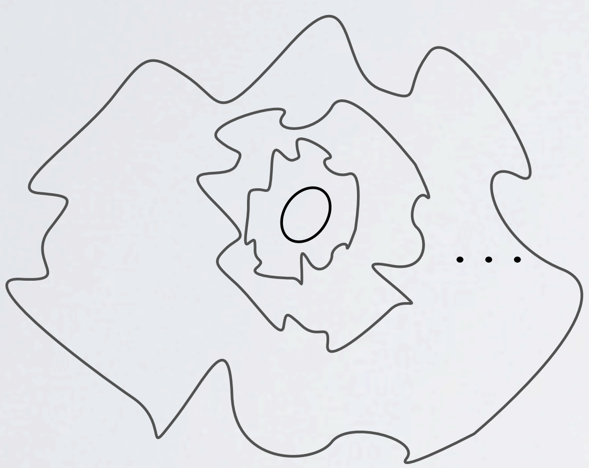
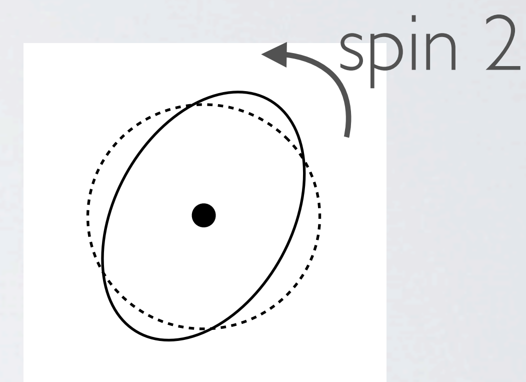




Since there is a.s. infinitely many loops around every point, these correlations mean such wave-like fluctuations must travel from small to large scales:

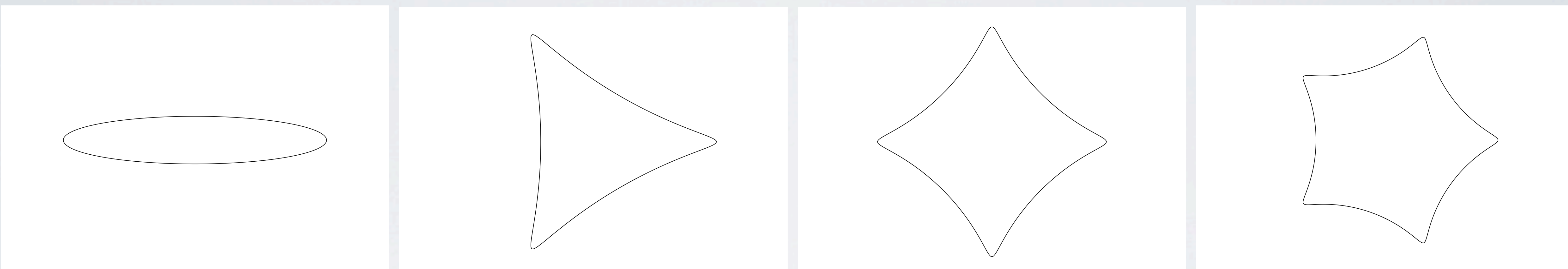


Tranfer of certain types of fluctuations from small to large scales is proportional to c

Hypotrochoids and Virasoro

BD (2011,2012,2013)

This can be generalized to higher number of crests using the *hypotrochoids* (natural generalizations of the ellipse):



$k = 2$ $k = 3$ $k = 4$ $k = 5$

$$\alpha_k(w, \theta, \epsilon, b) := \left\{ w + \epsilon e^{i\theta}(b e^{i\beta} + b^{1-k} e^{(1-k)i\beta}) : \beta \in [0, 2\pi) \right\}$$

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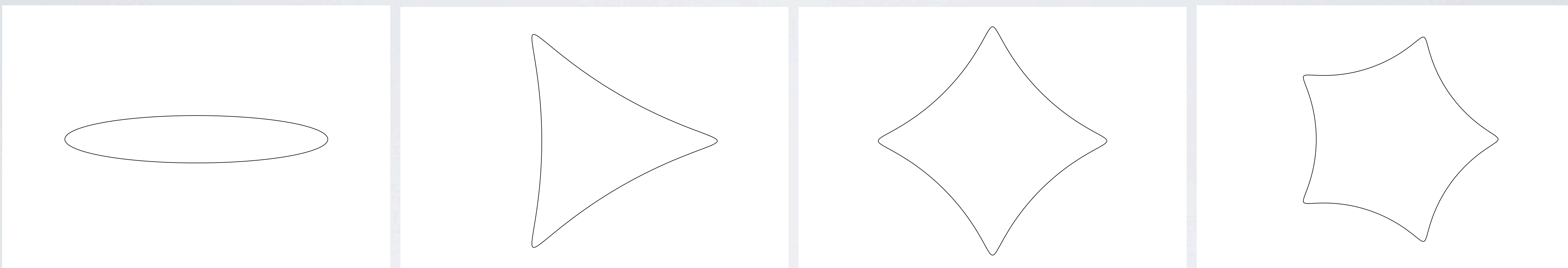

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It can also be generalized to higher spins, and to any correlation function on any domain.

$$\mathcal{T}_{k,m}(w) := \lim_{\epsilon \to 0} \frac{m!}{2\pi \epsilon^{km}} \int_0^{2\pi} d\theta \, e^{-kmi\theta} \, \mathtt{E}(\alpha_k(w,\theta,\epsilon,b))$$

$$\Big(\mathcal{T}_{2,1}(w) = \mathcal{T}(w) \Big)$$

$$\mathbb{E}\big[\mathcal{T}_{k_1,m_1}(w_1) \cdots \mathcal{T}_{k_n,m_n}(w_n)\big]_A$$

Hypotrochoids and Virasoro

BD (2011,2012,2013)

Result: these expectation values are equal to *correlation functions of descendants of the stress-energy tensor in CFT*:

$$\mathbb{E}\big[\mathcal{T}_{k_1,m_1}(w_1)\cdots\mathcal{T}_{k_n,m_n}(w_n)\big]_A = \langle T_{k_1,m_1}(w_1)\cdots T_{k_n,m_n}(w_n)\rangle_A$$

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They are explicit functions of c whenever the domain A is simply connected.

They include the *conformal Ward identities* of Belavin, Polyakov and Zamolodchikov (1984)

$$T(w_1)T(w_2) \sim \frac{c/2}{(w_1-w_2)^4} + \frac{2T(w_2)}{(w_1-w_2)^2} + \frac{\partial T(w_2)}{w_1-w_2}$$

as well as the *conformal boundary conditions* of Cardy (1984).

$SLE_{8/3}$ boundary: Friedrich and Werner (2003); $SLE_{8/3}$ bulk: BD, Riva and Cardy (2006)

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Stress-energy tensor and descendants have algebraic meaning via the *Virasoro algebra*, where c is the associated *central charge*:

$$[L_n, L_m] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\,\delta_{m+n,0}$$

By the state-operator correspondence, they are elements of the *identity module*, for instance we have:

$$T_{k,1} = L_{-k}\mathbf{1}$$

$$T_{k,2} = (L_{-k}^2 + (k-1)L_{-2k})\mathbf{1}$$

$$T_{k,3} = (L_{-k}^3 + 3(k-1)L_{-2k}L_{-k} + 2(k-1)(2k-1)L_{-3k})\mathbf{1}$$

Hypotrochoids and Virasoro

BD (2011,2012,2013)

This means that these small-scale wave-like fluctuations generate the structure of the *Virasoro vertex operator algebra*. For instance:

$$\frac{\partial}{\partial w} \quad T_{2,1}(w) \quad = \quad T_{3,1}(w)$$

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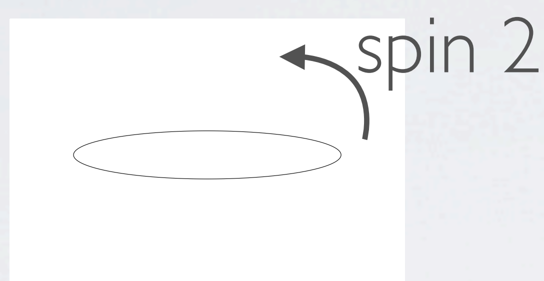

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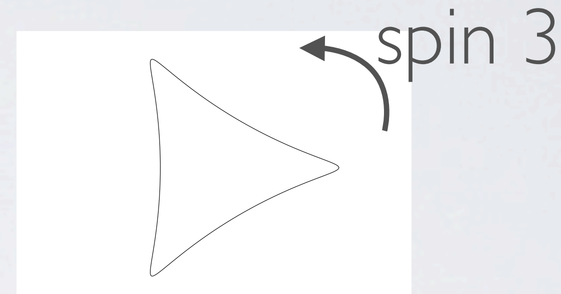

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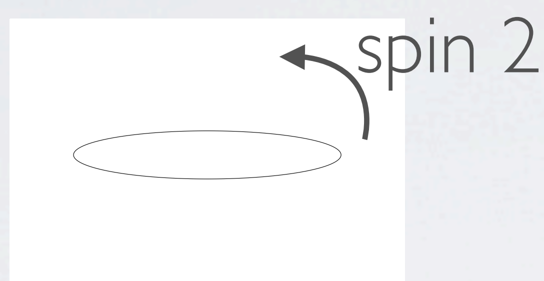

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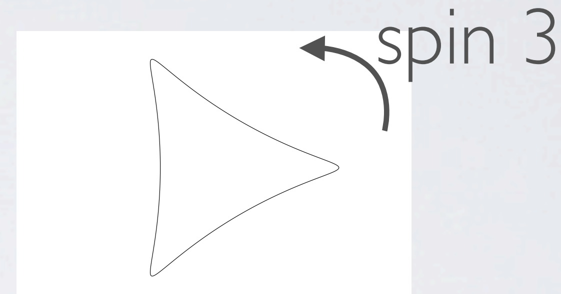

$$\Big[T_{2,1}(w)T_{2,1}(w') - \text{singular part}\Big]_{w=w'} \quad = \quad T_{2,2}(w) - T_{4,1}(w)$$

(Wilson's Operator Product Expansion)

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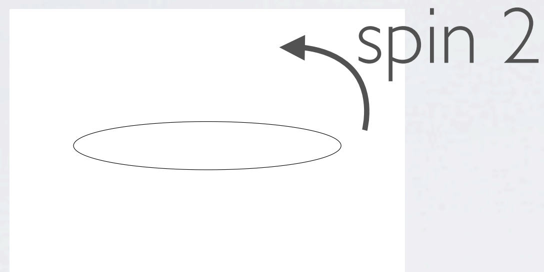
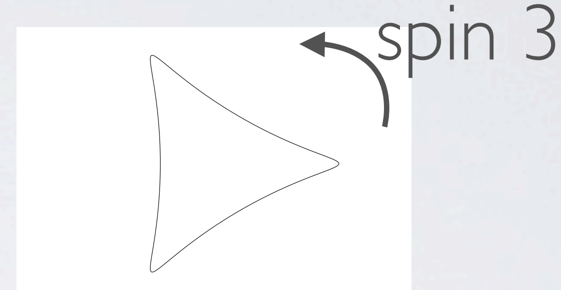
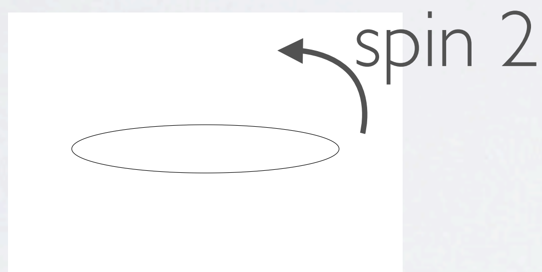
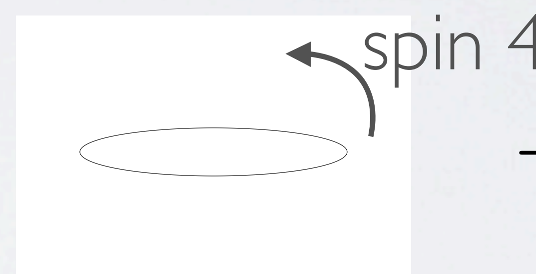
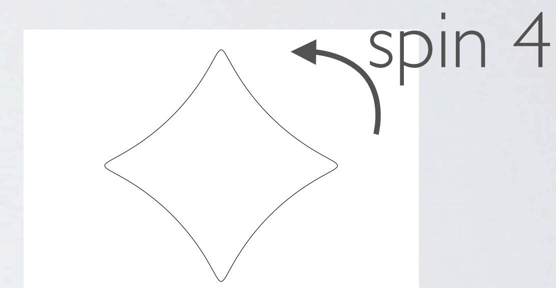
$\frac{\partial}{\partial w}$ 

$=$ 

Regular part of $\Big($ 

$\Big)^2$ $=$ 

$-$ 

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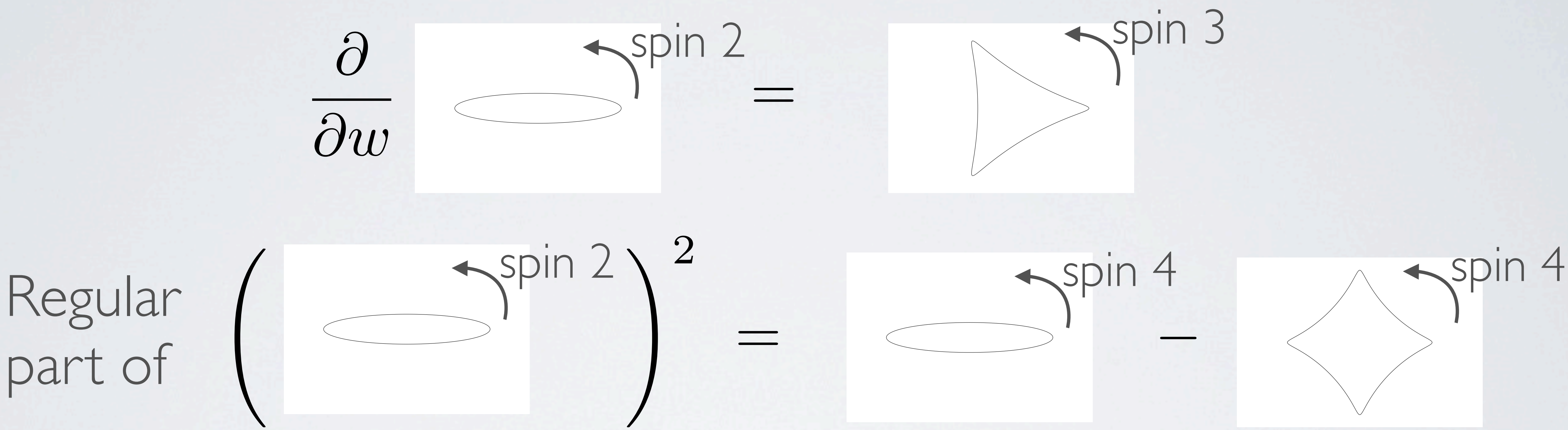
BD (2011,2012,2013)

This means that these small-scale wave-like fluctuations generate the structure of the *Virasoro vertex operator algebra*. For instance:



(Wilson's Operator Product Expansion)

Transfer of Fluctuations from small to large scales gives rise to the Virasoro (vertex operator) algebra

Hypotrochoids and Virasoro

Descendants of the stress-energy tensor were initially introduced in CFT as an algebraic structure useful in order to evaluate correlation functions of physical fields. Here we have for the first time a **statistical interpretation**: measuring fluctuations of small loops.

From small to large scales

BD (2011,2012,2013)

$$\mathbb{E}\big[\mathcal{T}(w)\,\mathcal{O}\big]_A = Z^{\rm id}\,\frac{\partial}{\partial\eta} Z^g\,\mathbb{E}\big[g\cdot\mathcal{O}\big]_{g\cdot A}\Big|_{\eta=0} \qquad g(z) = z + \frac{\eta}{w-z}$$

Random variable measuring macroscopic loops, e.g. $\mathtt{I}(N)$ with $g\cdot\mathtt{I}(N) = \mathtt{I}(g(N))$

The derivative is conjugated by the *relative partition function*:

$$Z^g := \frac{1}{\mathbb{E}\big[\mathtt{E}(g(\partial A))\big]_{g(\hat{\mathbb{C}}\setminus\text{neighborhood of } w)}}$$

This is a generalization of the conformal Ward identities to non-local observables.

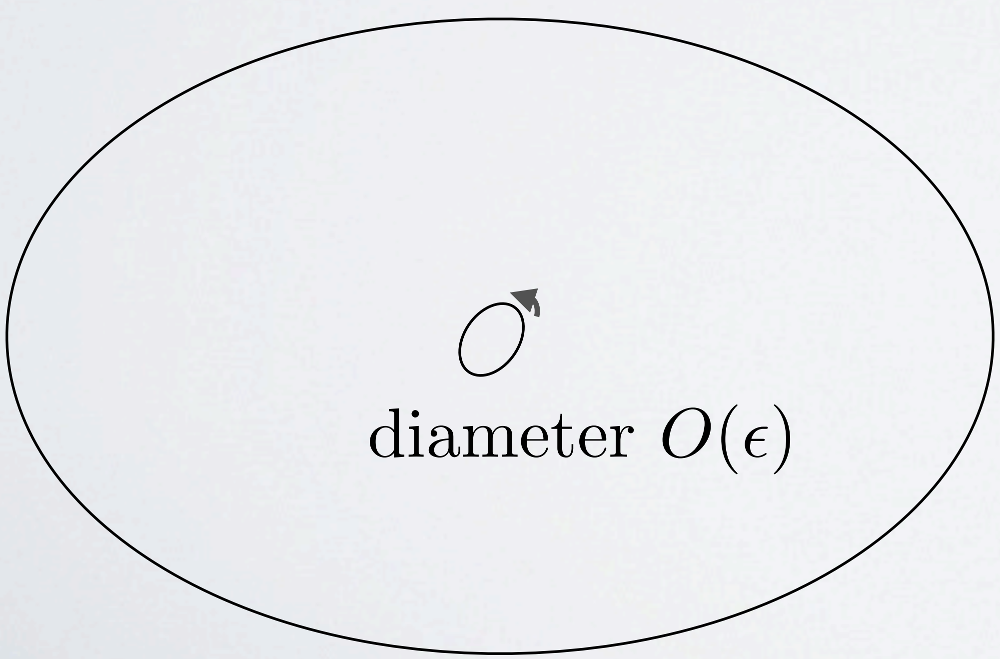
From small to large scales

$$\mathbb{E}\big[\mathcal{T}(w)\,\mathcal{O}\big]_A = Z^{\mathrm{id}}\;\frac{\partial}{\partial\eta} Z^{g}\,\mathbb{E}\big[g\cdot\mathcal{O}\big]_{g\cdot A}\Big|_{\eta=0} \qquad g(z) = z + \frac{\eta}{w-z}$$

Random variable measuring macroscopic loops, e.g. $\mathtt{I}(N)$

$$\mathcal{T}(w) := \lim_{\ell\to 0} \frac{1}{2\pi\epsilon^2}\int_0^{2\pi} d\theta\, e^{-2i\theta}\,\mathtt{E}(\alpha(w,\theta,\ell,e)) \qquad \left(\epsilon := \frac{\ell e}{2}\right)$$

fluctuations $O(\epsilon^2)$



From small to large scales

$$\mathbb{E}\big[\mathcal{T}(w)\,\mathcal{O}\big]_A = Z^{\mathrm{id}}\,\frac{\partial}{\partial\eta} Z^g\,\mathbb{E}\big[g\cdot\mathcal{O}\big]_{g\cdot A}\bigg|_{\eta=0} \qquad g(z) = z + \frac{\eta}{w-z}$$

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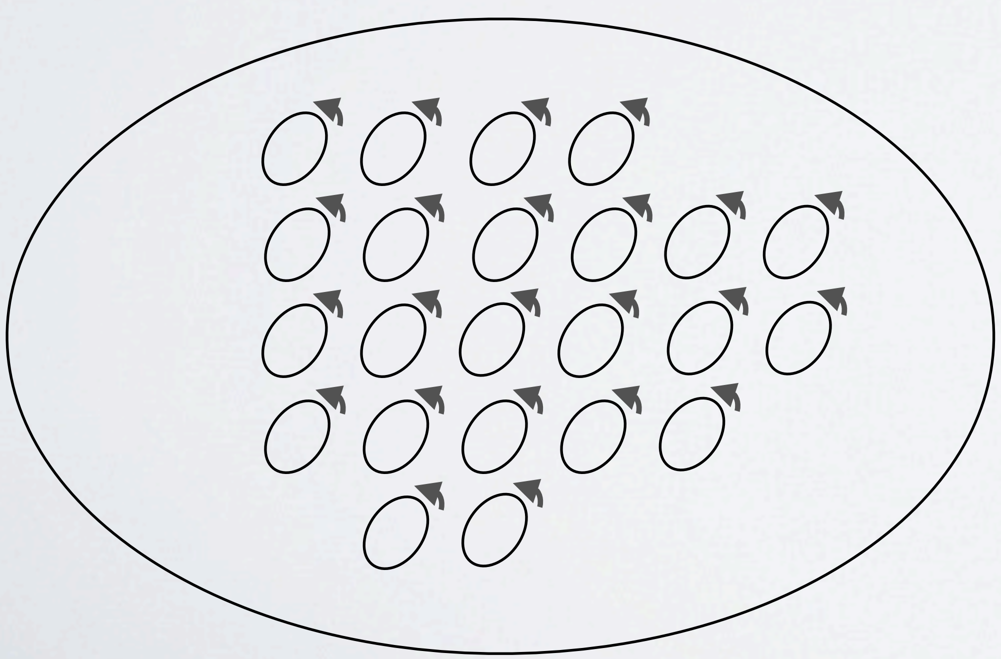
fluctuations $O(1)$



2-crest spin-2 small-loop fluctuations give macroscopic fluctuations of large loops.

From small to large scales

$$\mathbb{E}\big[\mathcal{T}(w)\,\mathcal{O}\big]_A = Z^{\mathrm{id}}\ \frac{\partial}{\partial\eta} Z^g\,\mathbb{E}\big[g\cdot\mathcal{O}\big]_{g\cdot A}\Big|_{\eta=0} \qquad g(z) = z + \frac{\eta}{w-z}$$

Random variable measuring macroscopic loops, e.g. $\mathtt{I}(N)$

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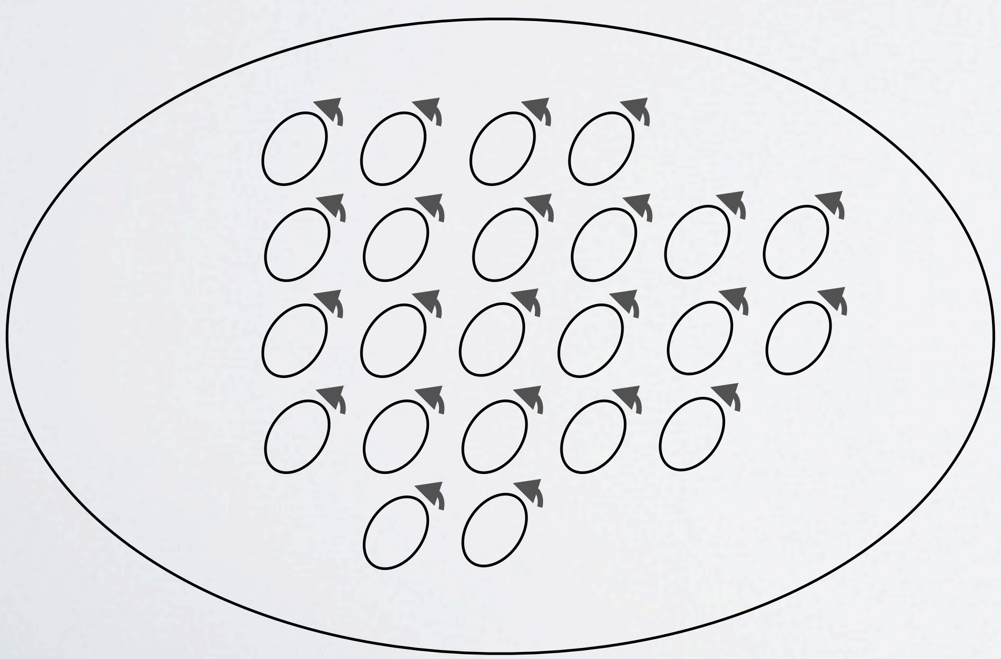
fluctuations $O(1)$



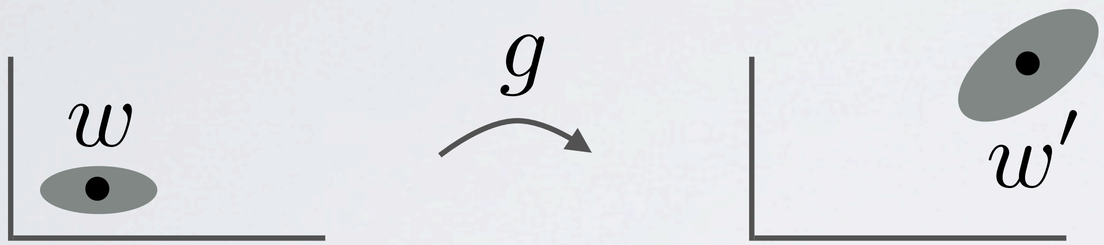
2-crest spin-2 small-loop fluctuations give macroscopic fluctuations of large loops

Higher number of crest / higher spin give finer fluctuations of large loops (fractals).

Conformal transformations

BD (2011,2012,2013)

The central charge appears because of the *renormalization* used in defining the variables $\mathcal{T}_{k,m}(w)$, which modifies their natural *conformal transformation properties*. Infinitesimally, we have:



$$w' = g(w)$$

$$\epsilon' e^{i\theta'} = \partial g(w) \epsilon e^{i\theta}$$

$$\alpha_k(w,\theta,\epsilon,b) \mapsto \alpha_k(w',\theta',\epsilon',b) + \text{corrections}$$

An extra factor is involved in the renormalized indicator variable:

$$\mathtt{E}(\alpha_k) \mapsto F\,\mathtt{E}(\alpha_k' + \text{corrections})$$

and the angle integral is further affected by the «corrections», giving, for instance, the extra term in

$$\mathcal{T}(w) \mapsto (\partial g(w))^2 \mathcal{T}(g(w)) + \frac{c}{12}\{g,w\}\mathbf{1}$$

Remarks

The construction is *not unique*: we may use a different definition (more like a Wilson loop) giving the same results; and any *local operator* transforming like the stress-energy tensor and vanishing on the unique disk should be a stress-energy tensor.

The construction gives more generally connections between loop fluctuations / shape correlations, the theory of manifold of conformal maps, and the theory of geometric vertex operator algebras of Y.-Z. Huang.

The formal derivation show that this may be applicable much more generally: CFT structure in other contexts?

Conclusion

At criticality in two dimensions, the bubbles representing boundaries between phases are well-defined objects at large scales; this picture is made precise thanks to conformal loop ensembles.

I have shown how the structures of conformal field theory can be extracted from these conformal random loops.

I have attempted to interpret my results as indicative of how universal aspects of small-scale fluctuations are transferred to larger scales.

To be done: generalization to other symmetry fields, understanding null-vector equations, applications beyond CLE....