The stress-energy tensor in conformal loop ensembles

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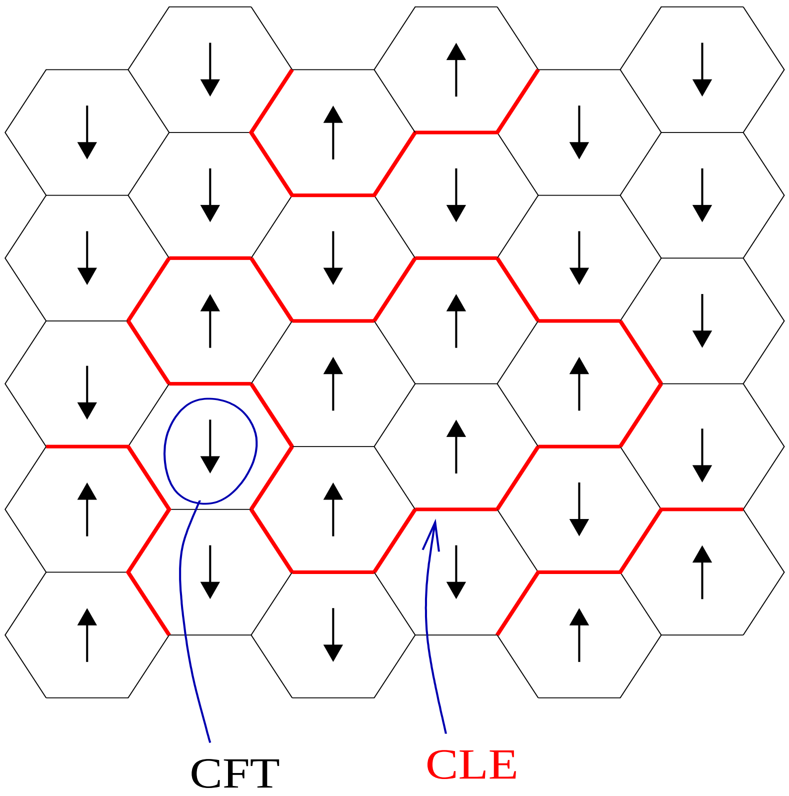
Statistical models (in two dimensions)

- Measures on functions σ from faces of a lattice (ex: hexagonal) to some set (ex: $\{\uparrow, \downarrow\}$).
- Locality, homogeneity: $\prod_{\text{faces } k} p(\sigma|_{\text{neighbourhood of } k})^{\beta}$.
- Criticality: $\beta = \beta_c$, correlation lengths become infinite. Two descriptions of large-distance behaviours:
  - conformal field theory, algebraic construction;
  - conformal loop ensembles, probabilistic/stochastic construction.



Conformal field theory

Consider $\uparrow = +1,\ \downarrow = -1$. At criticality:

$$\lim_{\varepsilon \to 0} \varepsilon^{-2d}\, \mathbb{E}[\sigma(x/\varepsilon)\sigma(y/\varepsilon)] = C(x,y)$$

(here, x, y are in $\mathbb{R}^2$). The coefficient $C(x,y)$ is a **correlation function in a CFT**

$$C(x,y) = \langle \mathcal{O}(x)\mathcal{O}(y) \rangle$$

The basic ingredients of CFT (and more generally QFT) are

- Local fields $\mathcal{O}(x) \Leftrightarrow$ local variables $1,\ \sigma(k),\ \sigma^2(k),\ \sigma(k)\sigma(\text{neighbour of } k),\ \ldots$
- correlation functions $\langle \cdot \rangle \Leftrightarrow$ expectations of products of local variables $\mathbb{E}[\cdot]$

- The main property of CFT is conformal invariance. Consider a transformation g conformal on a domain D. If $\mathcal{O}$ is "local enough," i.e. **primary**, it only feels the local translation, scaling and rotation, the latter two multiplicatively:

$$\langle \mathcal{O}(w)\mathcal{O}(z)\rangle_D = \left(g'(w)g'(z)\right)^h \left(\bar{g}'(\bar{w})\bar{g}'(\bar{z})\right)^{\bar{h}} \langle \mathcal{O}(g(w))\mathcal{O}(g(z))\rangle_{g(D)}$$

(here, w, z are in $\mathbb{C}$; $h, \bar{h}$ are in $\mathbb{R}^+$; and $'$ is the derivative). In particular, $h + \bar{h} = d$.

- This implies the existence of the **stress-energy tensor** $T(w)$, with **Ward identities**:

$$\langle T(w)\mathcal{O}(z)\cdots\rangle_D = \langle T(w)\rangle_D \langle \mathcal{O}(z)\cdots\rangle_D +$$
$$\left(\frac{h}{(w-z)^2} + \frac{1}{w-z}\frac{\partial}{\partial z} + \ldots\right)\langle \mathcal{O}(z)\cdots\rangle_D + \int_{\partial D} ds \frac{1}{w - \partial D(s)} \frac{\partial}{\partial(\partial D(s))}\langle \mathcal{O}(z)\cdots\rangle_D$$

- From explicit calculations in some models: T is not a primary field, there is a **central charge** $c \in \mathbb{R}$:

$$T(w) \mapsto g'(w)^2 T(g(w)) + \frac{c}{12}\{g, w\}, \quad \{g, w\} = \left(\frac{\partial^3 g(w)}{\partial g(w)} - \frac{3}{2}\left(\frac{\partial^2 g(w)}{\partial g(w)}\right)^2\right)$$

The algebraic structure of CFT [Kac, Lepowsky, ..., Cardy, Zamokodchikov, ...; 1980 –]

- Virasoro algebra:

$$T(w)\mathcal{O}(z) = \sum_{n\in\mathbb{Z}} (w-z)^{-n-2}(L_n\mathcal{O})(z)$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$$

- "Chiral part" of local fields: elements of **highest weight modules for the Virasoro algebra** – more generally, for a **vertex operator algebra** – characterised by the weight h

- Correlation functions: modular invariant chiral–anti-chiral pairings of "Clebsch-Gordon coefficients" of tensor products of VOA modules.

Conformal fields, SLE, and CLE

- Conformal fields as described correspond to **local observables of the statistical model**.
- All **correlation functions** of conformal fields can be obtained from SLE$_\kappa$: **appropriate martingales** [Bauer, Bernard 2002 –].
- But generically conformal fields are **not local observables of the random curve in SLE**.
- Some fields do correspond to local SLE observables: stress-energy tensor ($\kappa = 8/3$) and $U(1)$-current ($\kappa = 4$) [D., Riva, Cardy 2006], parafermions (some $\kappa > 4$) [Riva, Rajabpour, Cardy; 2006 –].
- For more general fields and arbitrary central charge, **we need all the loops: CLE**.
- Only for $\kappa = 8/3$ is there a local weight function describing the SLE curve.
- One can also **extend the CFT algebraic structure** in order to describe natural SLE observables [Cardy, Watts, Mathieu, Ridout, Simmons].

Constructing the stress-energy tensor [D., Riva, Cardy (2006)]

- Consider the algebraic definition of the stress-energy tensor from the **identity field** $\mathbf{1}$:

$$(L_{-2}\mathbf{1})(w) = T(w)$$

- Interpret geometrically:

The stress-energy tensor is the result of a conformal transformation that preserves ∞,
on a simply connected domain that excludes the point w,
whose extension to w has a simple pole at that point.

- Hence the stress-energy tensor should be obtained from the conformal transformation

$$f(z) = z + \frac{\varepsilon^2 e^{2i\theta}}{16(w-z)}$$

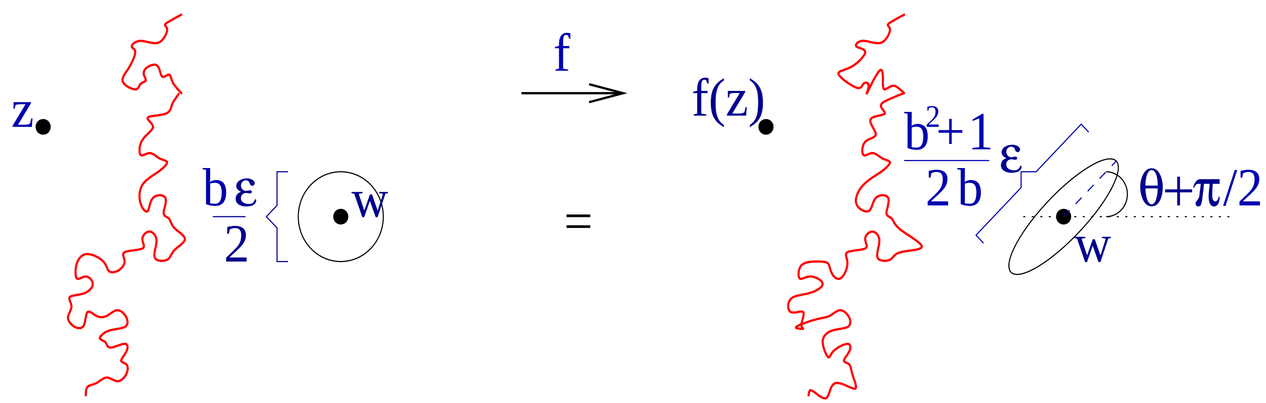
for ε small.

Conformal restriction?

Consider a statistical model on $\mathbb{C}$, with a random set Γ (e.g.: the loops of the $O(n)$ model).

If we restrict the model to nothing intersecting the boundary of a domain $B \subset \mathbb{C}$, then this should be equivalent to looking at the model on $\mathbb{C} \setminus B$.

- $D(w,\varepsilon)$: the disk of diameter $b\varepsilon/2$ centered at w, for some $b>1$
- $E(w,\varepsilon,\theta)$: the ellipse obtained from $f(\mathbb{C}\setminus D(w,\varepsilon)) = \mathbb{C}\setminus E(w,\varepsilon,\theta)$.
- $X(z,\ldots)$: e.g. event that at least one loop winds in a certain way around points $z,\ldots$
- $Y(w,\varepsilon,\theta)$: the event $\Gamma\cap\partial E(w,\varepsilon,\theta)=\emptyset$



$$\begin{aligned} P(X(z,\ldots))_{\mathbb{C}\setminus D(w,\varepsilon)} &= P(X(f(z),\ldots))_{\mathbb{C}\setminus E(w,\varepsilon,\theta)} \\ &= P(X(f(z),\ldots)\,|\,Y(w,\varepsilon,\theta))_{\mathbb{C}} \\ &= \frac{P(X(f(z),\ldots)\cap Y(w,\varepsilon,\theta))_{\mathbb{C}}}{P(Y(w,\varepsilon,\theta))_{\mathbb{C}}} \end{aligned}$$

$$\begin{aligned} &\Rightarrow P(Y(w,\varepsilon,\theta))_{\mathbb{C}} P(X(z,\ldots))_{\mathbb{C}\setminus D(w,\varepsilon)} \\ &\quad = \left(1 + \frac{\varepsilon^2 e^{2i\theta}}{16(w-z)} \frac{\partial}{\partial z} + \text{c.c.} + \ldots\right) P(X(z,\ldots)\cap Y(w,\varepsilon,\theta))_{\mathbb{C}} \\ &\quad = P(X(z,\ldots)\cap Y(w,\varepsilon,\theta))_{\mathbb{C}} + \left(\frac{\varepsilon^2 e^{2i\theta}}{16(w-z)} \frac{\partial}{\partial z} + \text{c.c.} + \ldots\right) P(X(z,\ldots))_{\mathbb{C}} \end{aligned}$$

Hence, the stress-energy tensor is the "insertion of a small rotating avoided ellipse":

$$\Rightarrow -\lim_{\varepsilon\to 0} \frac{8}{\pi\varepsilon^2} \int_0^{2\pi} d\theta e^{-2i\theta} P(X(z,\ldots)\cap Y(w,\varepsilon,\theta))_{\mathbb{C}} = \left(\frac{1}{w-z}\frac{\partial}{\partial z} + \ldots\right) P(X(z,\ldots))_{\mathbb{C}}$$

Problem: zero central charge!

For small ε, only translation, rotation and scaling affect the rotating ellipse:

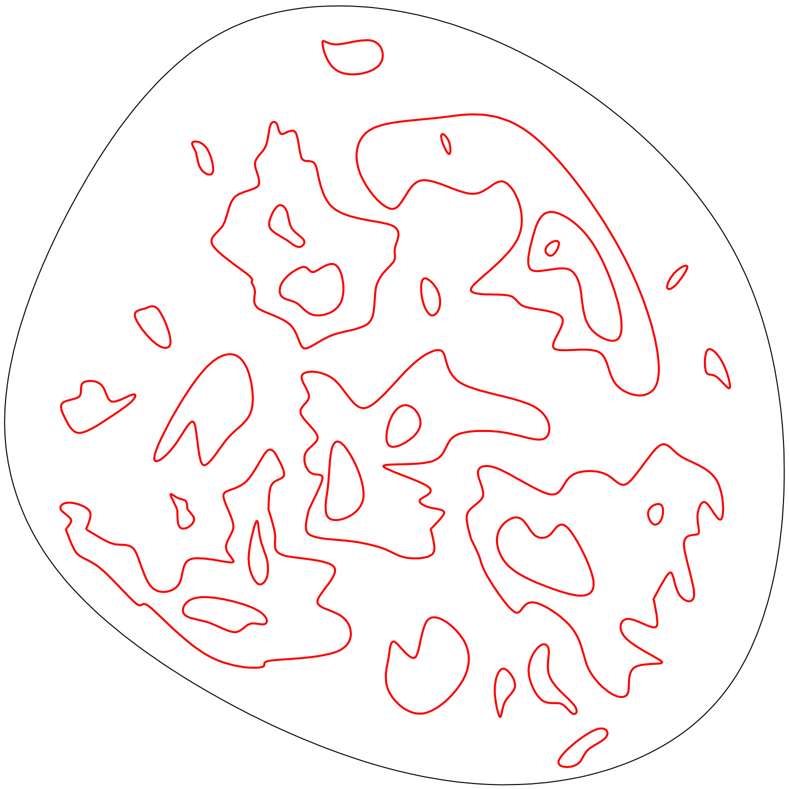
$$-\lim_{\varepsilon\to 0}\frac{8}{\pi\varepsilon^2}\int_0^{2\pi} d\theta e^{-2i\theta} P(X(z,\ldots)\cap Y(w,\varepsilon,\theta))_C$$
$$= -(\partial g(w))^2 \lim_{\varepsilon\to 0}\frac{8}{\pi\varepsilon^2}\int_0^{2\pi} d\theta e^{-2i\theta} P(X(g(z),\ldots)\cap Y(g(w),\varepsilon,\theta)))_{g(C)}$$

(can be shown by providing an appropriate modification of $f(z)$ in order to produce a conformally transformed ellipse from $\mathbb{C}\setminus D(w,\varepsilon)$).

Construction works for the **true restriction measure** [Lawler, Schramm, Werner 2003] SLE$_{8/3}$ [D., Riva, Cardy 2006], generalises the construction for the boundary stress-energy tensor [Friedrich, Werner (2003)]. In this case, the central charge indeed is zero.

Conformal loop ensembles [Sheffield, Werner 2005 –]

Consider the set $\mathcal{S}_D$ whose elements are collections of at most a countable infinity of self-avoiding, disjoint loops lying on a simply connected domain D.



A conformal loop ensemble can be seen as a family of measures μ_D on the sets $\mathcal{S}_D$ for all simply connected domains D, with the **three properties**.

1. Conformal invariance.

For any conformal transformation $f : D \longrightarrow D'$,

$$\mu_D = \mu_{D'} \cdot f$$

- Expected, but hard to prove from statistical models.
- The only non-trivial condition is for $f : D \longrightarrow D$ for one given canonical domain D. The rest is definition for other domains.

2. Nesting.

The measure μ_D restricted on a loop $\gamma \subset D$ and on all loops outside γ is equal to the CLE measure μ_{D_γ} on the domain $D_\gamma \subset D$ delimited by γ (i.e. with $\partial D_\gamma = \gamma$).

- This is (usually) simple to see from statistical models.
- It says that the inner boundary of a loop is like the boundary of a domain.
- It can be implemented by an iterative construction: first construct a measure on outer loops only, then inside every outer loop put the conformally transported measure, etc.

3. Conformal restriction.

Given a domain $B \subset D$ such that $D \setminus B$ is simply connected, consider $\tilde{B}$, the closure of the set of points of B and points that lie inside loops that intersect B. Then the measure on each component C_i of $D \setminus \tilde{B}$, obtained by restriction on loops that intersect B, is μ_{C_i}.

- This is again (usually) simple to see from statistical models.
- It is “trying” to say two things:

1. the outer boundary of a loop is like a domain boundary;
2. the measure restricted on no loop crossing ∂B is a product of independent CLE measures;

neither of which can be exactly implemented as conditions on μ_D. For the first: requires CLE on non-simply connected domains. For the second: impossible because around any point there is a.s. infinitely many loops.

Relations between CLE and SLE

- In both cases, there is a condition of conformal invariance and conditions saying that the curves are like domain boundaries.
- There is a construction of a family of CLE's parametrised by a parameter κ for $8/3 < \kappa < 4$ which have the property that locally, the loops look like SLE$_\kappa$.
- Both CLE and SLE$_{8/3}$ have (slightly different) conformal restriction properties. In both cases, curves/loops are everything there is.

The stress-energy tensor with non-zero central charge [D.]

Infinitely many small loops: modify conformal restriction, give rise to non-zero central charge.

Basic idea: suppose we have a "regularised" probability $P^{\rm reg}(\{z,\ldots\};A)_D$ depending on a simply connected domain A (disjoint from $z,\ldots$), which in a sense imposes that no loop crosses ∂A. Suppose it has the following properties:

$$\begin{aligned} P^{\rm reg}(X(g(z),\ldots);g(A))_{g(D)} &= f(g,A)P^{\rm reg}(X(z,\ldots);A)_D \\ P^{\rm reg}(X(g(z),\ldots);g(A))_{g(D)} &= P^{\rm reg}(X(z,\ldots);A)_D \quad \text{for } g \text{ conformal on } \mathbb{C}+\{\infty\} \\ \frac{P^{\rm reg}(X(z,\ldots);A)_D}{P^{\rm reg}(A)_D} &= P(X(z,\ldots))_{D\setminus A} \end{aligned}$$

Then, **we have the Ward identities...**

$$-\lim_{\varepsilon\to 0}\frac{8}{\pi\varepsilon^2}\int_0^{2\pi} d\theta e^{-2i\theta} P^{\rm reg}(X(z,\ldots);E(w,\varepsilon,\theta))_{\mathbb{C}} = \left(\frac{1}{w-z}\frac{\partial}{\partial z}+\ldots\right)P(X(z,\ldots))_{\mathbb{C}}$$

... and the correct transformation properties:

- Fourier decomposition: $f(g, E(w, \varepsilon, \theta)) = \sum_{n\in\mathbb{Z}} f_{2n}(g, w, \varepsilon) e^{2ni\theta}$
- Fourier transform of the transformation equation:

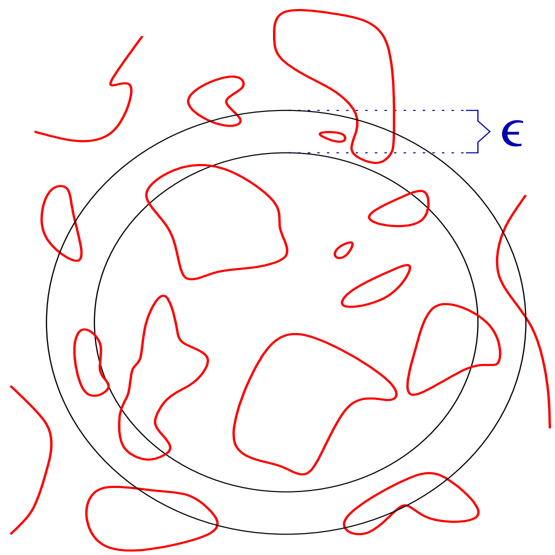
$$\int_0^{2\pi} d\theta e^{-2i\theta} P^{\mathrm{reg}}(X(g(z), \ldots); g(E(w, \varepsilon, \theta)))_{g(D)}$$
$$= \int_0^{2\pi} d\theta e^{-2i\theta} f(g, E(w, \varepsilon, \theta)) P^{\mathrm{reg}}(X(z, \ldots); E(w, \varepsilon, \theta))_D$$

- Deduce that $f_2(g, w, \varepsilon) \sim \varepsilon^2 f_2(g, w)$, with $f_2(g, w)$ holomorphic in w, and the correct transformation properties with central term $f_2(g, w)/4$.
- Automorphic factor: $f(h \circ g, A) = f(g, A) f(h, g(A))$
- Infinitesimally $f_2(h \circ g, w) = f_2(g, w) + (\partial g(w))^2 f_2(h, g(w))$, with constraint $f_2(g, w) = 0$ for g conformal on $\mathbb{C} + \{\infty\}$ (global conformal transformation), has solution $f_2(g, w) = (c/3)\{g, w\}$ (for some c) – the Schwarzian derivative.

Construction in CLE

Consider a family of events $\mathcal{E}(A, \epsilon)$ characterised by any simply connected domains A and any small enough real numbers $\epsilon > 0$, defined as follows:

- For $A = \mathbb{D}$, it is the event that no loop intersect both $(1-\epsilon)\partial\mathbb{D}$ and $\partial\mathbb{D}$.



- For $A \neq \mathbb{D}$, it is the event $g_A(\mathcal{E}(\mathbb{D}, \epsilon))$, where the conformal transformations g_A is chosen such that $A = g_A(\mathbb{D})$, and such that if $A = G(B)$ for some global conformal transformation G, then there is a global conformal transformation K with $K(B) = B$ such that $g_A = G \circ K \circ g_B$.

The regularised probability can be defined by

(for A simply connected and disjoint from $z, \ldots$):

$$P^{\mathrm{reg}}(X(z,\ldots);A)_D = \mathcal{N} \lim_{\epsilon\to 0} \frac{P(X(z,\ldots)\cap \mathcal{E}(A,\epsilon))_D}{P(\mathcal{E}(\mathbb{D},\epsilon))_{2\mathbb{D}}}$$

"Theorems:"

- The limit exists.
- The ratio $\dfrac{P^{\mathrm{reg}}(X(z,\ldots);A)_D}{P^{\mathrm{reg}}(A)_D}$, as a function of $z, \ldots$ and ∂A, is invariant under any transformation conformal on $D \setminus A$.
- The ratio $\dfrac{P^{\mathrm{reg}}(X(g(z),\ldots);g(A))_{g(D)}}{P^{\mathrm{reg}}(X(z,\ldots);A)_D}$ is independent of $z, \ldots$ and of D, and is 1 if g is a global conformal transformation.

“Theorem.” Any two “local” objects that are zero on the unit disk and transform like the stress-energy tensor, have the same “correlation functions”.

Corolary. Any “local” object that is zero on the unit disk and transforms like the stress-energy tensor, satisfies the conformal Ward identities.

Conclusion

We have constructed an object (the limit of an integral of the limit of a ratio of probabiilities...) that satisfies the conformal Ward identities, and that transforms like the stress-energy tensor.

- Can we re-construct the vertex operator algebra from this?
- What are the events / objects corresponding to rational modules?
- Can we repeat the construction on surfaces of arbitrary genus?