Exact low-energy results for non-equilibrium steady-states

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based on work in preparation with Denis Bernard

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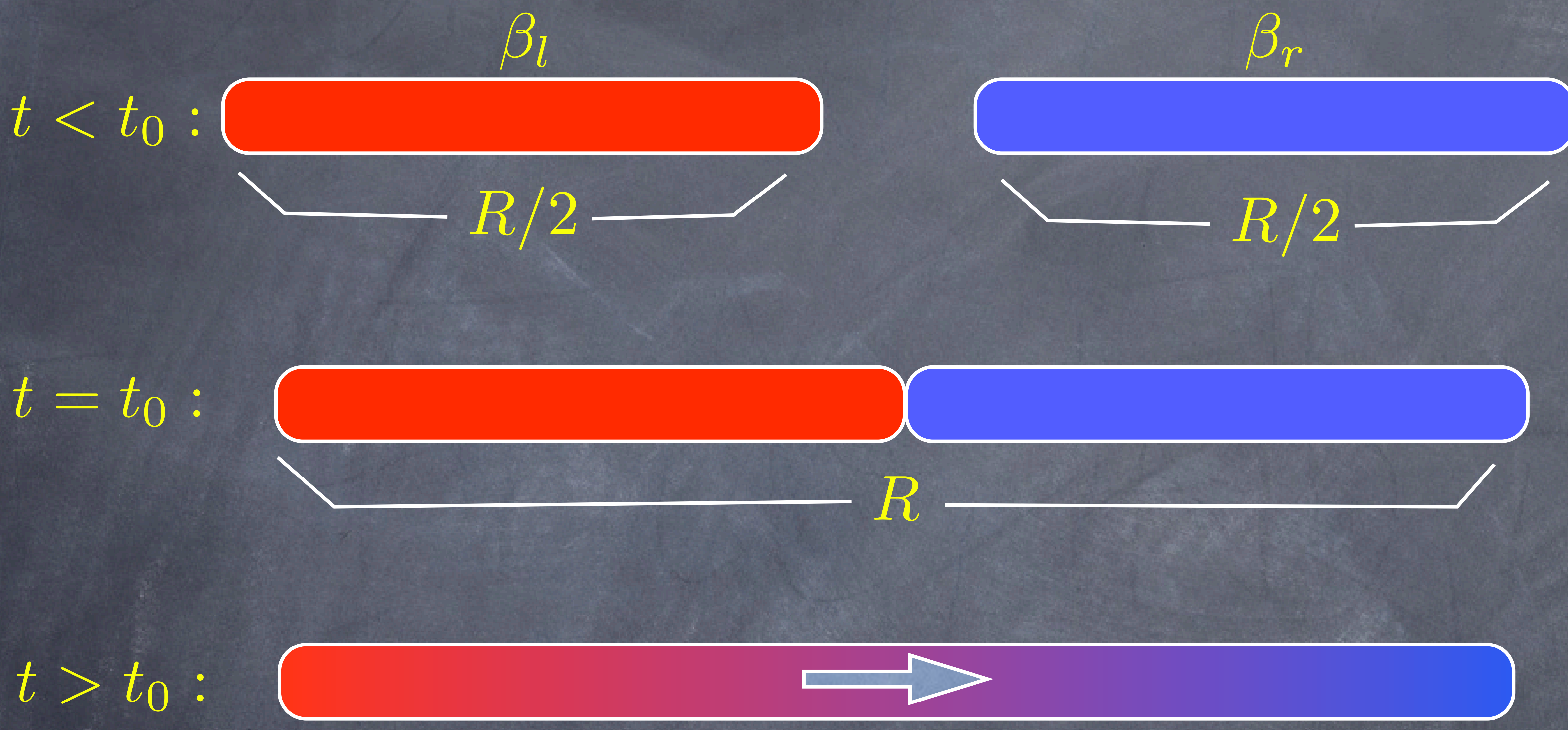
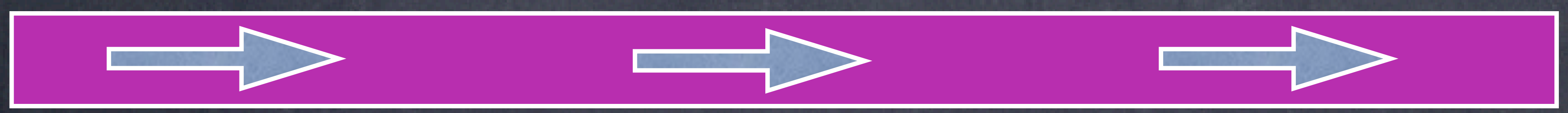
Physical situation



$$\frac{R}{v_F} \gg t - t_0 \gg \frac{\hbar \beta_{r,l}}{k_B}, \ldots :$$



Physical situation

$$\langle \cdots \rangle_{\text{ness}} = \lim_{t_0 \to -\infty} \lim_{R \to \infty} \frac{\mathrm{Tr}\left(e^{iHt_0}\rho_0 e^{-iHt_0} \ldots\right)}{\mathrm{Tr}\left(\rho_0\right)}$$

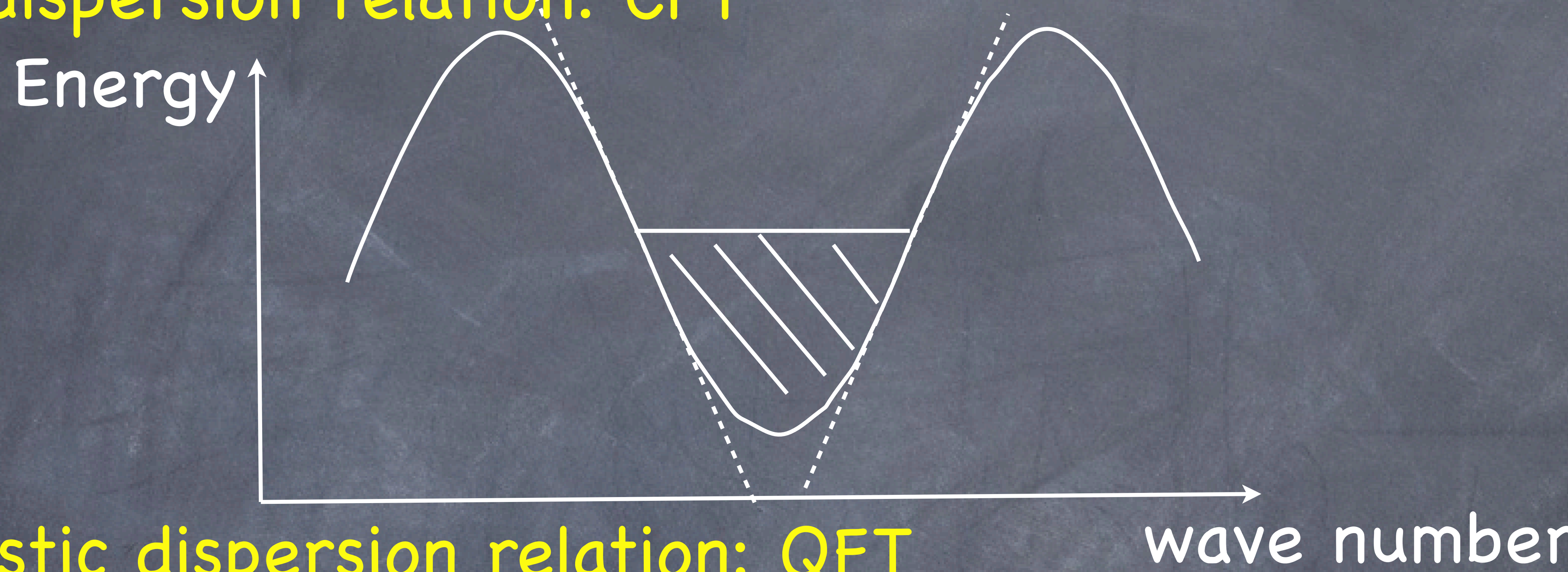
$$\rho_0 = e^{-\beta_l H^l - \beta_r H^r}$$

$$H = H^l + H^r + H_{\text{contact}}$$

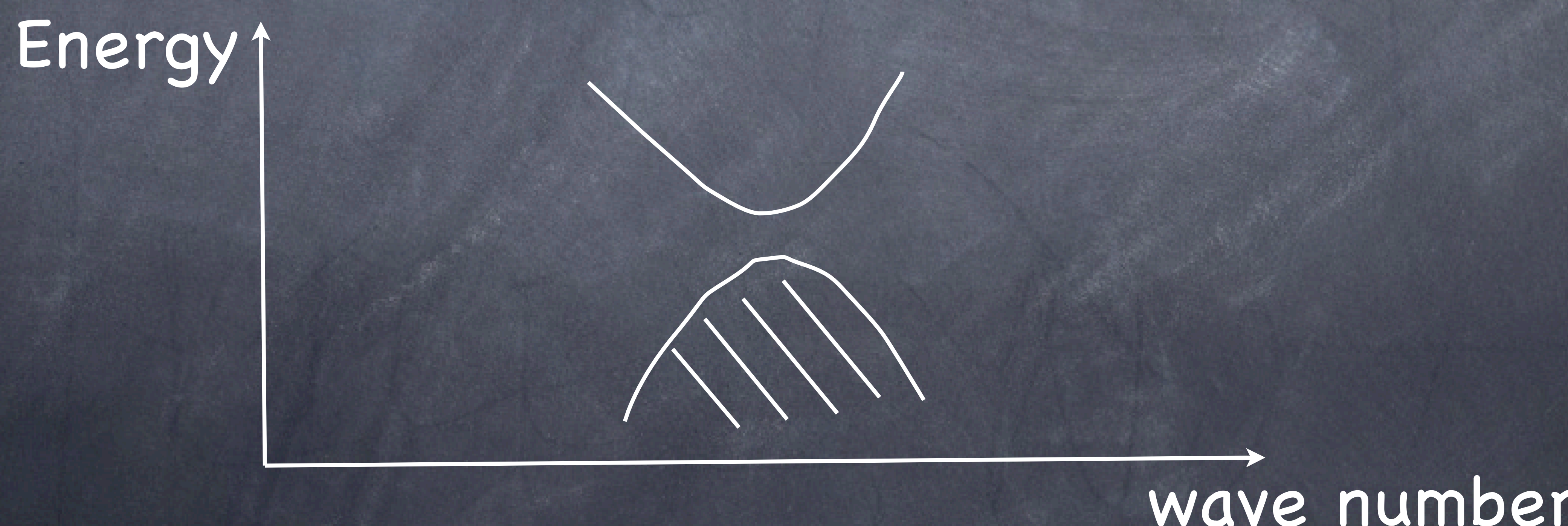
Observables supported on a finite region

Scaling limit: (relativistic) QFT

Linear dispersion relation: CFT



Relativistic dispersion relation: QFT



Description of the steady state

$$\langle \cdots \rangle_{\text{ness}} = \frac{\mathrm{Tr}\left(e^{-Y} \cdots\right)}{\mathrm{Tr}\left(e^{-Y}\right)}$$

Operator Y :

- Commutes with the Hamiltonian H
- «Asymptotically looks like» $\beta_l H^l + \beta_r H^r$

★ Formal definition first proposed by Hershfield (PRL 1993) (case where both temperatures are the same and something else is flowing, like a charge)

★ Studied widely for charge transfer in impurity systems

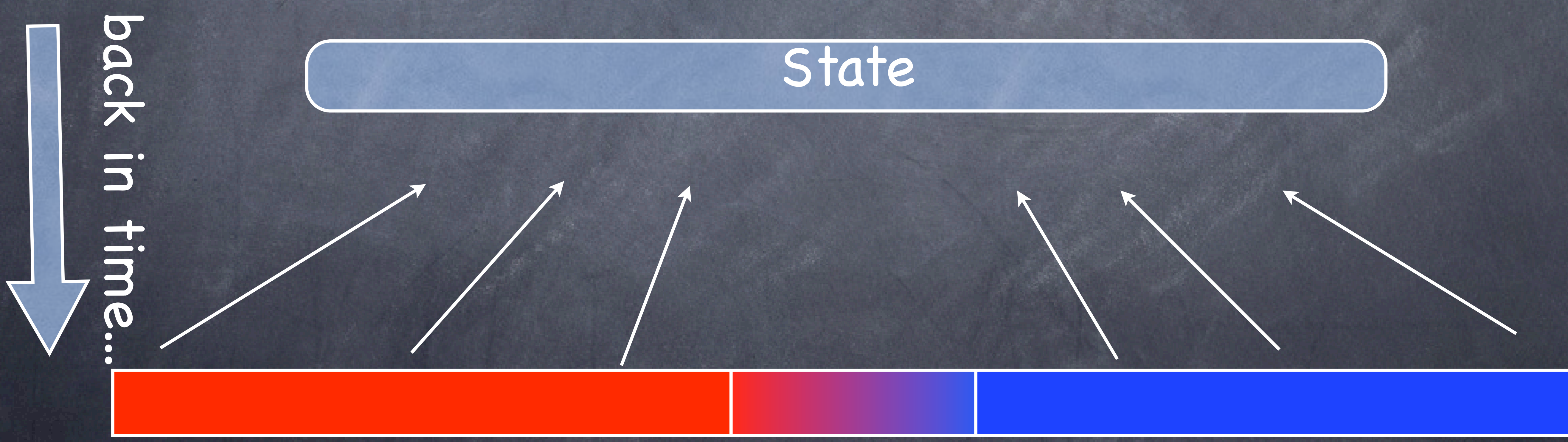
Description of the steady state

[DB & BD]

$$Y = \beta_l \int_0^\infty d\theta E_\theta A(\theta)^\dagger A(\theta) + \beta_r \int_{-\infty}^0 d\theta E_\theta A(\theta)^\dagger A(\theta)$$

Total energy of right-moving asymptotic particles

Total energy of left-moving asymptotic particles



The energy current

$$J = \langle p^1(x) \rangle_{\text{ness}}$$

$$= \lim_{R\to\infty} R^{-d} \frac{\mathrm{Tr}_R\left(e^{-Y} P^1\right)}{\mathrm{Tr}_R\left(e^{-Y}\right)}$$

$$= \lim_{R\to\infty} R^{-d} \frac{\mathrm{Tr}_R\left(e^{-Y_+} P^1_+\right)}{\mathrm{Tr}_R\left(e^{-Y_+}\right)} + \lim_{R\to\infty} R^{-d} \frac{\mathrm{Tr}_R\left(e^{-Y_-} P^1_-\right)}{\mathrm{Tr}_R\left(e^{-Y_-}\right)}$$

$$\mathcal{H} = \mathcal{H}_+ \otimes \mathcal{H}_-$$

$$Y = Y_+ + Y_-$$

$$P^1 = P^1_+ + P^1_-$$

The energy current

Using the fact that the energy is unchanged under change of sign of a momentum component:

$$J = f(\beta_l) - f(\beta_r)$$

Towards the conformal (gapless) point:

$$J = \alpha\, m^{d-1}(\beta_l^{-2} - \beta_r^{-2})$$

1D: the CFT central charge

[DB & BD]

$$J = \frac{\pi c}{12}(\beta_l^{-2} - \beta_r^{-2}) = \frac{\pi c k_B^2}{12\hbar}(T_l^2 - T_r^2)$$

central charge

$$T(x) = -\frac{c}{24} + \sum_{n \in Z} L_n e^{-\frac{2\pi i n x}{R}}$$

Virasoro

$$H = \int dx\,(h_+(x) + h_-(x)) \qquad h_+(x) = \frac{2\pi}{R^2}T(x)$$

$$H^{l,r} = \int dx\,(h_+^{l,r}(x) + h_-^{l,r}(x)) \qquad h_-(x) = \frac{2\pi}{R^2}\bar{T}(x)$$

$$J = \langle h_+(x) - h_-(x) \rangle_{\text{ness}}$$

1D: the CFT central charge

Using the fact that

$$h_{\pm}(x) = \begin{matrix} h^l_{\pm}(x) & (x < 0) \\ h^r_{\pm}(x) & (x > 0) \end{matrix}$$

and

$$\rho_0 = e^{-\beta_l H^l - \beta_r H^r}$$

we find

$$Y = \frac{2\pi\beta_l}{R} L_0 - \frac{2\pi\beta_r}{R} \bar{L}_0$$

1D: the CFT central charge

Hence:

$$J = f(\beta_l) - f(\beta_r), \quad f(\beta) = -\lim_{R\to\infty} \frac{1}{R}\frac{d}{d\beta}\log Z(\beta)$$

where

$$Z(\beta) = \mathrm{Tr}\left(e^{-\frac{2\pi\beta}{R}L_0}\right)$$

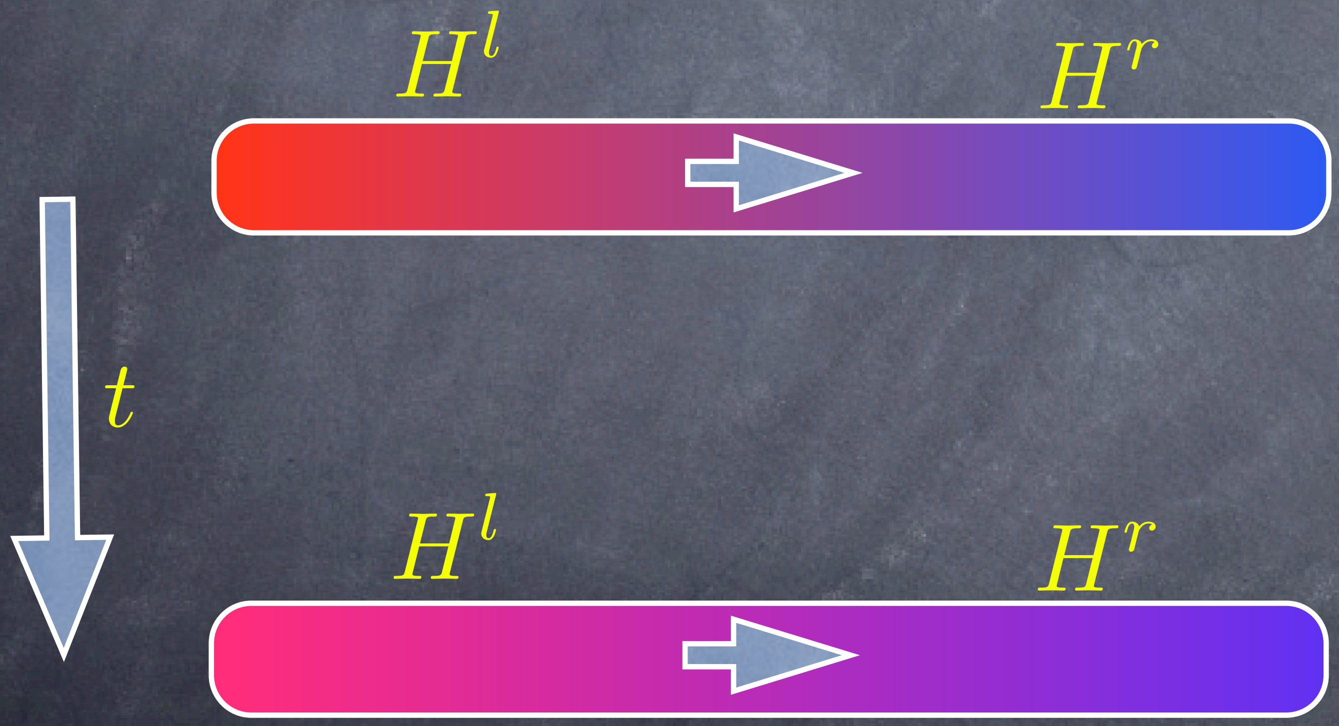
and we can use

$$Z(\beta) \sim N\, e^{\frac{\pi c R}{12\beta}}$$

Fluctuations of the energy transfer

We want to measure the fluctuations of the transfer of energy, whose «charge» can be taken as:

$$Q = \frac{1}{2}\left(H^l - H^r\right)$$



$$Q = q_0$$

$$Q = q_0 + q$$

Fluctuations of the energy transfer

$$\frac{e^{-Y}}{\mathrm{Tr}\,(e^{-Y})}$$

$$P(q,t) = \sum_{q_0} \mathrm{Tr}\left(P_{q_0+q} e^{-iHt} P_{q_0} \rho_{\mathrm{ness}} P_{q_0} e^{iHt} P_{q_0+q}\right)$$

$$P(\lambda,t) = \sum_{q} e^{i\lambda q} P(q,t)$$

$$\boxed{\log P(\lambda,t) \sim t F(\lambda) + O(1)}$$

Cumulant generating function

$$= -i\lambda J + \ldots$$

An expected fluctuation relation

$$F(\lambda) = F(i(\beta_l - \beta_r) - \lambda)$$

Equivalent to:

$$P(q, t \to \infty) = e^{(\beta_l - \beta_r)q} P(-q, t \to \infty)$$

Such a relation was argued for first measurement at $t = t_0$

Jarzynski, Wojcik (PRL 2004)

See the nice review by: Esposito, Harbola, Mukamel (RMP 2009)

More rigorous proof given in: Andrieux, Gaspard, Monnai, Tasaki (2008)

Basic ideas: Gallavoti, ...

The full counting statistics in CFT

Recall:

$$P(\lambda,t)=\sum_{q,q_0}e^{i\lambda q}\mathrm{Tr}\left(P_{q_0+q}e^{-iHt}P_{q_0}\rho_{\mathrm{ness}}P_{q_0}e^{iHt}P_{q_0+q}\right)$$

Use $\sum_q f(q)P_q=f(Q)$ and $P_q\propto\int d\mu\, e^{i\mu(Q-q)}$

$$F(\lambda)=\lim_{t\to\infty}t^{-1}\log\left[\lim_{t_0\to-\infty}\lim_{R\to\infty}\int d\mu\ \star\ \right]$$

$$\star = \frac{\mathrm{Tr}\left(\rho_0(t_0)\,e^{-i(\frac{\lambda}{2}+\mu)Q}e^{i\lambda Q(t)}e^{-i(\frac{\lambda}{2}-\mu)Q}\right)}{\mathrm{Tr}\,\rho_0(t_0)}$$

The full counting statistics in CFT

Parenthesis: charge transfer in free-fermion systems

- Cumulant generating function known in terms of transmission matrix: Lesovik-Levitov formula (1993,1994) (also: Klich, Schonhammer, DB & BD, . . .)
- It is observed that the same result is obtained with any fixed μ

Hence we expect to get the same result with:

$$\boxed{\frac{\mathrm{Tr}\left(\rho_0(t_0)\, e^{i\lambda Q(t)} e^{-i\lambda Q}\right)}{\mathrm{Tr}\,\rho_0(t_0)}}$$

The full counting statistics in CFT

$$e^{i\lambda Q(t)}e^{-i\lambda Q} = e^{i\lambda Q + i\lambda \int_0^t dx\,(h_-(x) - h_+(-x))}e^{-i\lambda Q}$$

Supported on a finite region

Finitely-supported observable, can use Y-operator, get factorization:

$$F(\lambda) = f(\lambda, \beta_l) + f(-\lambda, \beta_r)$$

$$f(\lambda, \beta) = \left\langle e^{i\lambda\left(-\frac{\pi}{R} + \frac{2}{R}\sum_{n\in Z} L_n \frac{\sin\frac{\pi n t}{R}}{n}\right)}\right\rangle_{\beta - \frac{i\lambda}{2}} \left\langle e^{i\lambda\frac{\pi L_0}{R}}\right\rangle_{\beta}$$

The full counting statistics in CFT

[DB & BD]

$$F(\lambda) = \frac{i\lambda\pi c}{12}\left(\frac{1}{\beta_r(\beta_r - i\lambda)} - \frac{1}{\beta_l(\beta_l + i\lambda)}\right)$$

Using dimensional analysis, unique solution to:

- Factorization $F(\lambda) = f(\lambda, \beta_l) + f(-\lambda, \beta_r)$
- Leading behaviour $F(\lambda) = O(\lambda)$
- Fluctuation relation

A stochastic interpretation

Independent Poisson processes for jumps of every energy E, positive or negative, with intensity

$$dE\, e^{-\beta_l E} \qquad (E > 0)$$
$$dE\, e^{\beta_r E} \qquad (E < 0)$$

Conclusion and perspectives

- Complete proof
- Generalization to presence of conformal impurity, to integrable models (thermodynamic Bethe ansatz), to charge currents, etc.
- Higher dimensions: same formula?
- Stochastic re-interpretation of CFT? of QFT?