Conformal field theory and Schramm-Loewner evolution

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Two possible approaches to relating CFT and SLE (or more generally, CFT and conformally invariant random processes in the continuum):

- Getting information about SLE from CFT [Bauer, Bernard (2002,...)]
- Constructing CFT from continuum random processes [Cardy, D., Riva (2006, ...)]

I will concentrate on the second.

Conformal field theory

A physical theory to describe the **scaling limit** of 2-dimensional statistical models at a **second order phase transition** (criticality).

Statistical models:

$$Z = \sum_{\{\sigma_i\}} e^{-\sum_{(i,j)} H(\sigma_i,\sigma_j)/T}\ , \langle\langle \sigma_k \sigma_l \rangle\rangle = Z^{-1} \sum_{\{\sigma_i\}} \sigma_k \sigma_l e^{-\sum_{(i,j)} H(\sigma_i,\sigma_j)/T}$$

For certain choices of H and T, the system is **critical**:

$$\langle\langle \sigma_{x/\varepsilon} \sigma_{y/\varepsilon} \rangle\rangle \overset{\varepsilon \to 0}{\sim} \varepsilon^{2d} C(x,y)$$

The coefficient $C(x,y)$ is a **correlation function in a CFT**

$$C(x,y) = \langle \mathcal{O}(x) \mathcal{O}(y) \rangle$$

The basic ingredients of CFT are

- Local fields $\mathcal{O}(x) \Leftrightarrow$ local variables of a statistical model $\sigma_i,\ \sigma_i^2,\ \sigma_i \sigma_{i+1},\ \ldots$
- correlation functions $\langle \cdot \rangle \Leftrightarrow$ averages of products of local variables $\langle\langle \cdot \rangle\rangle$

CFT possesses global conformal invariance

- Conformal group: group of transformations of 2-d domains, $D \longrightarrow D'$, that preserve the angles everywhere $\Rightarrow$ holomorphic/anti-holomorphic maps $f(z), f(\bar{z})$

- Symmetries: transformations preserve the domain D on which the system is defined

- Riemann sphere $\mathbb{R}^2 + \{\infty\}$ – global conformal invariance: symmetries are **translations** $z \mapsto z + \varepsilon$, **rotations, scaling** $z \mapsto z + \varepsilon z$, **special conformal transformation** $z \mapsto z + \varepsilon z^2$

- Conserved currents: stress-energy tensor components $T \equiv T_{zz}$, $\bar{T} \equiv T_{\bar{z}\bar{z}}$ satisfy $\partial_{\bar{z}} T = \partial_z \bar{T} = 0$

- Operator product expansion: non-conservation of currents at positions of local fields

$$\langle T(w)\mathcal{O}(z,\bar{z})\cdots\rangle = \left(\ldots + \frac{h}{(w-z)^2} + \frac{1}{w-z}\frac{\partial}{\partial z} + \ldots\right)\langle\mathcal{O}(z,\bar{z})\cdots\rangle$$

where $h = (d+s)/2$.

CFT possesses "local conformal invariance"

- With $\mathbb{R}^2 + \{\infty\} - \{D_1, D_2, D_3, \ldots\}$, may consider more conformal transformations that preserve the **topology** only
- Around a point z, it may look like $w - z \mapsto w - z + \varepsilon(w-z)^n$ for $n \geq 3$
- Assuming invariance under these higher-order transformations (**primary fields**):

$$\langle T(w)\mathcal{O}(z,\bar{z})\cdots\rangle = \left(\frac{h}{(w-z)^2} + \frac{1}{w-z}\frac{\partial}{\partial z} + \ldots\right)\langle\mathcal{O}(z,\bar{z})\cdots\rangle$$

- From explicit calculations in some models: T is not a primary field,

$$\langle T(w)T(z)\cdots\rangle = \left(\frac{c/2}{(w-z)^4} + \frac{h}{(w-z)^2} + \frac{1}{w-z}\frac{\partial}{\partial z} + \ldots\right)\langle T(z)\cdots\rangle$$

- Those are called **conformal Ward identities**.

The algebraic structure of CFT and additional symmetries

- Virasoro algebra:

$$T(w)\mathcal{O}(z,\bar{z}) = \sum_{n\in\mathbb{Z}} (z-w)^{-n-2} (L_n\mathcal{O})(z,\bar{z})$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}$$

- Local fields: **highest weight modules for the Virasoro algebra** – more generally, for a **vertex operator algebra** – characterised by the weight h

- Correlation functions: “Clebsch-Gordon coefficients” of tensor products of VOA modules.

- Reducibility: With $h = h_{1,2} \equiv \frac{6-\kappa}{2\kappa}$ and $c = \frac{(6-\kappa)(3\kappa-8)}{2\kappa}$, there is a **null-field**:

$$L_{-2}\phi_{1,2} - \frac{\kappa}{4} L_{-1}^2 \phi_{1,2} = 0$$

$\Rightarrow$ A certain transformation that is singular at the point z is a symmetry of the correlators $\langle \phi_{1,2}(z,\bar{z}) \cdots \rangle$

The powerful algebraic structure gives results for correlation functions of local fields, but:

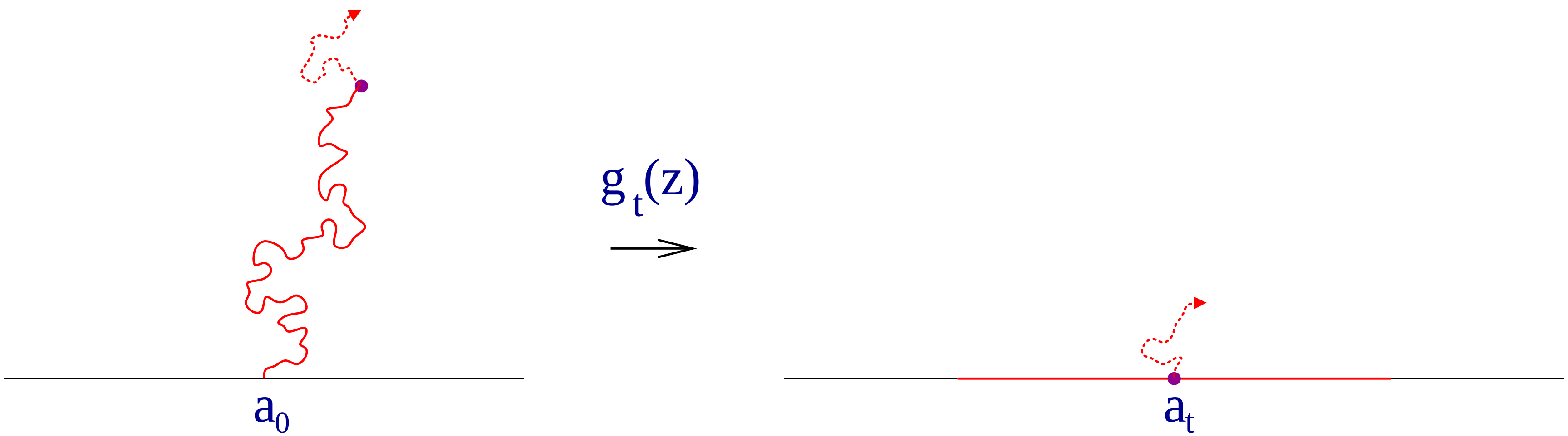
- Precise relations between statistical variables and local fields are conjectural / hard to get
- Non-local objects are not described easily
- Non-rational models are out of the range of applicability for now

Axiomatic CFT

Schramm-Loewner evolution [Schramm (1999), Lawler, Schramm, Werner (2001, ...)]

Tracing **random curves** in the upper half plane

using **stochastic conformal maps**:



$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - a_t} \quad , \quad g_0(z) = z \quad , \quad a_t = \sqrt{\kappa} B_t + a_0$$

B_t: standard Brownian motion, normalised by $E[B_t^2] = t$.

Defining **random curves** on any **simply connected domain** D

through **conformal transport** $f : \mathbb{H} \longrightarrow D$

$$\mu_{\mathbb{H}}[\gamma] = \mu_D[f(\gamma)]$$

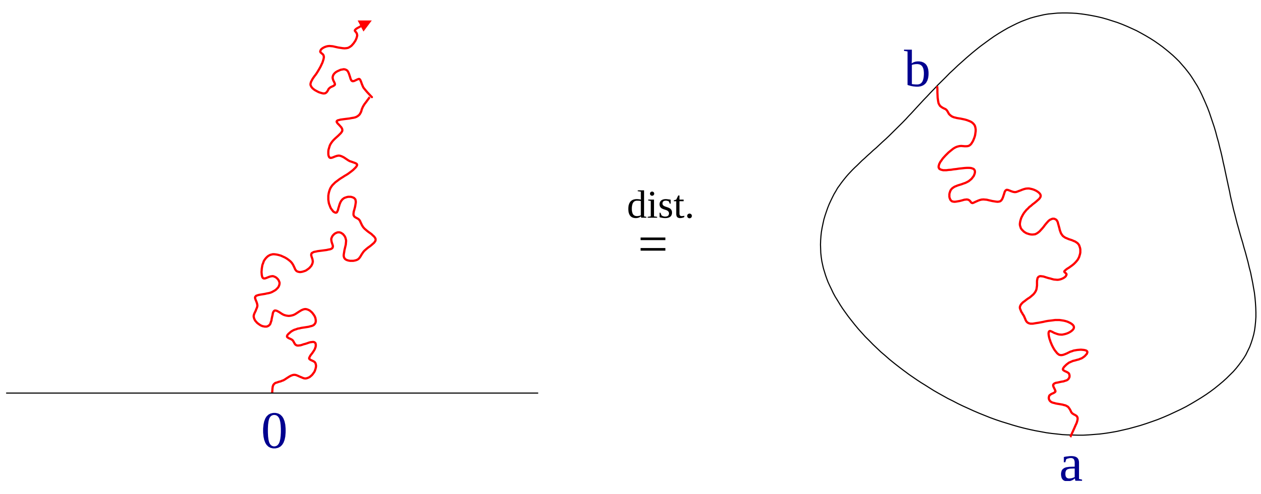
Conformally invariant family of measures

Family of **measures on curves**

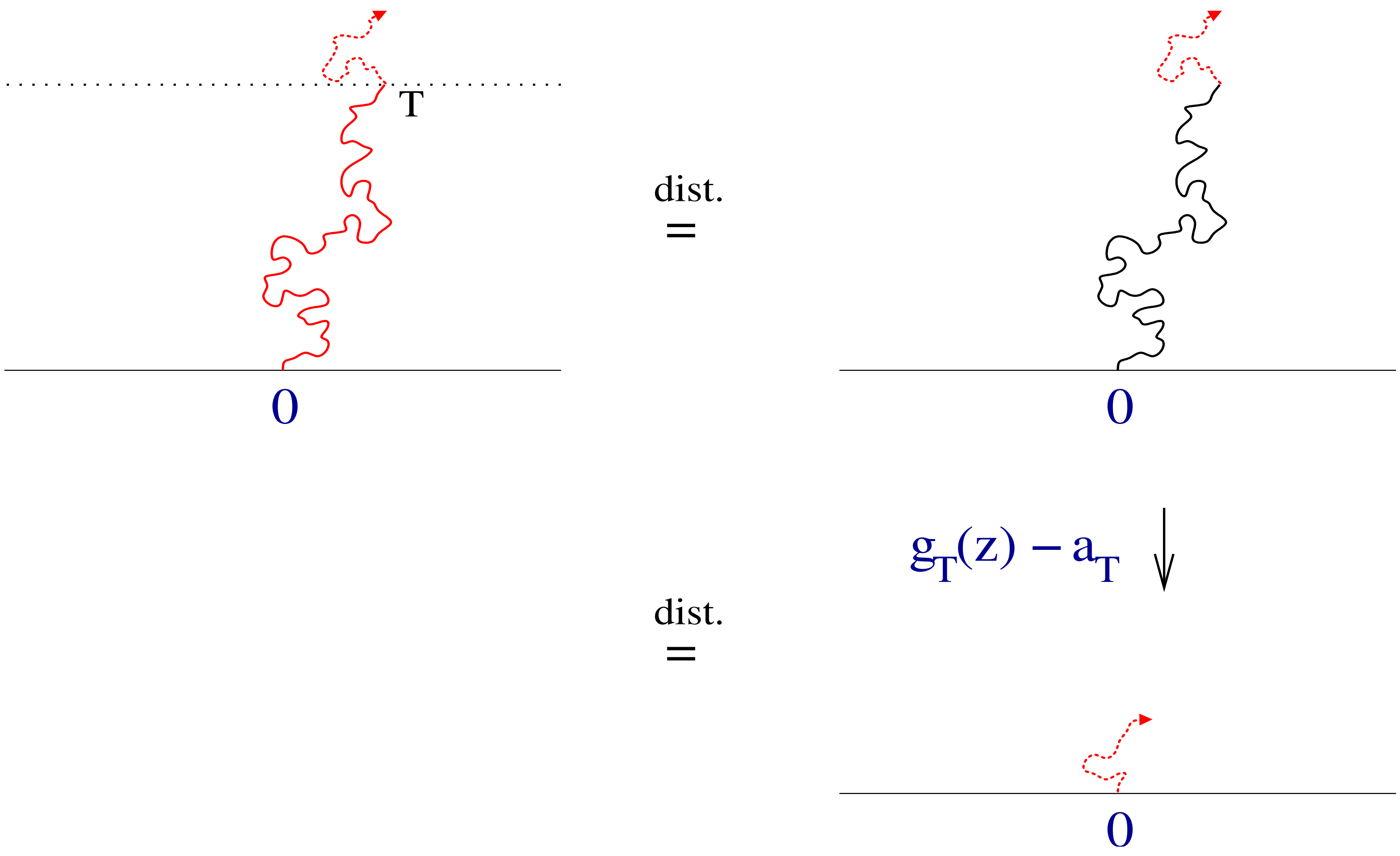
defined on any **simply connected domain** with any **starting and ending point**,

with **two properties**:

- Conformal transport (with $f : \mathbb{H} \longrightarrow D;\ 0 \mapsto a,\ \infty \mapsto b$)

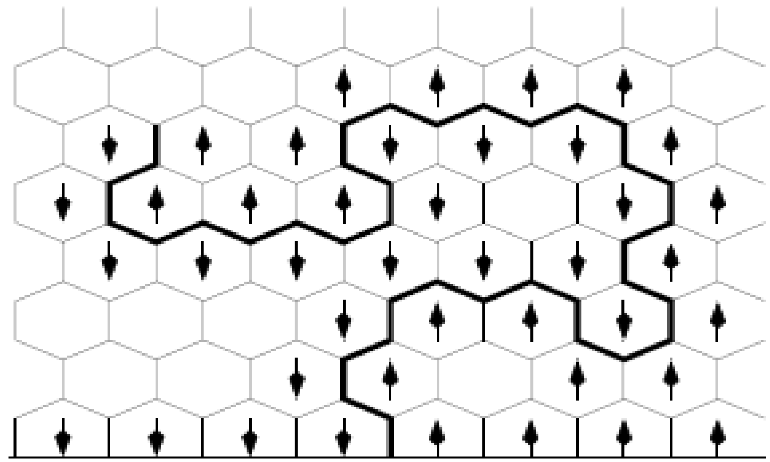


- Domain Markov property (the curve itself is like the boundary of a domain)



Domain walls in statistical models at criticality and other non-local critical objects

- We expect that critical curves, like domain walls (walls of clusters that are connected to the boundary) in statistical models at criticality, are conformally invariant in the continuum limit $\Rightarrow$ described by some SLE$_\kappa$



- Proofs: relatively little is required; proofs of conformally invariant scaling limit exist for domain wall in gaussian field, percolation, Ising model... [S. Smirnov (2001,...)]

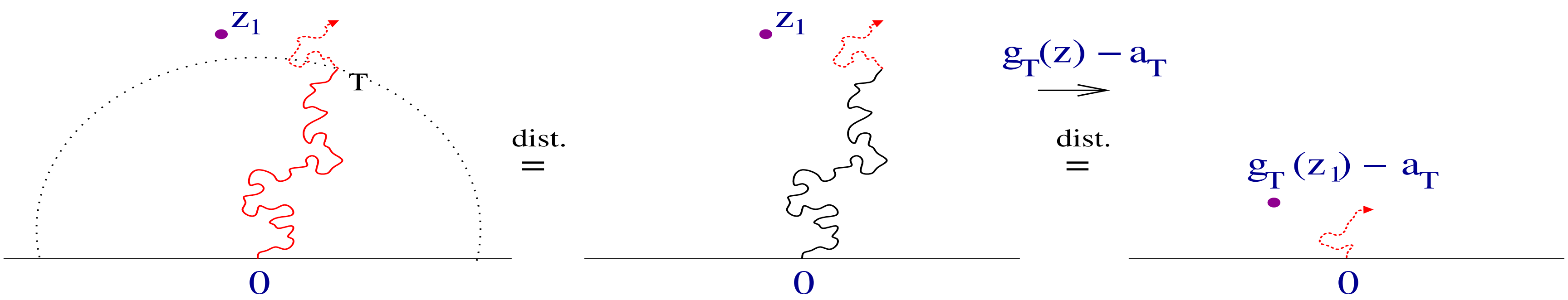
- SLE$_\kappa$ gives precise description of these non-local objects in the scaling limit

Constructing CFT from SLE

- What events correspond to “known” local fields of CFT?
- What fields or Virasoro modules correspond to other events in SLE?
- What does the algebraic structure means in the probabilistic setting? What becomes of it in non-rational cases?

Constructive CFT?

The SLE equation and level-2 boundary null field



With $T = dt$:

$$P(z_1, \bar{z}_1, \ldots) = E[P(g_{dt}(z_1) - \sqrt{\kappa}\, dB_t,\, \bar{g}_{dt}(\bar{z}_1) - \sqrt{\kappa}\, dB_t, \ldots)]\,, \quad dB_t^2 = dt$$

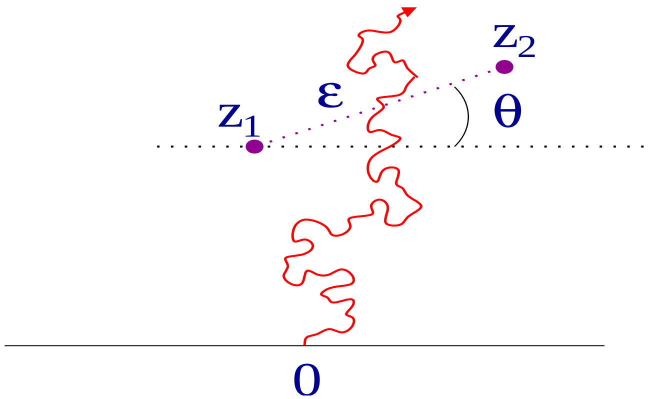
Equation obtained: equivalent to null-vector equation for

$$\frac{\langle \mathcal{O}(z_1, \bar{z}_1) \cdots \phi_{1,2}(0)\phi_{1,2}(\infty)\rangle}{\langle \phi_{1,2}(0)\phi_{1,2}(\infty)\rangle}$$

with

$$\mathcal{O} \;:\; d = s = 0,\ \text{primary}\,, \quad c = \frac{(6-\kappa)(3\kappa-8)}{2\kappa}\,, \quad h_{1,2} = \frac{6-\kappa}{2\kappa}$$

Holomorphic bulk fields



Solving the null-vector equation in the case of:

The curve being on the right of z_1 and on the left of z_2

with $z_1 \longrightarrow z_2 \longrightarrow w$ gives

$$\lim_{\varepsilon \to 0} \varepsilon^{-s} \int d\theta \, e^{-is\theta} \; P(z_1, \bar{z}_1, z_2, \bar{z}_2) \; \propto \; w^{-s} \; \propto \; \frac{\langle \mathcal{O}_s(w) \phi_{1,2}(0) \phi_{1,2}(\infty) \rangle}{\langle \phi_{1,2}(0) \phi_{1,2}(\infty) \rangle}$$

if and only if the following condition is satisfied:

$$\kappa = \frac{8}{s+1}$$

Particular cases

$$s = 1$$

$\kappa = 4$, $c = 1$: spin-1 holomorphic $U(1)$ current in the gaussian field – simple proof for one insertion [Cardy]

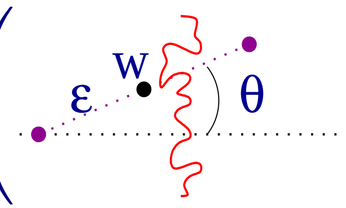
$$s = \frac{1}{2}$$

$\kappa = \frac{16}{3}$, $c = \frac{1}{2}$, holomorphic fermion in the Ising model (the domain wall is in the FK representation) – proof of lattice holomorphicity for one insertion [Riva, Cardy (2006)]

$$s = 2$$

$\kappa = \frac{8}{3}$, $c = 0$, holomorphic stress-energy tensor in the $O(0)$ "loop model": a domain wall which is a self-avoiding random walk, and no loops – proof: [D., Riva, Cardy (2006), cf. Friedrich, Werner (2003)]

$$\langle T(w) \cdots \rangle \propto \lim_{\varepsilon \to 0} \varepsilon^{-2} \int d\theta \, e^{-2i\theta} \, P \Bigg($$



$$, \dots \Bigg)$$

Why $\kappa = \frac{8}{3}, c = 0$

- SLE does not give direct information on the loops. In SLE, we measure only the energy on the domain wall, not on cluster boundaries in the bulk.
- Must have a model where no energy is in the bulk. All energy must be on the domain wall, there should be no "vacuum energy".
- Central charge must be zero, since the theory cannot "feel the boundary" of the domain where it lies.
- Should correspond to $O(n)$ loop model at $n = 0$. The partition function of the $O(n)$ loop model is

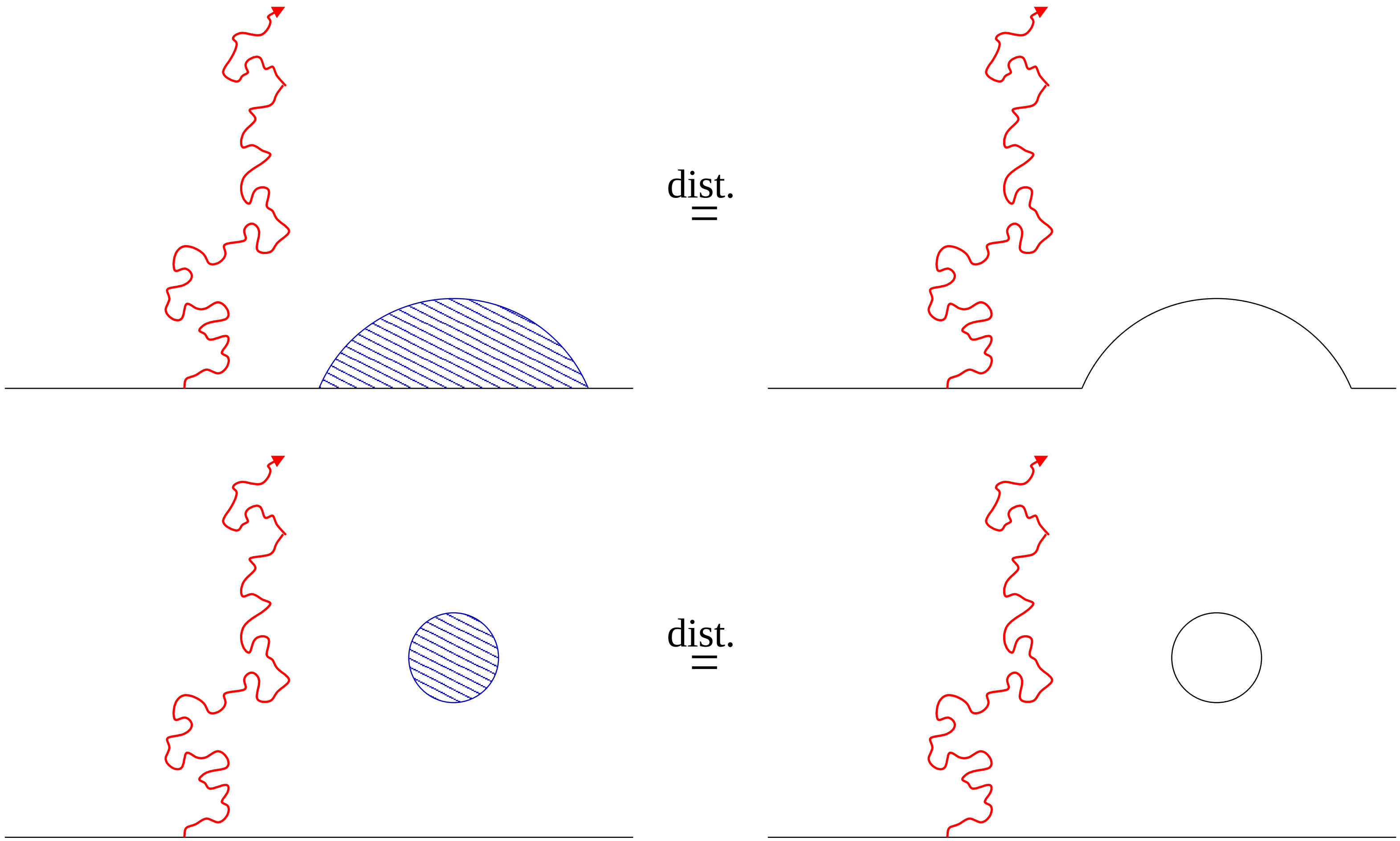
$$Z = \sum_{\text{configurations}} \mathrm{x}_c^{\text{total length}} n^{\text{number of loops}}$$

and one has (from Coulomb gas arguments)

$$n = -2 \cos \pi \left(\frac{4}{\kappa} \right)$$

Conformal restriction at $\kappa = \frac{8}{3}$ [LSW (2003); Bauer, Friedrich (2004)]

$$\mu(\gamma | \gamma \cap A = \emptyset) = \mu_{\mathbb{H}\setminus A}(\gamma)$$



Deriving the conformal Ward identities

- Consider the algebraic definition of the stress-energy tensor:

$$(L_{-2}\mathbf{1})(w) = T(w)$$

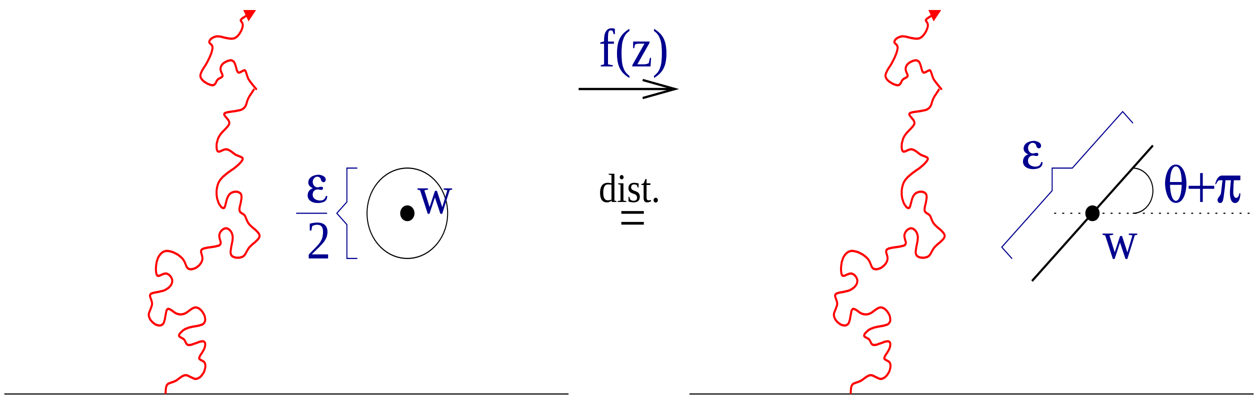
- Interpret geometrically:

The stress-energy tensor is the result of a conformal transformation
that preserves ∞ and that has a simple pole at a point w

Hence, consider the conformal transformation

$$f(z) = z + \frac{\varepsilon^2 e^{2i\theta}}{16(w-z)} + \frac{\varepsilon^2 e^{-2i\theta}}{16(\bar{w}-z)} - \frac{\varepsilon^2 e^{2i\theta}}{16w} - \frac{\varepsilon^2 e^{-2i\theta}}{16\bar{w}}$$

Then, we have



Generalisations to $c \neq 0$

We need CLE$_\kappa$ – conformal loop ensemble [Werner, Sheffield (2007)]

Stress-energy tensor:

- The **anomaly term** $\frac{c/2}{(w-z)^4}$ in $\langle T(w)T(z)\rangle$ is due to **loops connecting both slits**
- A certain kind of "random conformal restriction" holds, but the difficulty is in the **normalisation of measures** because of the **infinitely many small loops**

Other holomorphic fields:

- For a boundary changing condition that corresponds to the action of a symmetry, one insertion of the current associated to this symmetry is supported on the corresponding domain wall
- Many insertions involve also loops connecting them $\Rightarrow$ anomaly terms

Generalisations to other fields

n-changing fields in the $O(n)$ model

- Field $\mathcal{O}_{n'}(x)$ such that loops around x are counted with n' instead of n
- Scaling dimension given (from Coulomb gas arguments) by $d_{n',n} = \frac{(\kappa'-\kappa)(\kappa\kappa'-2\kappa-2\kappa')}{\kappa(\kappa')^2}$
- For $\kappa' = \frac{4\kappa}{4-\kappa}$, $d_{n',n} = 2h_{2,1} \Rightarrow$ differential equations for four-point functions [Gamsa, Cardy (2006)]
- How to prove from CLE$_\kappa$? Use of geometric meaning of L_{-2}?

N-leg fields and "descendants" in the $O(n)$ model

- Besides the loops, put N curves from the boundary of the disk to the center
- What are the possible dimensions of the corresponding field at the center? [D., Cardy (2007)]

Perspectives

- OPE's, vertex operator algebra
- Null-fields and modules for VOA
- Analysis of $c = 0$ cases: logarithmic companion t of stress-energy tensor....