Identification of the stress-energy tensor through conformal restriction in SLE and related processes

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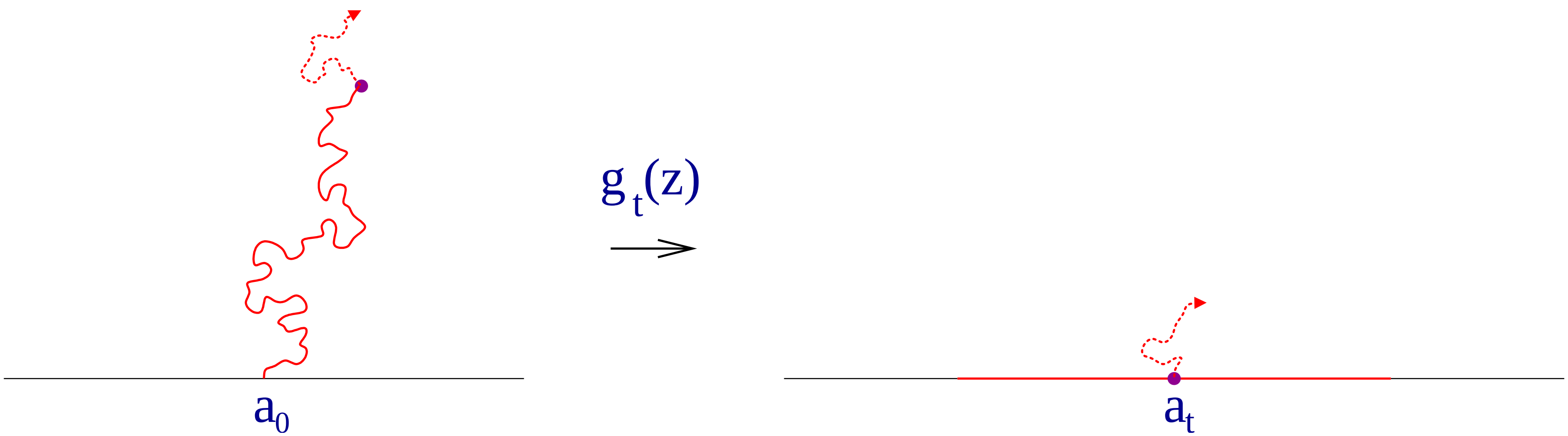
Goal: constructing CFT from continuum random processes

Schramm-Loewner evolution [Schramm (1999), Lawler, Schramm, Werner (2001, ...)]

Tracing **random curves** in the upper half plane

using **stochastic conformal maps**:



$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - a_t} \quad , \quad g_0(z) = z \quad , \quad a_t = \sqrt{\kappa} B_t + a_0$$

B_t: standard Brownian motion, normalised by $E[B_t^2] = t$.

Defining **random curves** on any **simply connected domain** D

through **conformal transport** $f : \mathbb{H} \rightarrow D$

$$\mu_{\mathbb{H}}[\gamma] = \mu_D[f(\gamma)]$$

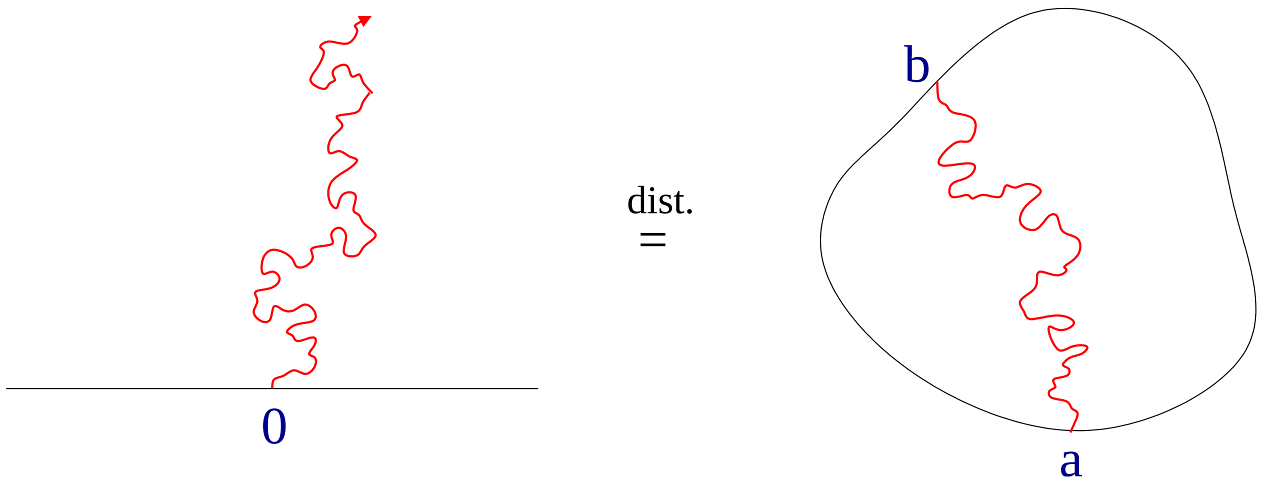
Conformally invariant family of measures

Family of **measures on curves**

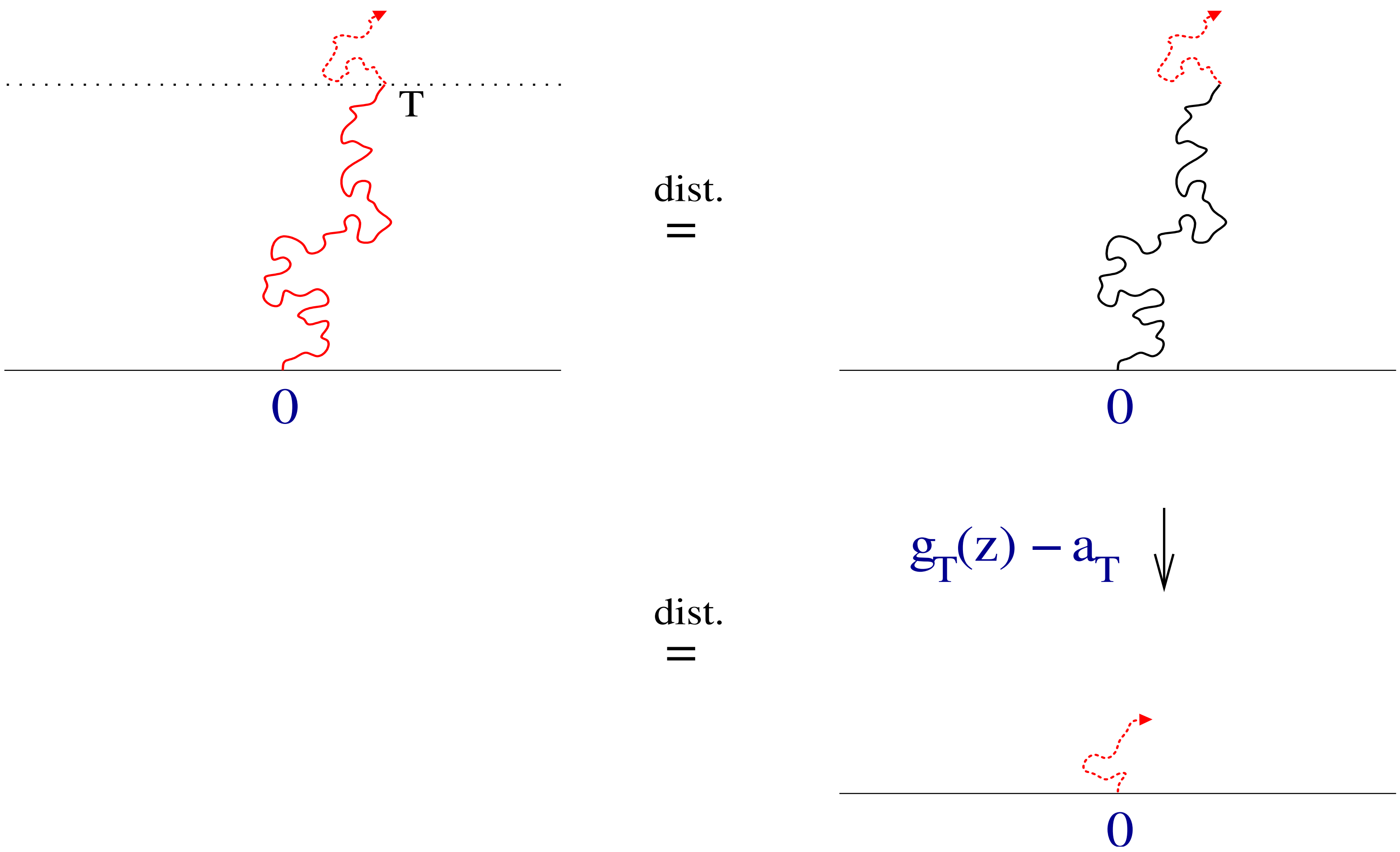
defined on any **simply connected domain** with any **starting and ending point**,

with **two properties**:

- Conformal transport (with $f : \mathbb{H} \rightarrow D;\ 0 \mapsto a,\ \infty \mapsto b$)



- Domain Markov property (the curve itself is like the boundary of a domain)



Domain walls in statistical models at criticality and other non-local critical objects

SLE$_\kappa$ description (constructive CFT?):

- We expect that critical curves, like domain walls (walls of clusters that are connected to the boundary) in statistical models at criticality, are conformally invariant in the continuum limit $\Rightarrow$ described by some SLE$_\kappa$
- Proofs: relatively little is required; proofs of conformally invariant scaling limit exist for domain wall in gaussian field, percolation, Ising model... [S. Smirnov (2001,...)]
- SLE$_\kappa$ gives precise description of these non-local objects in the scaling limit

Algebraic description (axiomatic CFT)

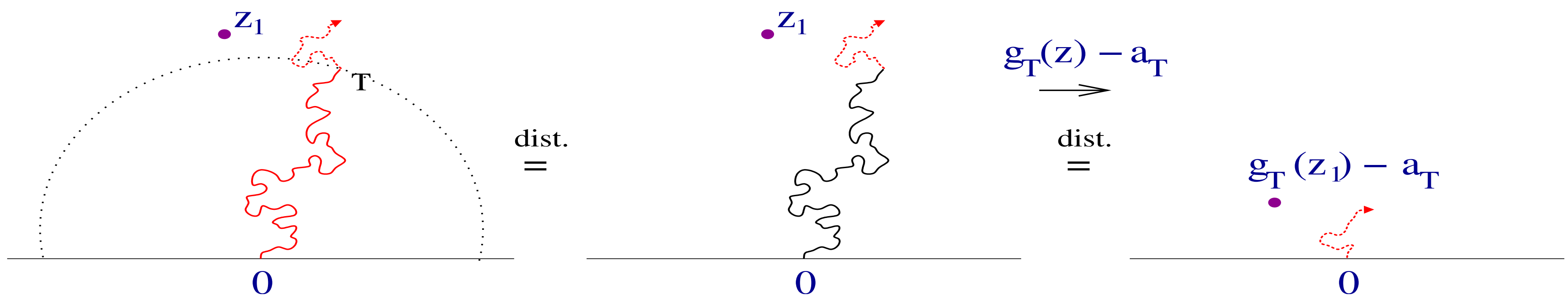
- We expect algebraic description: Virasoro algebra / vertex operator algebra, Verma modules, null vectors...
- Powerful for correlation functions of local variables
- But: no proof of algebraic description from statistical model

How to relate both?

One approach: assuming CFT, coupling it with SLE [Bauer, Bernard (2002,...)]

Other approach: constructin CFT from SLE (and eventually other random processes)

The SLE equation and level-2 boundary null field



With $T = dt$:

$$P(z_1, \bar{z}_1, \ldots) = E[P(g_{dt}(z_1) - \sqrt{\kappa}\, dB_t,\, \bar{g}_{dt}(\bar{z}_1) - \sqrt{\kappa}\, dB_t, \ldots)]\,, \quad dB_t^2 = dt$$

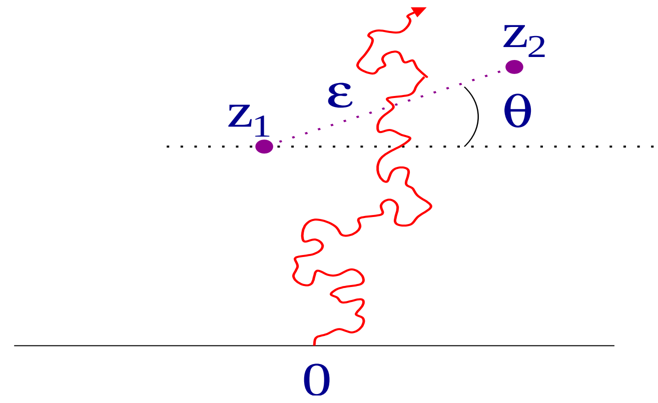
Equation obtained: equivalent to null-vector equation for

$$\frac{\langle \mathcal{O}(z_1, \bar{z}_1) \cdots \phi_{1,2}(0)\phi_{1,2}(\infty)\rangle}{\langle \phi_{1,2}(0)\phi_{1,2}(\infty)\rangle}$$

with

$$\mathcal{O} \;:\; d = s = 0,\ \text{primary}\,, \quad c = \frac{(6-\kappa)(3\kappa-8)}{2\kappa}\,, \quad h_{1,2} = \frac{6-\kappa}{2\kappa}$$

Holomorphic bulk fields



Solving the null-vector equation in the case of:

The curve being on the right of z_1 and on the left of z_2

with $z_1 \longrightarrow z_2 \longrightarrow w$ gives

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-s} \int d\theta \, e^{-is\theta} \; P(z_1, \bar{z}_1, z_2, \bar{z}_2) \; \propto \; w^{-s} \; \propto \; \frac{\langle \mathcal{O}_s(w) \phi_{1,2}(0) \phi_{1,2}(\infty) \rangle}{\langle \phi_{1,2}(0) \phi_{1,2}(\infty) \rangle}$$

if and only if the following condition is satisfied:

$$\kappa = \frac{8}{s+1}$$

Particular cases

$$s = 1$$

$\kappa = 4$, $c = 1$: spin-1 holomorphic $U(1)$ current in the gaussian field

$$s = \frac{1}{2}$$

$\kappa = \frac{16}{3}$, $c = \frac{1}{2}$, holomorphic fermion in the Ising model (the domain wall is in the FK representation)

$$s = 2$$

$\kappa = \frac{8}{3}$, $c = 0$, holomorphic stress-energy tensor in the $O(0)$ loop model (where the domain wall is a self-avoiding random walk, and there are no loops remaining) (cf. [Friedrich, Werner (2003)])

A case for the stress-energy tensor

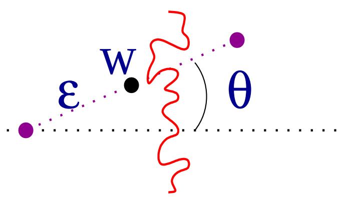
Why a spin-2 rotating slit

- A generic stress-energy tensor T_{ij} measures the flow in the direction j of energy locally stored in distortions in the direction i
- Distortions where energy is stored are at cluster or domain walls in statistical models.
- If the wall is vertical: x-distortion. If the wall is horizontal: y-distortion.
- In complex coordinates $z = x + iy$, we have

$$T \equiv T_{zz} = \frac{1}{4}\left(T_{xx} - T_{yy} - 2iT_{xy}\right)$$

- “Morally”, this agrees with

$$\langle T(w)\cdots\rangle \propto \lim_{\varepsilon\to 0} \varepsilon^{-2} \int d\theta\, e^{-2i\theta}\, P\Bigg($$



$$, \ldots \Bigg)$$

Why $\kappa = \frac{8}{3}, c = 0$

- SLE does not give direct information on the loops. In SLE, we measure only the energy on the domain wall, not on cluster boundaries in the bulk.
- Must have a model where no energy is in the bulk. All energy must be on the domain wall, there should be no "vacuum energy".
- Central charge must be zero, since the theory cannot "feel the boundary" of the domain where it lies.
- Should correspond to $O(n)$ loop model at $n = 0$. The partition function of the $O(n)$ loop model is

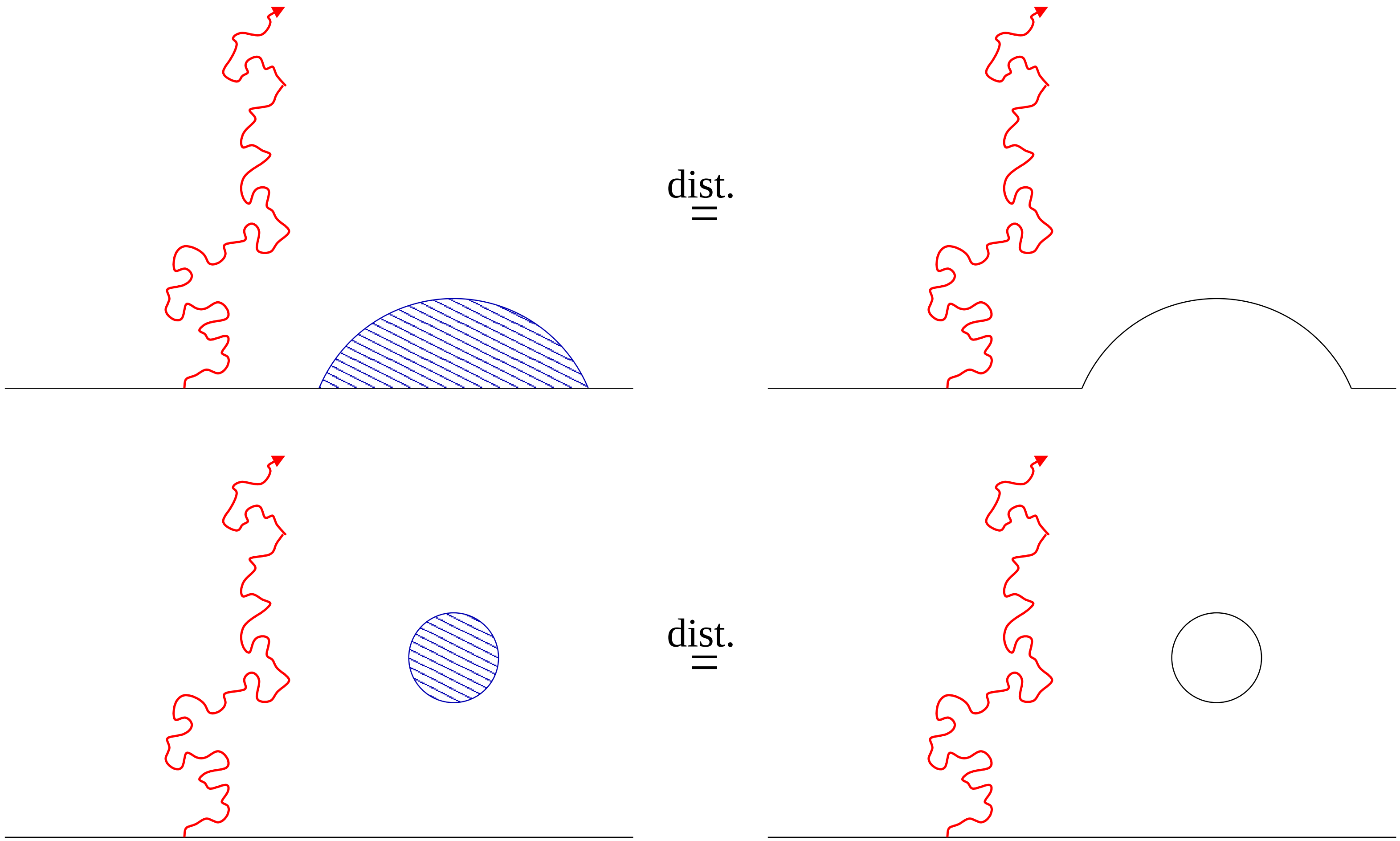
$$Z = \sum_{\text{configurations}} \mathrm{x}_c^{\text{total length}} n^{\text{number of loops}}$$

and one has (from Coulomb gas arguments)

$$n = 2\cos\pi\left(1 - \frac{4}{\kappa}\right)$$

Conformal restriction at $\kappa = \frac{8}{3}$ [LSW (2003); Bauer, Friedrich (2004)]

$$\mu(\gamma | \gamma \cap A = \emptyset) = \mu_{\mathbb{H} \setminus A}(\gamma)$$



Deriving the conformal Ward identities

- Consider now with the algebraic definition of the stress-energy tensor:

$$L_{-2}(w)\mathbf{1} = T(w)$$

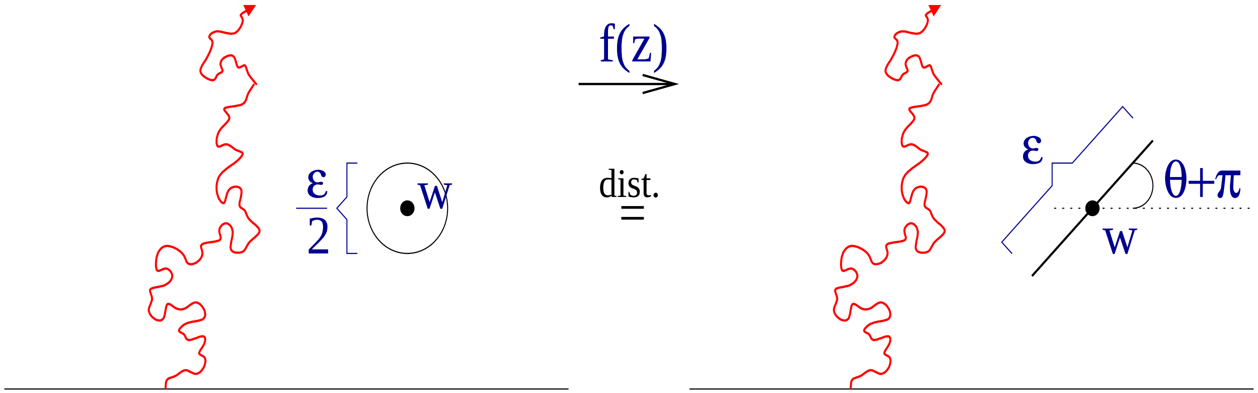
- Interpret geometrically:

The stress-energy tensor is the result of a conformal transformation
that preserves ∞ and that has a simple pole at a point w

Hence, consider the conformal transformation

$$f(z) = z + \frac{\varepsilon^2 e^{2i\theta}}{16(w-z)} + \frac{\varepsilon^2 e^{-2i\theta}}{16(\bar{w}-z)} - \frac{\varepsilon^2 e^{2i\theta}}{16w} - \frac{\varepsilon^2 e^{-2i\theta}}{16\bar{w}}$$

Then, we have



In equations:

$$P_{\mathbb{H}\setminus D_{w,\varepsilon}}(z,\bar{z}) = P_{\mathbb{H}\setminus S_{w,\varepsilon,\theta}}(f(z),\bar{f}(\bar{z}))$$

$$= P_{\mathbb{H}\setminus S_{w,\varepsilon,\theta}}(z,\bar{z}) + \left(\frac{\varepsilon^2 e^{2i\theta}}{16(w-z)} - \frac{\varepsilon^2 e^{2i\theta}}{16w}\right)\frac{\partial}{\partial z} P_{\mathbb{H}\setminus S_{w,\varepsilon,\theta}}(z,\bar{z}) + \ldots$$

$$P(z,\bar{z}|\gamma \cap S_{w,\varepsilon,\theta} = \emptyset)$$

$$= \frac{P(z,\bar{z};\gamma \cap S_{w,\varepsilon,\theta} = \emptyset)}{P(\gamma \cap S_{w,\varepsilon,\theta} = \emptyset)}$$

$$= \frac{P(z,\bar{z}) - P(z,\bar{z};T_{w,\varepsilon,\theta})}{1 - P(T_{w,\varepsilon,\theta})}$$

$$= P(z,\bar{z}) - P(z,\bar{z};T_{w,\varepsilon,\theta}) + P(z,\bar{z})P(T_{w,\varepsilon,\theta}) + \ldots$$

$$\left(\frac{\varepsilon^2 e^{2i\theta}}{16(w-z)} - \frac{\varepsilon^2 e^{2i\theta}}{16w}\right)\frac{\partial}{\partial z} P(z,\bar{z}) + \ldots$$

where

$$T_{w,\varepsilon,\theta} = \{\gamma \cap S_{w,\varepsilon,\theta} \neq \emptyset\}$$

Integrating over the angle θ:

$$Q(z,\bar{z};w) = \lim_{\varepsilon\to 0} \varepsilon^{-2} \int d\theta \, e^{-2i\theta} P(z,\bar{z};T(w,\epsilon,\theta))$$

gives

$$\begin{aligned} Q(z,\bar{z};w) &= P(z,\bar{z})Q(w) + \frac{\pi}{8}\left(\frac{1}{w-z} - \frac{1}{w}\right)\frac{\partial}{\partial z}P(z,\bar{z}) + \\ &\quad + \frac{\pi}{8}\left(\frac{1}{w-\bar{z}} - \frac{1}{w}\right)\frac{\partial}{\partial \bar{z}}P(z,\bar{z}) \\ &= \frac{\pi}{8}\left[\frac{5/8}{w^2} + \left(\frac{1}{w-z} - \frac{1}{w}\right)\frac{\partial}{\partial z} + \left(\frac{1}{w-\bar{z}} - \frac{1}{w}\right)\frac{\partial}{\partial \bar{z}}\right] P(z,\bar{z}) \\ &= \frac{\langle T(w)\mathcal{O}(z,\bar{z})\phi_{1,2}(0)\phi_{1,2}(\infty)\rangle}{\langle \phi_{1,2}(0)\phi_{1,2}(\infty)\rangle} \end{aligned}$$

Generalisations

- It is possible to prove that the stress-energy tensor transforms like a spin-2 holomorphic field $\Rightarrow$ multi-point correlation functions of stress-energy tensors

- Method works for any conformal restriction measure with appropriate smoothness properties:

  - Similar method for boundary stress-energy tensor [Friedrich, Werner (2003)]: there it is possible to get to $c \neq 0$ theories, using conformal restriction measures that reproduce *connected* correlation functions

  - Generalisation to certain sub-measures of CLE_6 (percolation), again $c = 0$ theory

- Other holomorphic fields: we don't expect multi-point correlation functions to be defined in SLE, because of anomalies

- Proof of lattice holomorphicity of parafermions in Potts models...

Perspectives

- Stress-energy tensor and other local bulk fields in CLE
- OPE's, vertex operator algebra
- Null-vectors and modules for VOA
- Analysis of $c = 0$ cases: logarithmic companion t of stress-energy tensor....