Calculus on infinite-dimensional manifolds, conformal field theory, and its probabilistic descriptions

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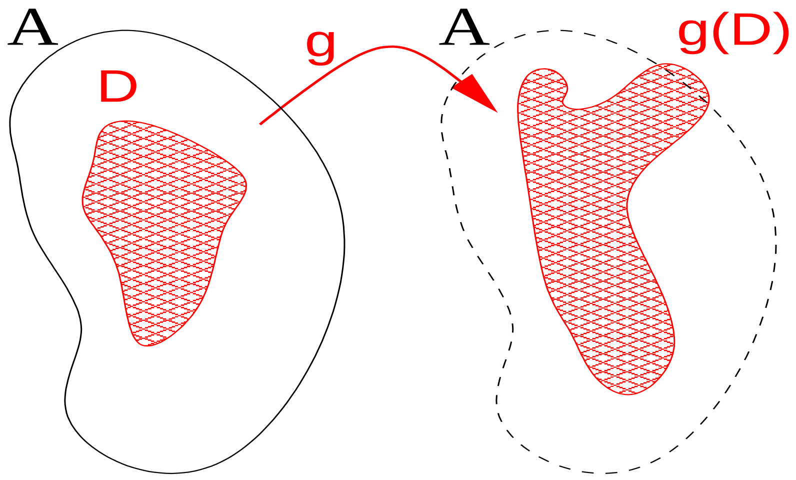
Manifold of conformal maps

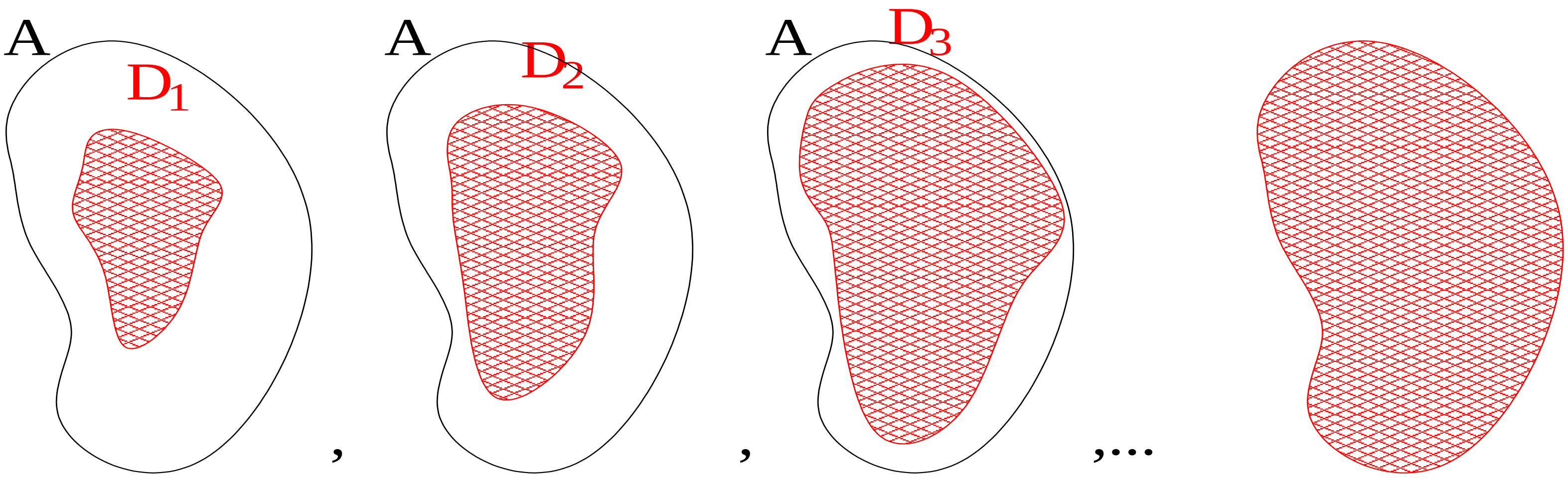
Consider a simply connected bounded domain A and the set of maps g that are conformal on some domain (below: the domain D) inside A.



Local topology around the **identity map**: what sequence of conformal maps $(g_1, g_2, g_3, \ldots)$ can be said to **converge** to the identity?

A-topology:

- domains D_n tend to A
- compact convergence: uniform convergence on any compact subset



$$\lim_{n\to\infty} \sup(g_n(z) - z : z \in D_n) = 0$$

- Topology **preserved under conformal maps** $G : A \longrightarrow B$ between simply connected domains $A,\ B$.
- By **conformal transport**, define the A-topology for any domain A of the Riemann sphere (bounded or not).

A manifold structure?

Take a family $(g_\eta : \eta > 0)$ with $\lim_{\eta \to 0} g_\eta = \mathrm{id}$. We have that $g_\eta - \mathrm{id}$ is holomorphic on D_η (for A bounded).

Is A-topology **locally like the vector space $\mathtt{H}(A)$ of holomorphic functions** on A with compact convergence topology?

Not quite... need to **restrict to A^*-topology** :

$$A^*\text{-topology} \;=\; A\text{-topology} \cap \{g : \exists h \in \mathtt{H}(A) \mid h\partial g = g \circ h\}$$

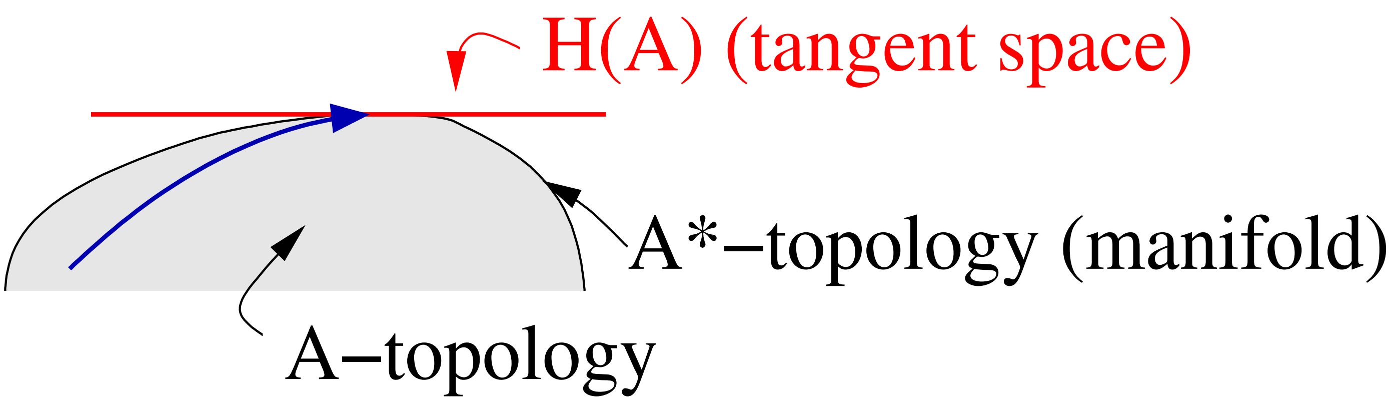
$$\text{homeomorphism:} \quad A^*\text{-topology} \;\leftrightarrow\; \mathtt{H}(A)$$

$$\mathrm{id} \;\leftrightarrow\; 0$$

“smooth” approach to id in A^*-topology $\Rightarrow$ **“smooth” approach in A-topology**:

$$\lim_{\eta\to 0} g_\eta = \mathrm{id} \quad (A\text{-topology}), \quad \lim_{\eta\to 0} \frac{g_\eta(z) - z}{\eta} = h(z) \quad \exists \quad (\text{compactly for } z \in A).$$

We say: $(g_\eta : \eta > 0) \in \mathtt{F}(A)$.



What about unbounded (still simply connected) domains A?

Define likewise A^*-topology by conformal transport. Restriction:

$$\{g : \exists h \in \mathtt{H}^>(A) \mid h\partial g = g \circ h\}$$

where $\mathtt{H}^>(A)$: holomorphic functions $h(z)$ on A except for $O(z^2)$ as $z \longrightarrow \infty$ (if $\infty \in A$).

- Tangent space $\cong \mathtt{H}^>(A)$
- Choosing any $a \in \hat{\mathbb{C}} \setminus A$, approach to identity in $\mathtt{F}(A)$ described by

$$g_\eta(z) = a + \frac{z-a}{1 - \frac{\eta}{z-a} h_\eta^{(a)}(z)}.$$

with $h_\eta^{(a)}(z)/(z-a)^2$ holomorphic on D_η, compactly convergent to $h(z)/(z-a)^2$

Derivatives

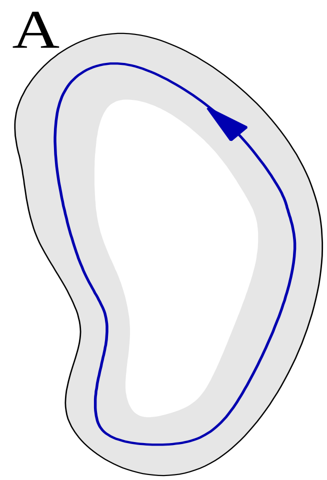
Derivative of a function on A^*-manifold at id = element of the **cotangent space** at id.

Need **continuous dual** $\mathtt{H}^{>*}(A)$ **(space of continuous linear functionals) of** $\mathtt{H}^{>}(A)$.

Any continuous linear functional $\Upsilon : \mathtt{H}^{>}(A) \to \mathbb{R}$ is of the form

$$\Upsilon(h) = \oint_{\partial A^-} \mathrm{d}z\, \alpha(z) h(z) + \oint_{\partial A^-} \bar{\mathrm{d}}\bar{z}\, \bar{\alpha}(\bar{z}) \bar{h}(\bar{z})$$

for some α holomorphic on an annular neighbourhood of ∂A inside A.



Arbitrariness of α: functional Υ is characterised by a **class of functions:**

$$\mathcal{C} = \left\{ \alpha + u : u \in \mathtt{H}^{<}(A) \right\}$$

where $\mathtt{H}^{<}(A)$: holomorphic functions $h(z)$ on A with $O(z^{-4})$ as $z \longrightarrow \infty$ (if $\infty \in A$).

- $\exists\ \gamma \in \mathcal{C} \mid \gamma$ is holomorphic on $\hat{\mathbb{C}} \setminus A$, except possibly for a pole of order 3 at some point $a \in \hat{\mathbb{C}} \setminus A$ if $\infty \in A$, and 0 at a if $\infty \notin A$.

- For any given a, γ is unique.

- Function $f:\Omega\to\mathbb{R}$
- Point $\Sigma\in\Omega$
- Action $g(\Sigma)\in\Omega$ for any g in A-neighbourhood of id.

A**-differentibility: for any** $(g_\eta:\eta>0)\in\mathtt{F}(A)$**,**

$$\lim_{\eta\to 0}\frac{f(g_\eta(\Sigma))-f(\Sigma)}{\eta}=\nabla^A f(\Sigma)h,\qquad \nabla^A f(\Sigma)\in\mathtt{H}^{>*}(A)$$

Case $A=\mathbb{D}$: can use basis $H_{n,s}(z)=e^{i\pi s/4}z^n,\ n=0,1,2,3,\ldots,\ s=\pm$,

$$f_{n,s}(\Sigma):=\lim_{\eta\to 0}\frac{f((\mathrm{id}+\eta H_{n,s})(\Sigma))-f(\Sigma)}{\eta}\quad\exists$$

$$\lim_{\eta\to 0}\frac{f(g_\eta(\Sigma))-f(\Sigma)}{\eta}=\sum_{n\geq 0,s=\pm}c_{n,s}f_{n,s}(\Sigma)\quad\text{converges}$$

with $h=\sum_{n,s}c_{n,s}H_{n,s}$.

Definitions and notation:

- $\nabla^A f(\Sigma)$: the **conformal A-derivative of f at Σ**
- $\nabla_h f(\Sigma) = \nabla^A f(\Sigma) h$: the **directional derivative of f at Σ in the direction h**
- $\Delta^A f(\Sigma)$: the **holomorphic A-class of f at Σ**, the corresponding class of holomorphic functions
- $\Delta^A_{a;z} f(\Sigma)$: the **holomorphic A-derivative of f at Σ**, the unique member (almost) holomorphic on $\hat{\mathbb{C}} \setminus A$ with a pole of maximal order 3 at $z = a$ or the value 0 at $z = a$

Main properties

How much depends on the domain A?

- If f is A- and B-differentiable and $h \in \mathtt{H}^{>}(A) \cap \mathtt{H}^{>}(B)$, then

$$\nabla^{A} f(\Sigma) h = \nabla^{B} f(\Sigma) h =: \nabla_{h} f(\Sigma)$$

- If f is A- and B-differentiable and $A \cup B \neq \hat{\mathbb{C}}$, then

$$\Delta^{A}_{a;z} f(\Sigma) \cong \Delta^{B}_{a;z} f(\Sigma) \quad \forall \quad a \in \hat{\mathbb{C}} \setminus (A \cup B)$$

(equality possibly up to pole of order 3...) and also in general

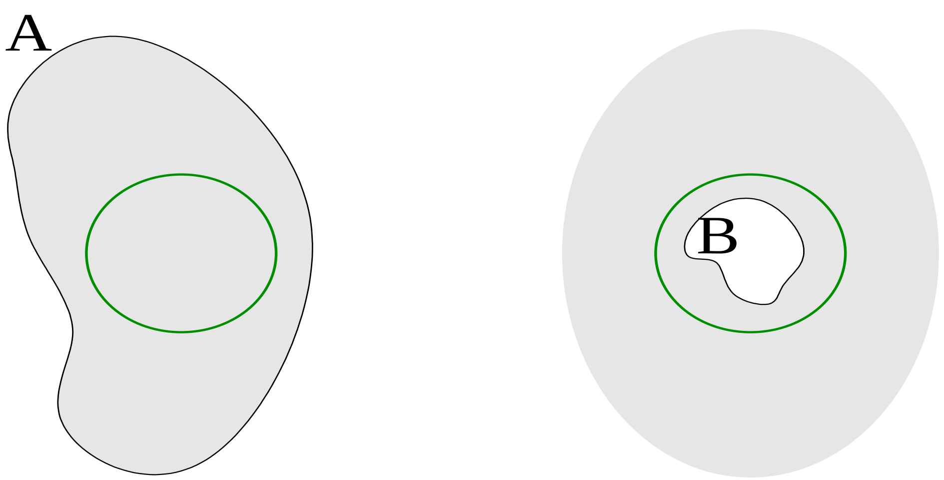
$$\Delta^{A}_{a;z} f(\Sigma) \cong \Delta^{A}_{b;z} f(\Sigma) \quad \forall \quad a, b \in \hat{\mathbb{C}} \setminus A$$

(equality up displacement of zero or of pole of order 3...)

- Consider set Ξ of all domains A such that f is A-differentiable.
- Equivalence relation: domains with intersecting complements are equivalent, complete by transitivity.
- Denote by $[A]$ the equivalence class, or **sector** containing A

$\Rightarrow$ **Ξ is divided into sectors where holomorphic derivatives are "the same" under $\cong$**

Example: $\Sigma =$ a circle, $\Omega =$ a space of smooth loops. Two natural sectors: $[A] =$ bounded sector, $[B] =$ another sector:



How does the derivative transform under conformal maps?

- A-differentiability of f at $\Sigma \quad \Leftrightarrow \quad g(A)$-differentiability of $f \circ g^{-1}$ at $g(\Sigma)$
- "Holomorphic dimension-2" transformation property for the holomorphic A-class:

$$\Delta^A f(\Sigma) = (\partial g)^2 \left(\Delta^{g(A)} (f \circ g^{-1})(g(\Sigma)) \right) \circ g.$$

Global stationarity

The global holomorphic A-derivative

If f is **globally stationary:** derivative $= 0$ along 1-parameter subgroups of the group of global conformal maps (möbius maps), then:

$$\Delta_z^{[A]} f(\Sigma) := \begin{cases} \Delta^A_{\infty;z} f(\Sigma) & (\infty \in \hat{\mathbb{C}} \setminus A) \\ \Delta^A_{a;z} f(\Sigma) & (\infty \in A, \text{ any } a \in \hat{\mathbb{C}} \setminus A) \end{cases}$$

- Well defined, and only depends on the sector
- Holomorphic for $z \in \hat{\mathbb{C}} \setminus \cap[A]$ for both bounded and unbounded sectors
- $O(z^{-4})$ as $z \to \infty$ in bounded sector
- "Holomorphic dimension-2" transformation property for G a möbius map:

$$\Delta_z^{[A]} f(\Sigma) = (\partial G(z))^2 \Delta_{G(z)}^{[G(A)]} (f \circ G^{-1})(G(\Sigma))$$

The A-connection

For a conformal transformation $g : A \longrightarrow B$, define:

$$\Gamma^{[A]}_{z;g} f(\Sigma) := \Delta^{[A]}_{z} f(\Sigma) - (\partial g(z))^2 \Delta^{[g(A)]}_{g(z)} (f \circ g^{-1})(g(\Sigma)).$$

- $\Gamma^{[A]}_{z;g} f(\Sigma)$ is in $\mathtt{H}^{<}(A)$ as function of z
- $\Gamma^{[A]}_{z;G} f(\Sigma) = 0$ for G global conformal map
- Transformation property:

$$\Gamma^{[A]}_{z;g_1 \circ g_2} f(\Sigma) = \Gamma^{[A]}_{z;g_2} f(\Sigma) + (\partial g_2(z))^2 \, \Gamma^{[g_2(A)]}_{g_2(z);g_1} (f \circ g_2^{-1})(g_2(\Sigma))$$

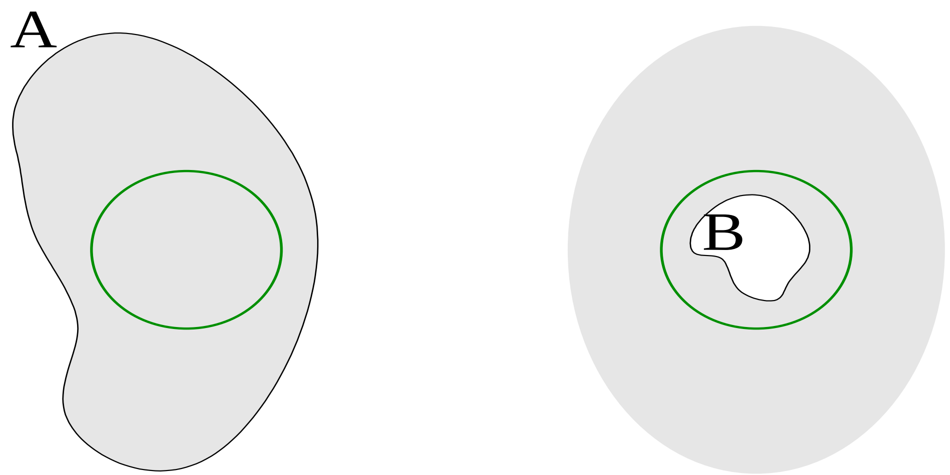
- It is like a **connection 1-form** but lives in the **class fiber bundle** instead of the cotangent space. Could it be that the functional dependence is multiplicative? Then,

$$\Gamma^{[A]}_{z;g} f(\Sigma) = \{g, z\} c(f)(\Sigma)$$

where $\{g, z\}$ is the Schwarzian derivative.

General transformation of global derivatives

Consider two domains A and B such that $\hat{\mathbb{C}} \setminus A \subset B$.



Consider a conformal map $g : A \to A'$. Then

$$\Delta_z^{[B]} f(\Sigma) - (\partial g(z))^2 \, \Delta_{g(z)}^{[\hat{\mathbb{C}} \setminus g(\hat{\mathbb{C}} \setminus B)]} (f \circ g^{-1})(g(\Sigma)) = \Gamma_{z;g}^{[A]} f(\Sigma)$$

In particular, if f is A-stationary (A-derivative $= 0$), then the r.h.s. is 0.

Application to CFT

Conformal Ward identities with boundary

Consider a CFT correlation function of n fields $\mathcal{O}_j$ at positions z_j in a domain C:

$$\langle \prod_{j=1}^{n} \mathcal{O}_j(z_j) \rangle_C$$

Covariant under conformal maps $g : C \to C'$

$$\langle \prod_{j=1}^{n} (g \cdot \mathcal{O}_j)(g(z_j)) \rangle_{g(C)} = \langle \prod_{j=1}^{n} \mathcal{O}_j(z_j) \rangle_C$$

$g \cdot \mathcal{O}_j$ is a linear transformation (over a ring of functions). Example: primary fields

$$(g \cdot \mathcal{O})(g(z)) = (\partial g(z))^{\delta} (\bar{\partial} \bar{g}(\bar{z}))^{\tilde{\delta}} \mathcal{O}(g(z))$$

Consider

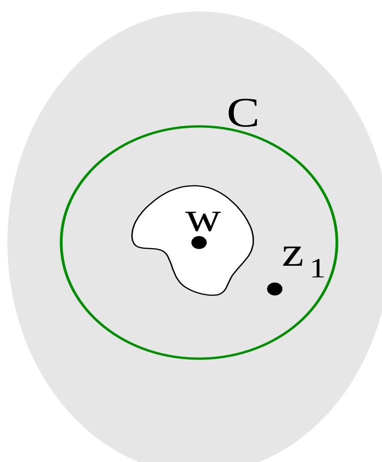
$$\Sigma = (\partial C; z_1, \ldots, z_n; \mathcal{O}_1, \ldots, \mathcal{O}_n) \in \text{domains} \times \mathbb{C}^n \times \text{fields}^{\otimes n}$$

$$g(\Sigma) = (g(\partial C); g(z_1), \ldots, g(z_n); g \cdot \mathcal{O}_1, \ldots, g \cdot \mathcal{O}_n)$$

$$f(\Sigma) = \langle \prod_{j=1}^{n} \mathcal{O}_j(z_j) \rangle_C : \text{ globally stationary, and } A\text{-stationary for any } A \supset \overline{C}$$

Insertion of stress-energy tensor is given by global derivative:

$$\langle T(w) \prod_{j=1}^{n} \mathcal{O}_j(z_j) \rangle_C - \langle T(w) \rangle_C \langle \prod_{j=1}^{n} \mathcal{O}_j(z_j) \rangle_C = \Delta_w^{[\hat{\mathbb{C}} \setminus N(w)]} f(\Sigma)$$



One-point average and partition functions

Consider the ratio of partition functions

$$Z(C|D) = \frac{Z_C Z_{\hat{\mathbb{C}}\setminus\overline{D}}}{Z_{C\setminus\overline{D}} Z_{\hat{\mathbb{C}}}}.$$

Using Liouville action for transformation of partition functions as well as the basic formula

$$\delta \log Z_A = \frac{1}{2}\int_{\overline{A}} d^2x \, \langle \delta\eta_{ab}(x) T^{ab}(x)\rangle_A$$

we find

$$\langle T(w)\rangle_C = \Delta_{w\,|\,\partial C\cup\partial D}^{[\hat{\mathbb{C}}\setminus N(w)]} \log Z(C|D)$$

Application to CLE

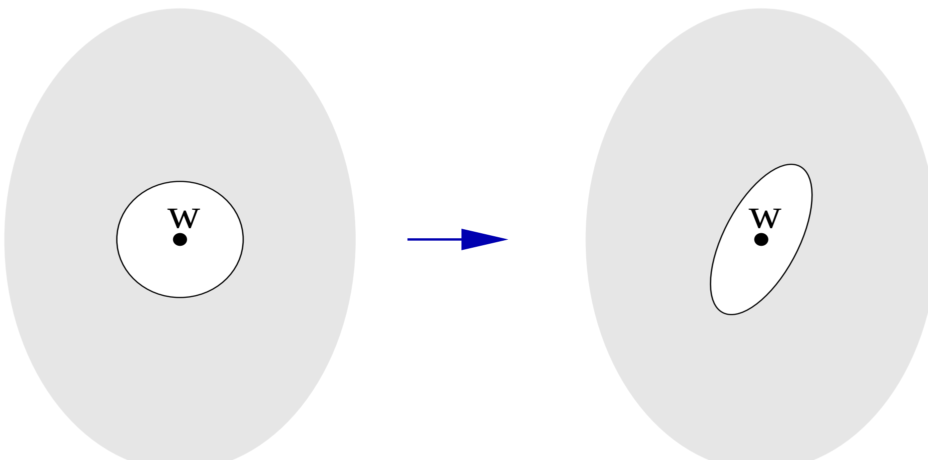
Consider

$$h_w = \frac{e^{2\pi i\theta}}{w - z}$$

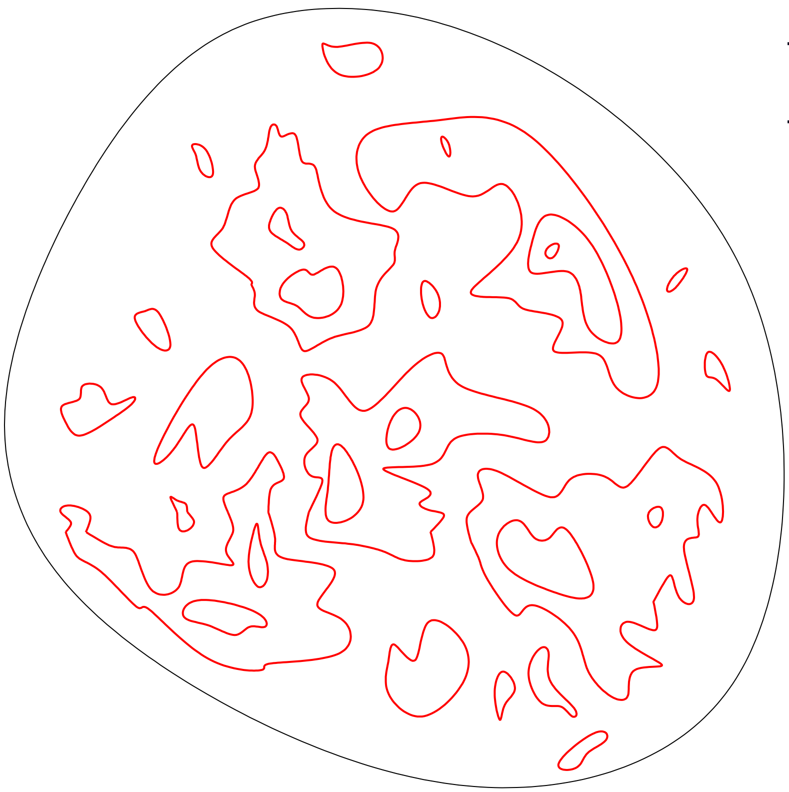
It is such that

$$\frac{1}{2\pi}\int d\theta\, e^{-2\pi i\theta}\,\nabla_{h_w} f(\Sigma) = \Delta_w^{[\hat{\mathbb{C}}\setminus N(w)]} f(\Sigma)$$

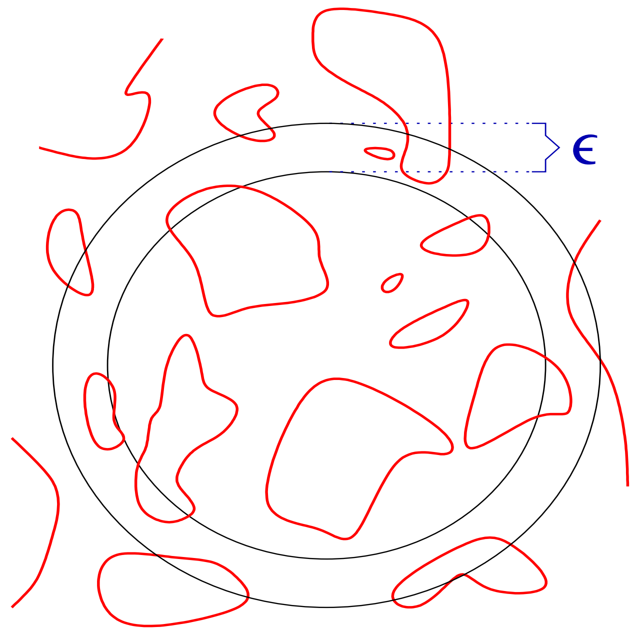
Interpret $\nabla_{h_w} f(\Sigma)$ **geometrically**: $\mathrm{id} + \eta h_w$ gives



CLE:



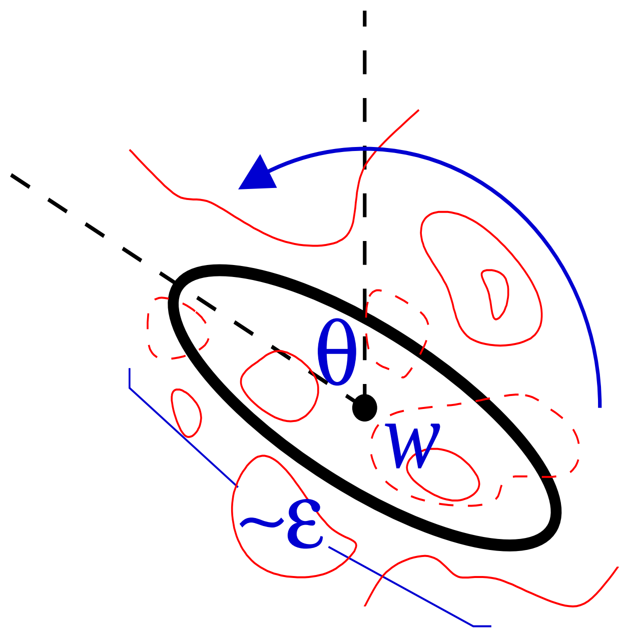
Renormalised probabilities $P(X;A)_C$ that no loop crosses boundary ∂A of thickness $\epsilon \longrightarrow 0$:



Theorems:

- $P(X)_A = P(X|A)_C := P(X;A)_C / P(A)_C$ (X supported inside A)
- $P(X)_{C \setminus A} = P(X|A)_C$ (X supported inside $C \setminus \overline{A}$)
- $P(A)_C$ is **global conformally invariant**, but in general conformally **covariant**
- ...

$T(w)$ =



This reproduces **conformal Ward identities**

$$\langle T(w)\mathcal{O}_X\rangle_C - \langle T(w)\rangle_C P(X)_C = \Delta_w^{[\hat{\mathbb{C}}\setminus N(w)]} P(X)_C$$

as well as **one-point average with identification:**

$$Z(C|D) = \frac{P(\hat{\mathbb{C}}\setminus \overline{C})_{\hat{\mathbb{C}}}}{P(\hat{\mathbb{C}}\setminus \overline{C})_{\hat{\mathbb{C}}\setminus\overline{D}}}$$

Perspectives

- Applications to other probability models of CFT
- Descendants: derivative $\partial/\partial z$ of $\Delta_z^{[\hat{\mathbb{C}}\setminus N(w)]} f(\Sigma)$; multiple conformal derivatives
- Other symmetry currents when internal symmetries are present...