Linear integral equations for finite-temperature dynamical correlation functions in the quantum Ising model

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The problem of finite-temperature correlation functions in real time

- Near quantum criticality, at **temperatures, energies and momenta of the order of the gap**, what is observed is described by **finite-temperature real-time correlation functions of QFT**

$$\langle \mathcal{O}(x,t)\mathcal{O}(0,0)\rangle_T = \frac{\mathrm{Tr}\left(e^{-H/T}\mathcal{O}(x,t)\mathcal{O}(0,0)\right)}{\mathrm{Tr}\left(e^{-H/T}\right)}$$

- Neutron scattering experiments $\Rightarrow$ dynamical structure factor $\overset{F.T.}{\Leftrightarrow} \langle \mathcal{O}(x,t)\mathcal{O}(0,0)\rangle_T$

What does the dynamical structure factor look like at low energies
(non-perturbative regime of QFT)?

Main idea of the talk

- The correlation functions of the quantum Ising model at finite temperature form a solution to an **integrable non-linear partial differential equation (sine/sinh-Gordon equation)**.
- There is a method to solve such equations: **the inverse scattering method**. It gives the solution at all times for any given initial condition. The initial condition is encoded into **scattering data**. A way of representing the solution is in terms of linear integral equations (Gelfand-Levitan-Marchenko equations), which take the scattering data as input.
- We show that the scattering data that corresponds to the Ising correlation functions are obtained from the **finite-temperature form factors** introduced and calculated some time ago [BD 2005, 2006]. This solves the problem in the Ising model.

Integrable QFT: results from exact form factors?

- The Hilbert space of QFT is described by **asymptotically free particles** with fixed rapidities θ_j.
- Integrable QFT: in many cases, **matrix elements are known** (form factors)

$$\langle\theta_1,\ldots,\theta_m|\mathcal{O}(0,0)|\theta_1',\ldots,\theta_n\rangle$$

- Direct calculation:

$$\langle\mathcal{O}(x,t)\mathcal{O}(0,0)\rangle_T \propto \sum_{\substack{\text{state } v\\ \text{state } w}} e^{-E_v/T}\langle v|\mathcal{O}(x,t)|w\rangle\langle w|\mathcal{O}(0,0)|v\rangle$$

an infinite series of plane waves with coefficients given by squares of form factors

Two problems:

- Poles in form factors need to be regularised (normalisation of fields and coefficients of plane waves are not given by form factors)
- The expansions in the space-like and time-like regions must be very different: the continuation from one region to another must involve an infinite re-summation

A paradigmatic example: the quantum Ising model

Quantum spin-$1/2$ chain in a transverse magnetic field:

$$H = -\sum_j (J\sigma_j^z \sigma_{j+1}^z + h\sigma_j^x)$$

There is a quantum critical point at a special value h_c of h

$\Rightarrow$ QFT of free massive Majorana fermions

Twist fields in Majorana QFT	Operators in quantum spin chain
$\sigma(x)$	$a^{-1/8}\sigma^z_{x/a}$ for $h < h_c$ (ordered regime)
$\mu(x)$	$a^{-1/8}\sigma^z_{x/a}$ for $h > h_c$ (disordered regime)

The finite-temperature expansion in space-like region

- **Form factors on the cylinder**:
  - large x expansion at $t = 0$ from form factors on the cylinder [Bugrij 2000, 2001]
  - analytically continued to $t^2 < x^2$ [Altshuler, Konik, Tsvelik 2005, 2006]
- **Finite-temperature form factors** [BD 2005, 2006] (directly gives $t^2 < x^2$)
  - Liouville space $\mathcal{L}$: space of operators, with $\{A^+(\theta), A^-(\theta')\} = \delta(\theta - \theta')$

$$|\theta, \pm\rangle^{\mathcal{L}} \equiv (1 - e^{\mp \frac{m \cosh \theta}{T}}) A^{\pm}(\theta)$$

$$|\theta, \pm; \theta', \pm'\rangle^{\mathcal{L}} \equiv (1 - e^{\mp \frac{m \cosh \theta}{T}})(1 - e^{\mp' \frac{m \cosh \theta'}{T}}) A^{\pm}(\theta) A^{\pm'}(\theta')$$

  - inner product: ${}^{\mathcal{L}}\langle v|w\rangle^{\mathcal{L}} = \frac{\mathrm{Tr}\left(e^{-H/T}\mathcal{U}\ V^\dagger W\right)}{\mathrm{Tr}\left(e^{-H/T}\mathcal{U}\right)}$ for $|v\rangle^{\mathcal{L}} \equiv V,\ |w\rangle^{\mathcal{L}} \equiv W$
  - right-action of fields

$${}^{\mathcal{L}}\langle v|\hat{\mathcal{O}}(x,t)|w\rangle^{\mathcal{L}} = \frac{\mathrm{Tr}\left(e^{-H/T}\mathcal{U}\ V^\dagger \mathcal{O}(x,t) W\right)}{\mathrm{Tr}\left(e^{-H/T}\mathcal{U}\right)}$$

– finite-temperature form factors

$$F^{\sigma_\pm}_{\epsilon_1,\ldots,\epsilon_k}(\theta_1,\ldots,\theta_k) = {}^{\mathcal{L}}\langle \mathrm{vac}|\hat{\sigma}_\pm(0,0)|\theta_1,\epsilon_1;\ldots;\theta_k,\epsilon_k\rangle^{\mathcal{L}} =$$

$$\prod_{j=1}^{k}\left(1-e^{-\frac{\epsilon_j\, m\cosh\theta_i}{T}}\right)\frac{\mathrm{Tr}\left(e^{-H/T}\mathcal{U}\;\sigma_\pm(0,0)A^{\epsilon_1}(\theta_1)\cdots A^{\epsilon_k}(\theta_k)\right)}{\mathrm{Tr}\left(e^{-H/T}\mathcal{U}\right)}$$

– finite-temperature two-point function as “vacuum expectation value”

$$\langle\sigma(x,t)\sigma(0,0)\rangle_T = {}^{\mathcal{L}}\langle \mathrm{vac}|\hat{\sigma}_+(x,t)\mathbf{1}^{\mathcal{L}}\hat{\sigma}_-(0,0)|\mathrm{vac}\rangle^{\mathcal{L}}$$

– expansion from decomposition of the identity

$$\mathbf{1}^{\mathcal{L}} = \sum_{k=0}^{\infty}\sum_{\substack{\epsilon_1,\ldots,\epsilon_k\\ =\pm}}\int_{\mathrm{Im}(\theta_j)=\epsilon_j 0^+} d\theta_1\cdots d\theta_k \frac{|\theta_1,\epsilon_1;\ldots;\theta_k,\epsilon_k\rangle^{\mathcal{L}}\,{}^{\mathcal{L}}\langle\theta_1,\epsilon_1;\ldots;\theta_k,\epsilon_k|}{\prod_{j=1}^{k}\left(1-e^{-\frac{\epsilon_j\, m\cosh\theta_i}{T}}\right)}$$

Exact finite-temperature form factors are obtained by solving a Riemann-Hilbert problem [BD 2005, 2006]

$$F^{\sigma\pm}_{\epsilon_1,\ldots,\epsilon_k}(\theta_1,\ldots,\theta_k) \propto \prod_{j=1}^{k} h_{\pm\epsilon_j}(\theta_j) \prod_{1\leq i<j\leq k} \left(\tanh\left(\frac{\theta_j-\theta_i}{2}\right)\right)^{\epsilon_i\epsilon_j}$$

$$h_\pm(\theta) = e^{\pm\frac{i\pi}{4}} \exp\left[\pm\int_{-\infty\mp i0^+}^{\infty\mp i0^+} \frac{d\theta'}{2\pi i}\frac{1}{\sinh(\theta-\theta')} \ln\left(\frac{1+e^{-\frac{m\cosh\theta'}{T}}}{1-e^{-\frac{m\cosh\theta'}{T}}}\right)\right]$$

$$h_-(\theta) \quad = -h_-(\theta+2\pi i)$$

has zeroes at $\theta = \frac{i\pi}{2} + \text{arcsinh}\left(\frac{2\pi n T}{m}\right),\ n\in\mathbb{Z}$

has poles at $\theta = \frac{i\pi}{2} + \text{arcsinh}\left(\frac{2\pi n T}{m}\right),\ n\in\mathbb{Z}+\frac{1}{2}$

Going to time-like region?

The expansion is space-like only

$$\cdots \int_{\mathrm{Im}(\theta_j)=\epsilon_j 0^+} d\theta_1 \cdots d\theta_k e^{\sum_{j=1}^{k} i\epsilon_j m(x\sinh\theta_j - t\cosh\theta_j)} \cdots \Rightarrow \text{convergence for } t^2 < x^2$$

Obtaining a time-like expansion requires infinite re-summations

- Semi-classical approximation to go around this problem ($T \ll m$ only) [Sachdev 1996, Sachdev, Young 1997]
- Partial resummation, valid (conjecturally) for $T \ll m$ [Altshuler, Konik, Tsvelik 2005, 2006]
- Other ways to go around the problem ($T \ll m$ only) [Reyes, Tsvelik 2006]

Our method: correlation functions from classical integrability

The dynamical correlation functions of the quantum Ising chain satisfy

integrable partial differential equations

$$\langle \sigma(x,t)\sigma(0,0)\rangle_T = e^{\chi/2}\cosh(\varphi/2)\,, \quad \langle \mu(x,t)\mu(0,0)\rangle_T = e^{\chi/2}\sinh(\varphi/2)$$

$$\begin{aligned}
(\partial_x^2 - \partial_t^2)\varphi &= \frac{m^2}{2}\sinh(2\varphi) \\
(\partial_x^2 - \partial_t^2)\chi &= \frac{m^2}{2}(1-\cosh(2\varphi)) \\
(\partial_x^2 + \partial_t^2)\chi &= -(\partial_x\varphi)^2 - (\partial_t\varphi)^2 \\
\partial_x\partial_t\chi &= -\partial_x\varphi\partial_t\varphi
\end{aligned}$$

The inverse scattering method

initial condition $\varphi(x,0)$ $\xrightarrow{\text{scattering problem}}$ initial scattering data $a(\theta), b(\theta)$

$\downarrow$ $\downarrow$

solution $\varphi(x,t)$ $\xleftarrow{\text{GLM integral equations}}$ $a(\theta,t) = a(\theta)\ , \quad b(\theta,t) = b(\theta)e^{itm\cosh\theta}$

Two problems to solve:

- Find initial scattering data $a(\theta), b(\theta)$
- Obtain large-t asymptotics of $\varphi(x,t)$ from GLM equations

Zero-curvature formulation and scattering data

The compatibility condition of the equations

$$(\partial_x - A_x)\Psi(x,t;\theta) = (\partial_t - A_t)\Psi(x,t;\theta) = 0$$

(or zero-curvature condition of the connections A_x, A_t), with

$$A_x = \frac{i}{4}\begin{pmatrix} 2i\partial_t\varphi & m(e^{\theta-\varphi} - e^{\varphi-\theta}) \\ m(e^{\varphi+\theta} - e^{-\varphi-\theta}) & -2i\partial_t\varphi \end{pmatrix}$$

$$A_t = \frac{i}{4}\begin{pmatrix} 2i\partial_x\varphi & -m(e^{\theta-\varphi} + e^{\varphi-\theta}) \\ -m(e^{\varphi+\theta} + e^{-\varphi-\theta}) & -2i\partial_x\varphi \end{pmatrix}$$

for all $\theta \in \mathbb{R}$, is equivalent to the sinh-Gordon equation for φ

The scattering problem is

$$(\partial_x - A_x)\Psi(x;\theta) = 0$$

The scattering data are coefficients in the Jost solutions to the scattering problem: independent solutions analytic in the strip $\mathrm{Im}(\theta) \in [0,\pi]$:

	$x \to \infty$	$x \to -\infty$
$\Psi_{J_+}(x;\theta)$	$v_+(x;\theta)$	$a(\theta)v_+(x;\theta) - b(\theta)v_-(x;\theta)$
$\Psi_{J_-}(x;\theta)$	$c(\theta)v_+(x;\theta) - d(\theta)v_-(x;\theta)$	$v_-(x;\theta)$

$$d = -a,\ b^* = -b,\ |a|^2 + bc^* = 1$$

$$v_+(x;\theta) = e^{\frac{ixm\sinh\theta}{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_-(x;\theta) = e^{\frac{-ixm\sinh\theta}{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Wronskian equations imply that $a(\theta)$ is analytic in the strip $\mathrm{Im}(\theta) \in [0,\pi]$

A special solution to the scattering problem

With $\varphi(x)$ given by the finite-temperature correlation functions at $t = 0$, a solution is

$$\Psi = \Psi_{\rm sym} \equiv e^{-\chi/2} \left(\begin{array}{c} \tilde{F} - iF \\ \tilde{F} + iF \end{array} \right)$$

$$\begin{array}{rcl} F(x;\theta) & = & {}^{\mathcal{L}}\langle {\rm vac}|\hat{\sigma}_+(x/2,0)\hat{A}^+(\theta)\hat{\mu}_-(-x/2,0)|{\rm vac}\rangle^{\mathcal{L}} \\ \tilde{F}(x;\theta) & = & {}^{\mathcal{L}}\langle {\rm vac}|\hat{\mu}_+(x/2,0)\hat{A}^+(\theta)\hat{\sigma}_-(-x/2,0)|{\rm vac}\rangle^{\mathcal{L}} \end{array}$$

Generalisation of the zero-temperature case showed by Fonseca and Zamolodchikov [2003]. Two copies of the Majorana theory, a and b; resulting conserved $U(1)$ charge Z_0; consequences of the conserved charge $[P_a - P_b, Z_0]$ on the objects above.

This solution is invariant under the symmetry transformations

- $\Psi^{\rm v}(x;\theta) = \sigma^z \Psi(x;\theta + i\pi)$
- $\bar{\Psi}(x;\theta) = \Psi^*(-x;\theta)$

The asymptotics of this special solution can be obtained from the **finite-temperature form factors** by using the resolution of the identity $\mathbf{1}^{\mathcal{L}}$:

$$
\begin{aligned}
&{}^{\mathcal{L}}\langle \mathrm{vac}|\hat{\sigma}_+(x/2,0)\hat{A}^+(\theta)\hat{\mu}_-(-x/2,0)|\mathrm{vac}\rangle^{\mathcal{L}} \\
&\qquad \overset{x\to\infty}{\sim} \; {}^{\mathcal{L}}\langle \mathrm{vac}|\hat{\sigma}_+(x/2,0)|\mathrm{vac}\rangle^{\mathcal{L}} \; {}^{\mathcal{L}}\langle \mathrm{vac}|\hat{A}^+(\theta)\hat{\mu}_-(-x/2,0)|\mathrm{vac}\rangle^{\mathcal{L}}
\end{aligned}
$$

We then obtain the following asymptotics:

	$x \longrightarrow \infty$	$x \longrightarrow -\infty$
$\Psi_{\rm sym}(x;\theta)$	$g_+ h_+ v_+(x;\theta) - g_- h_- v_-(x;\theta)$	$i g_+ h_- v_+(x;\theta) - i g_- h_+ v_-(x;\theta)$

$$g_\pm(\theta) = \frac{1}{1 - e^{\mp \frac{m \cosh\theta}{T}}}$$

$h_\pm(\theta) =$ one-particle finite-temperature form factors

The scattering data

Inspired by this explicit solution, we make the following ansatz for the scattering data

$$a(\theta) = \alpha(\theta)\frac{h_-(\theta)}{h_+(\theta)}\ , \quad b(\theta) = i\beta(\theta)\frac{g_-(\theta)}{g_+(\theta)}$$

- x-independence of the wronskian $\det(\Psi_{\rm sym}, \Psi_{J_+})$

- $\Psi^{\rm v}_{J_+}$ and $\bar{\Psi}_{J_+}$ can be written as linear combinations of $\Psi_{\rm sym}$ and Ψ_{J_+}

- analyticity of $a(\theta)$ in the strip $\mathrm{Im}(\theta) \in [0, \pi]$

- large-θ analysis

$$\Downarrow$$

$$\beta(\theta) = 1 + \alpha(\theta)$$

$\alpha(\theta) \in \mathbb{R}$ for $\theta \in \mathbb{R}$

$$\alpha(\theta + i\pi) = -\alpha(\theta)$$

$\alpha(\theta) \sim 1$ as $\theta \longrightarrow \pm\infty$

$\alpha(\theta)$ has zeroes at $\theta = \frac{i\pi}{2} + \mathrm{arcsinh}\,\left(\frac{2\pi n T}{m}\right),\ n \in \mathbb{Z} + \frac{1}{2}$

$\alpha(\theta)$ is analytic for $\mathrm{Im}(\theta) \in [0, \pi]$ except maybe for poles at $\theta = \frac{i\pi}{2} + \mathrm{arcsinh}\,\left(\frac{2\pi n T}{m}\right),\ n \in \mathbb{Z}$

The unique solution is

$$\alpha(\theta) = \frac{1 + e^{-\frac{m\cosh\theta}{T}}}{1 - e^{-\frac{m\cosh\theta}{T}}}\,, \quad \beta(\theta) = \frac{2}{1 - e^{-\frac{m\cosh\theta}{T}}}$$

The Gelfand-Levitan-Marchenko linear integral equations

$$e^{2\varphi(x)} = 1 + \frac{4i}{m} W(x,x)^- - \frac{4i}{m} W(x,x)^+ + \frac{16}{m^2}\left(U(x,x)^- - U(x,x)^+\right) U(x,x)^+ - \frac{16}{m^2}\left(\partial_x U(x,y)^+ + \partial_y U(x,y)^-\right)|_{x=y}$$

$$-\frac{2}{m}\sigma^z U(x,y) = F_0(x+y)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \int_x^\infty \Big[F_0(y+z)U(x,z) + F_{-1}(y+z)W(x,z)\Big] dz$$

$$\frac{2}{m}\sigma^z W(x,y) = F_{-1}(x+y)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \int_x^\infty \Big[F_{-1}(y+z)U(x,z) + F_{-2}(y+z)W(x,z)\Big] dz$$

$$F_j(x) = \frac{1}{4\pi}\int_{-\infty}^{\infty} d\theta e^{(j+1)\theta}\left(e^{\frac{ixm\sinh\theta}{2}}\frac{b(\theta+i\pi)}{a(\theta)} + (-1)^j e^{\frac{-ixm\sinh\theta}{2}}\frac{b(\theta)}{a(\theta+i\pi)}\right)$$

Conclusions and perspectives

We derived linear integral equations that determine the finite-temperature dynamical correlation functions in the quantum Ising model near its critical point

- We have checked that it reproduces the known finite-temperature form factor expansion in the space-like region $t^2 < x^2$, up to (including) three-particle terms
- Calculation of the near-light-cone time-like asymptotic $t \longrightarrow \infty,\ x \longrightarrow \infty$ with $0 < t - x \ll t, x$, for all m, T, is in progress – check of unrigorous proposed asymptotics will be possible, with extension to $T \sim m$
- This is a systematic method to evaluate any expansion of the finite-temperature Ising correlators; numerical solution could also be useful
- Structure of expansion:
  - Wick's theorem $\longrightarrow$ classical integrable PDE
  - Two-particle form factors $\longrightarrow$ structure of linear problem
  - One-particle form factor (leg-factors) $\longrightarrow$ scattering data