Branch-point twist fields and entanglement entropy in integrable quantum field theory

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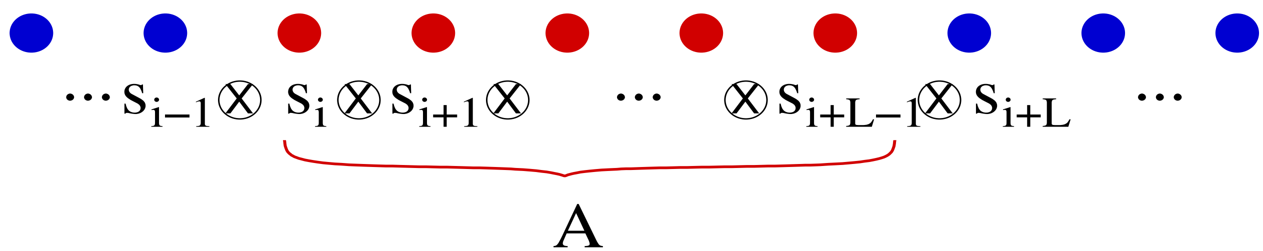
Entanglement entropy

A measure of the **quantity of entanglement** between different parts of a quantum system (here: in its ground state).

- Reduced density matrix:

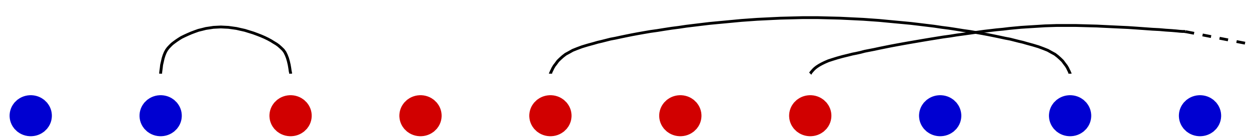
$$\rho_A = \mathrm{Tr}_{\bar{A}}(|\mathrm{gs}\rangle\langle\mathrm{gs}|)$$



- Entanglement entropy:

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A))$$

It is the "number of links between A and $\bar{A}$ in the ground state" $\Rightarrow$ $S_A = S_{\bar{A}}$.



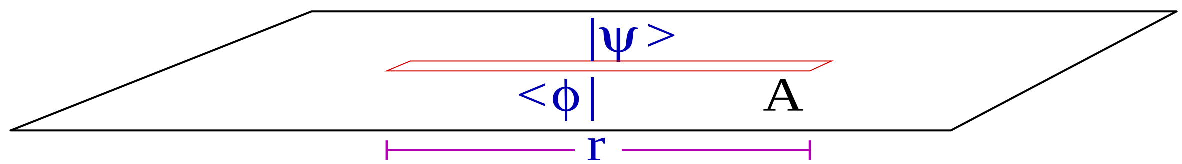
Scaling limit and partition functions on multi-sheeted Riemann surfaces

- Scaling limit: correlation length $\xi \to \infty,\ L/\xi = mr$ fixed

QFT, mass m, lagrangian density $\mathcal{L}[\varphi]$

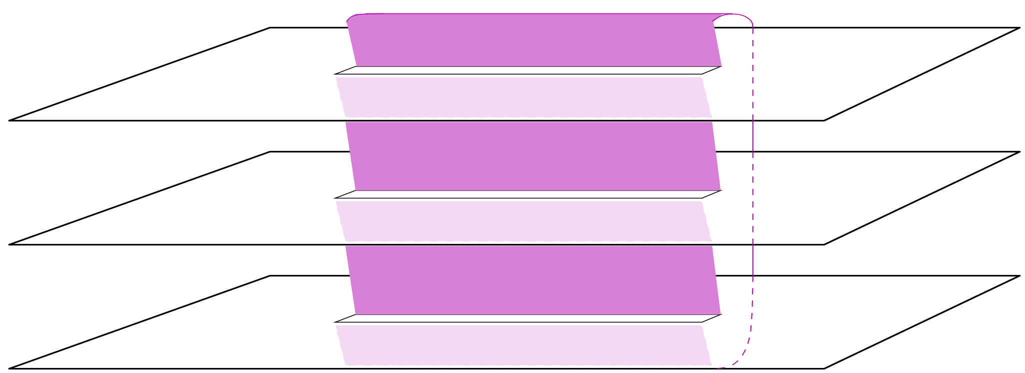
- "Replica trick:" $S_A = -\lim_{n\to 1} \frac{d}{dn}\mathrm{Tr}_A(\rho_A^n)$
- Partition function on Riemann surfaces for $n \in \mathbb{N}$ in the scaling limit:

$${}_A\langle\phi|\rho_A|\psi\rangle_A \quad \sim$$

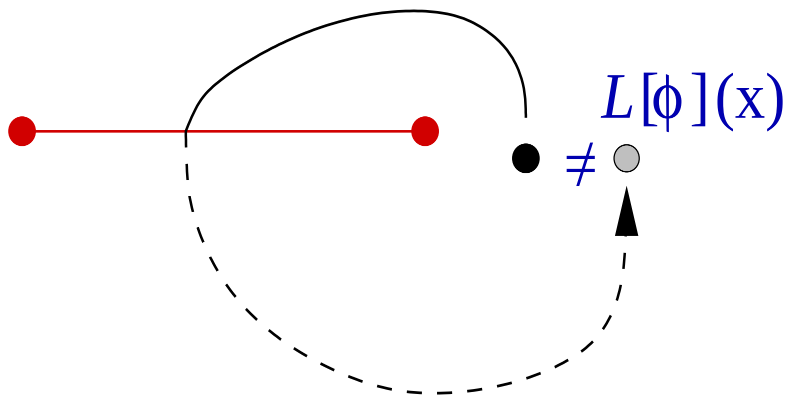


$$\mathrm{Tr}_A(\rho_A^n) \quad \sim \quad Z_n = \int [d\varphi]_{\mathcal{M}_n} \exp\left[-\int_{\mathcal{M}_n} d^2x\, \mathcal{L}[\varphi](x)\right]$$

$\mathcal{M}_n$:



Branch points are not local fields in the QFT $\mathcal{L}$



$$Z_n \not\propto \langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}}$$

Branch-point twist fields

Local twist fields associated to cyclic permutation symmetry of the n-copy model

- Multi-copy model on $\mathbb{R}^2$:

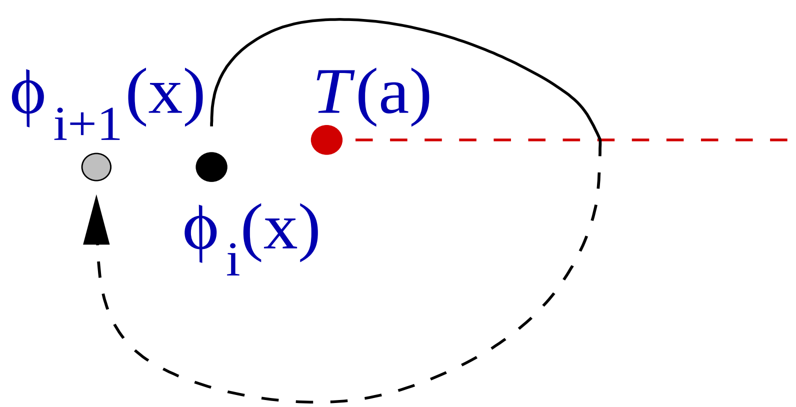
$$\mathcal{L}^{(n)}[\varphi_1,\ldots,\varphi_n](x) = \mathcal{L}[\varphi_1](x) + \ldots + \mathcal{L}[\varphi_n](x)$$

- Symmetry $\mathcal{L}^{(n)}[\sigma\varphi_1,\ldots,\sigma\varphi_n] = \mathcal{L}^{(n)}[\varphi_1,\ldots,\varphi_n]$, with $\sigma\varphi_i = \varphi_{i+1 \bmod n}$

- Associated twist fields $\mathcal{T}$:

$$\langle \mathcal{T}(a)\cdots\rangle_{\mathcal{L}^{(n)}} \propto \int_{C_a} [d\varphi_1\cdots d\varphi_n]_{\mathbb{R}^2} \exp\left[-\int_{\mathbb{R}^2} \mathcal{L}^{(n)}[\varphi_1,\ldots,\varphi_n](x)\right]$$

C_a:

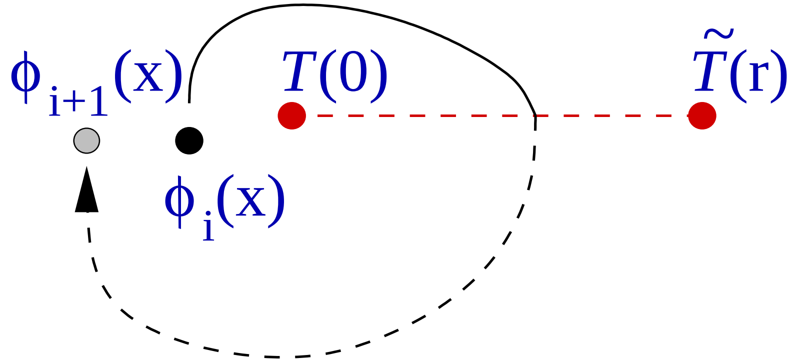


Branch points are local fields in the QFT $\mathcal{L}^{(n)}$

With additional twist field $\tilde{\mathcal{T}}$ associated to the inverse symmetry σ^{-1}, we have

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} \propto \int_{C_{0,r}} [d\varphi_1 \cdots d\varphi_n]_{\mathbb{R}^2} \exp\left[-\int_{\mathbb{R}^2} \mathcal{L}^{(n)}[\varphi_1,\ldots,\varphi_n](x)\right] = Z_n$$

$C_{0,r}$:



Short- and large-distance entanglement entropy

$$Z_n = \varepsilon^{2d_n} \langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} \;, \quad S_A = -\lim_{n\to 1} \frac{d}{dn} Z_n$$

where ε is a non-universal short-distance cutoff and d_n is the scaling dimension of $\mathcal{T}$:

$$d_n = \frac{c}{12}\left(n - \frac{1}{n}\right)$$

[Calabrese and Cardy, 2004]

- Short distance: logarithmic behavior

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} \sim r^{-2d_n} \;\Rightarrow\; S_A \sim -\frac{c}{3}\log\left(\frac{\varepsilon}{r}\right)$$

- Large distance: saturation

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} \sim \langle \mathcal{T}\rangle^2_{\mathcal{L}^{(n)}} \;\Rightarrow\; S_A \sim -\frac{c}{3}\log(m\varepsilon) - U$$

$$U = \frac{d}{dn}\left(m^{-2d_n}\langle \mathcal{T}\rangle^2_{\mathcal{L}^{(n)}}\right)\Big|_{n=1}$$

Our result [Cardy, Castro Alvaredo, D.], [Castro Alvaredo, D.]: for any massive integrable QFT, the entropy with its first correction to saturation at large distances is

$$S_A \sim -\frac{c}{3}\log(m\varepsilon) - U - \frac{1}{8}\sum_{\alpha=1}^{\ell} K_0(2rm_\alpha) + O\left(e^{-3rm_1}\right)$$

where ℓ is the number of particles in the spectrum of the QFT, and m_α are the masses of the particles, with $m_1 \leq m_\alpha \, \forall\alpha$.

Scattering matrix in integrable quantum field theory

In scattering:

- the number of particles and the set of their momenta are conserved
- the scattering matrix **factorises** into a product of two-particle scattering matrices, as if particles were interacting by pairs at space-time points that are far apart

Analytic properties and Yang-Baxter equation for the two-particle scattering matrix gives a Riemann-Hilbert problem that can be solved

Form factors of branch-point twist fields

For an integrable QFT $\mathcal{L}$ with a spectrum of one particle, no bound state, and S-matrix $S(\theta)$

- Scattering matrix of $\mathcal{L}^{(n)}$:

$$
\begin{aligned}
S_{ii}(\theta) &= S(\theta) \qquad \forall \quad i = 1, \ldots, n, \\
S_{ij}(\theta) &= 1, \qquad \forall \quad i, j = 1, \ldots, n \quad \text{and} \quad i \neq j,
\end{aligned}
$$

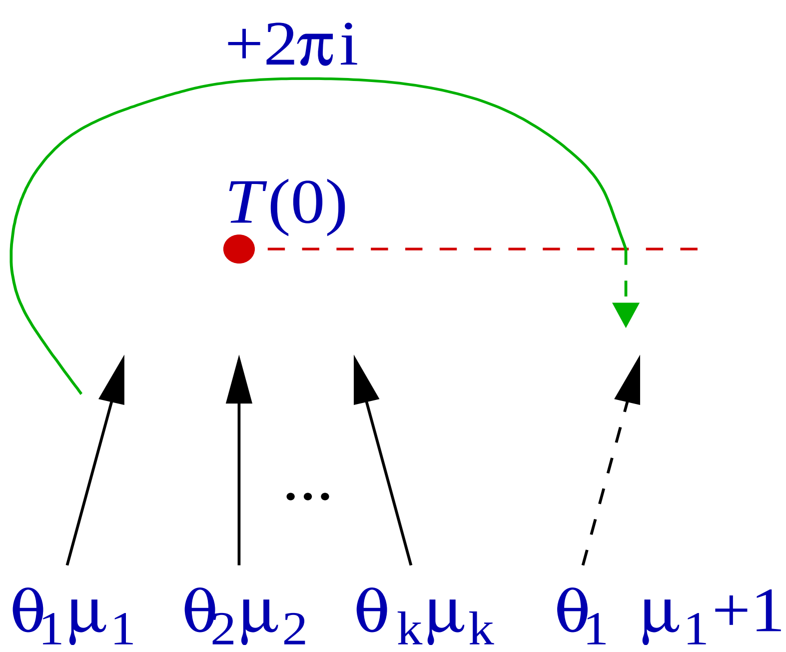
- Form factors of branch-point twist field in $\mathcal{L}^{(n)}$:

$$
F_k^{\mu_1 \ldots \mu_k}(\theta_1, \ldots, \theta_k) := \langle \mathrm{gs} | \mathcal{T}(0) | \theta_1, \ldots, \theta_k \rangle^{\text{in}}_{\mu_1, \ldots, \mu_k}
$$

$$
\begin{aligned}
&F_k^{\ldots \mu_i \mu_{i+1} \ldots}(\ldots, \theta_i, \theta_{i+1}, \ldots) = S_{\mu_i \mu_{i+1}}(\theta_i - \theta_{i+1}) F_k^{\ldots \mu_{i+1} \mu_i \ldots}(\ldots, \theta_{i+1}, \theta_i, \ldots) \\
&F_k^{\mu_1 \mu_2 \ldots \mu_k}(\theta_1 + 2\pi i, \ldots, \theta_k) = F_k^{\mu_2 \ldots \mu_k \; \mu_1 + 1}(\theta_2, \ldots, \theta_k, \theta_1) \\
&-i\text{Res}_{\bar{\theta}_0 = \theta_0} F_{k+2}^{\mu \mu \mu_1 \ldots \mu_k}(\bar{\theta}_0 + i\pi, \theta_0, \theta_1 \ldots, \theta_k) = F_k^{\mu_1 \ldots \mu_k}(\theta_1, \ldots, \theta_k) \\
&-i\text{Res}_{\bar{\theta}_0 = \theta_0} F_{k+2}^{\mu \; \mu+1 \; \mu_1 \ldots \mu_k}(\bar{\theta}_0 + i\pi, \theta_0, \theta_1 \ldots, \theta_k) = -\prod_{i=1}^{k} S_{\mu \mu_i}(\theta_{0i}) F_k^{\mu_1 \ldots \mu_k}(\theta_1, \ldots, \theta_k)
\end{aligned}
$$

The quasi-periodicity relation

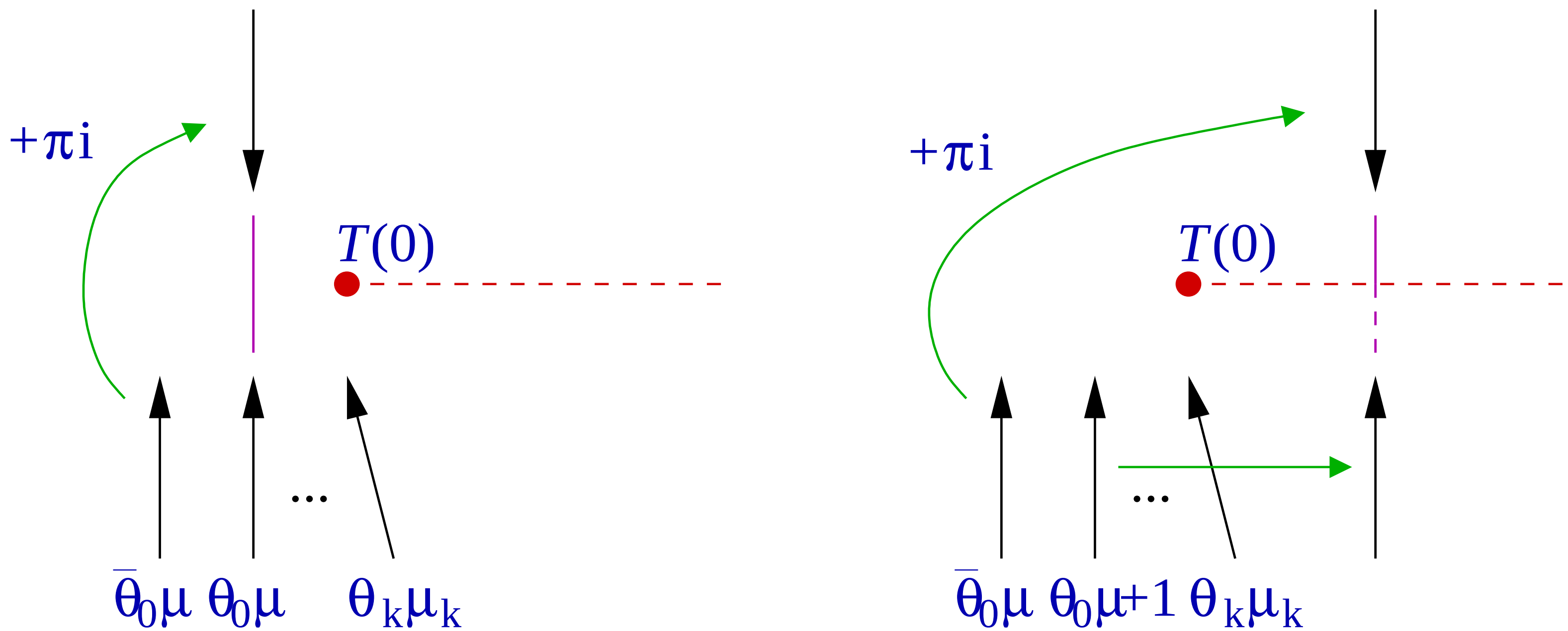
$$F_k^{\mu_1\mu_2\cdots\mu_k}(\theta_1+2\pi i,\ldots,\theta_k)=F_k^{\mu_2\cdots\mu_k\mu_1+1}(\theta_2,\ldots,\theta_k,\theta_1)$$



The kinematic residue equations

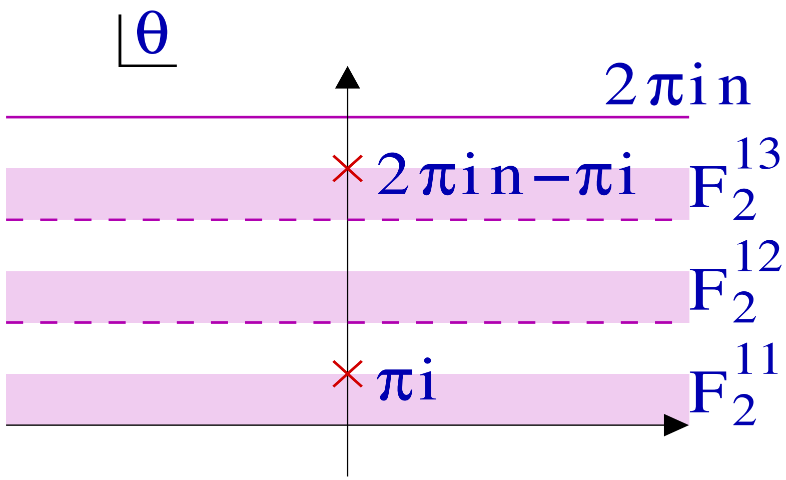
$$-i\text{Res}_{\bar{\theta}_0=\theta_0} F_{k+2}^{\mu\mu\mu_1\ldots\mu_k}(\bar{\theta}_0 + i\pi, \theta_0, \theta_1 \ldots, \theta_k) = F_k^{\mu_1\ldots\mu_k}(\theta_1, \ldots, \theta_k)$$

$$-i\text{Res}_{\bar{\theta}_0=\theta_0} F_{k+2}^{\mu\ \mu+1\ \mu_1\ldots\mu_k}(\bar{\theta}_0 + i\pi, \theta_0, \theta_1 \ldots, \theta_k) = -\prod_{i=1}^{k} S_{\mu\mu_i}(\theta_{0i}) F_k^{\mu_1\ldots\mu_k}(\theta_1, \ldots, \theta_k)$$



The structure of the two-particle form factors

- **Basic properties:** $F_2^{ij}(\theta_1,\theta_2) = F_2^{1\;1+j-i}(\theta_1-\theta_2)$
- **Only** $F_2^{11}(\theta)$ **matters:** $F_2^{1j}(\theta) = F_2^{11}(2\pi i(j-1)-\theta)\,, \quad j=2,\dots,n$
- **Non-trivial constraints:** $F_2^{11}(\theta) = S(\theta)F_2^{11}(-\theta) = F_2^{11}(2\pi i n-\theta)$



The exact two-particle form factors

With the integral representation for the scattering matrix:

$$S(\theta) = \exp\left[\int_0^\infty \frac{dt}{t} g(t) \sinh\left(\frac{t\theta}{i\pi}\right)\right]$$

the solution is

$$F_2^{11}(\theta) = \frac{\langle \mathcal{T} \rangle \sin\left(\frac{\pi}{n}\right)}{2n \sinh\left(\frac{i\pi-\theta}{2n}\right) \sinh\left(\frac{i\pi+\theta}{2n}\right)} \frac{F_{\text{min}}^{11}(\theta)}{F_{\text{min}}^{11}(i\pi)}$$

where

$$F_{\text{min}}^{11}(\theta) = \exp\left[\int_0^\infty \frac{dt}{t \sinh(nt)} g(t) \sin\left(\frac{it}{2}\left(n + \frac{i\theta}{\pi}\right)\right)^2\right]$$

Ising and sinh-Gordon cases

- Ising case:

$$S(\theta) = -1\,, \quad F^{11}_{\text{min}}(\theta) = -i\sinh\frac{\theta}{2n}$$

- sinh-Gordon case:

$$S(\theta) = \frac{\tanh\frac{1}{2}\left(1-\frac{i\pi B}{2}\right)}{\tanh\frac{1}{2}\left(1+\frac{i\pi B}{2}\right)}\,, \quad g(t) = \frac{8\sinh\frac{tB}{4}\sinh\frac{t}{2}\left(1-\frac{B}{2}\right)\sinh\frac{t}{2}}{\sinh t}$$

Checks:

- Evaluating the scaling dimension using Cardy-Delfino-Simonetti formula and Fring-Mussardo form factors of the stress-energy tensor in sinh-Gordon: exact formula in the Ising case, good numerical accuracy in the sinh-Gordon case
- Evaluating the form factors directly in the angular quantisation using Brazhnikov-Lukyanov's angular quantisation for integrable models

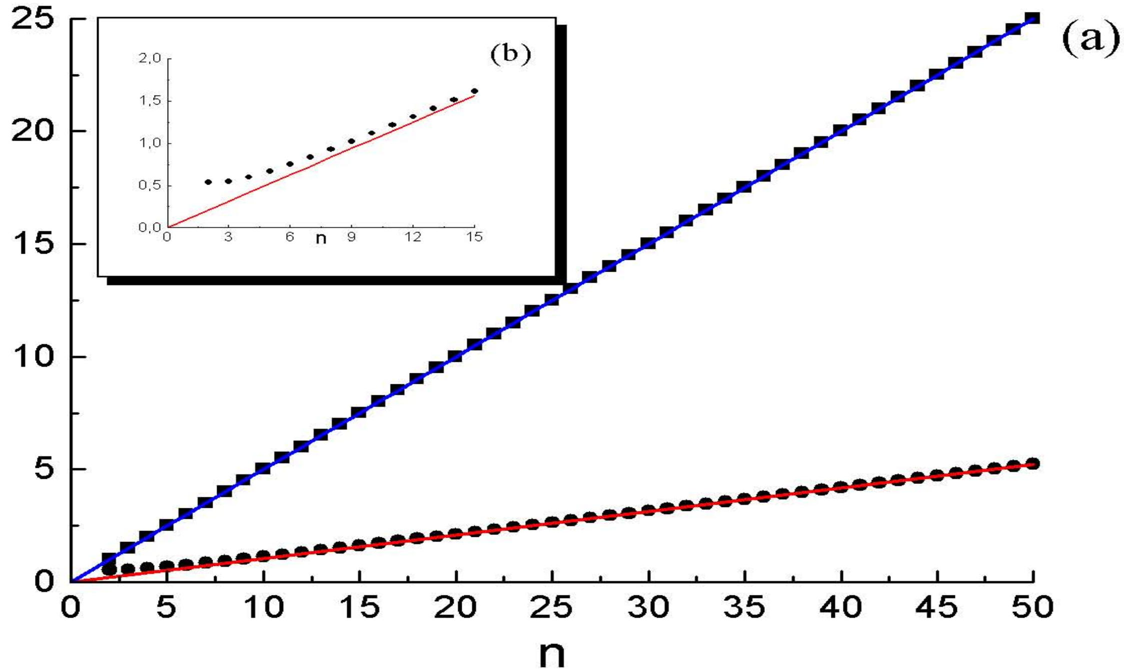
Two-point correlation functions

$$\begin{aligned}\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle &= \langle \mathrm{gs}|\mathcal{T}(0)\tilde{\mathcal{T}}(r)|\mathrm{gs}\rangle \\ &= \sum_{\text{state } k} \langle \mathrm{gs}|\mathcal{T}(0)|k\rangle\langle k|\tilde{\mathcal{T}}(r)|\mathrm{gs}\rangle \\ &= \langle \mathcal{T}\rangle^2 + n\sum_{j=1}^{n}\int d\theta_1 d\theta_2 e^{-mr(\cosh\theta_1+\cosh\theta_2)}|F_2^{1j}(\theta_1-\theta_2)|^2 + \ldots \\ &= \langle \mathcal{T}\rangle^2 \left(1 + \frac{n}{4\pi^2}\int_{-\infty}^{\infty} d\theta f(\theta,n) K_0(2rm\cosh(\theta/2)) + \ldots\right)\end{aligned}$$

where

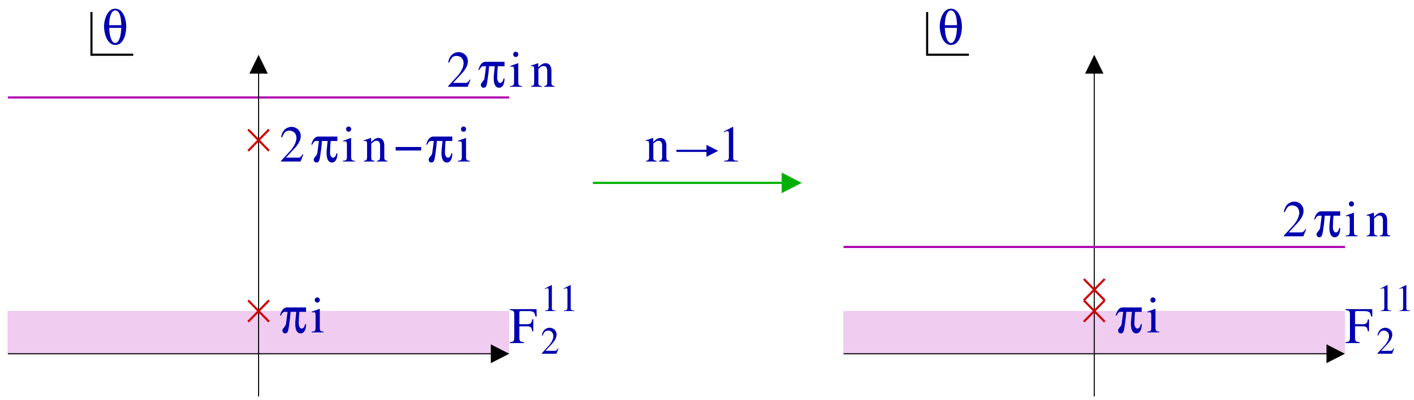
$$\langle \mathcal{T}\rangle^2 f(\theta,n) = |F_2^{11}(\theta)|^2 + \sum_{j=1}^{n-1}|F_2^{11}(-\theta+2\pi i j)|^2$$

We would like to evaluate $\lim_{n\to 1} \frac{d}{dn}(nf(\theta,n)) \Rightarrow$ analytic continuation $\tilde{f}(\theta,n)$ of $f(\theta,n)$ from $n = 1, 2, 3, \ldots$ to $n \in [1, \infty)$



The analytic continuation $\tilde{f}(\theta,n)$ of $f(\theta,n)$ does not converge unformely as $n \to 1$ on $\theta \in \mathbb{R}$, that is, $\tilde{f}(0,1) \neq f(0,1) = 0$

The non-zero value of $\tilde{f}(0,1)$ comes from the collision of poles of $|F_2^{11}(2\pi i j)|^2 = F_2^{11}(2\pi i j)^2$ as function of j as $n \to 1$, as can be seen from Poisson's re-summation formula



Poisson re-summation formula:

$$\sum_{j=1}^{n-1} s(\theta, j) = \sum_{k\in\mathbb{Z}} (s_{nk} - s_k)$$

$$s(\theta,j) = |F_2^{11}(-\theta + 2\pi i j)|^2 \,, \quad s_k = \int_0^n dj\; e^{-\frac{2\pi i j k}{n}} s(\theta, j)$$

Extracting the poles:

$$s(\theta, j) \sim \frac{iF_2^{11}(-2\theta + 2\pi in - i\pi)}{-\theta - 2\pi ij + 2\pi in - i\pi} - \frac{iF_2^{11}(-2\theta + i\pi)}{-\theta - 2\pi ij + i\pi} + \text{c.c.}$$

and re-summing them exactly gives

$$\tilde{f}(\theta, n) \sim \tilde{f}(0,1) \left(\frac{i\pi(n-1)}{2(\theta + i\pi(n-1))} - \frac{i\pi(n-1)}{2(\theta - i\pi(n-1))} \right) , \quad \tilde{f}(0,1) = \frac{1}{2}$$

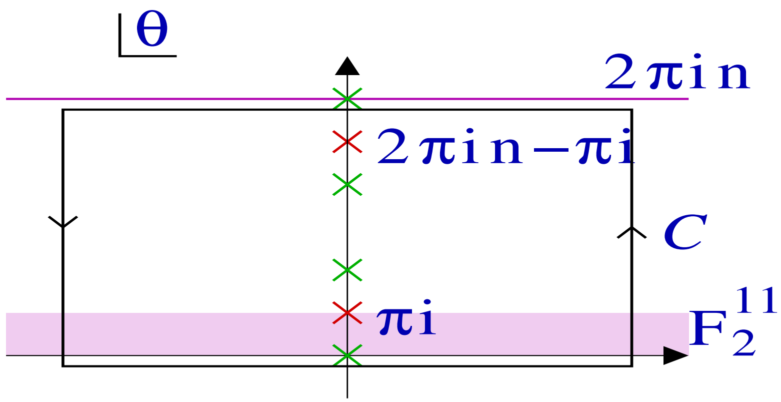
Hence the derivative is supported at $\theta = 0$:

$$\left(\frac{\partial}{\partial n} \tilde{f}(\theta, n) \right)_{n=1} = \pi^2 \tilde{f}(1) \delta(\theta)$$

There is an exact analytic continuation:

Consider the closed-contour integral

$$\int_{\mathcal{C}} \frac{dj}{2\pi i} \pi \cot \pi j \ F_2^{11}(2\pi i j)^2$$



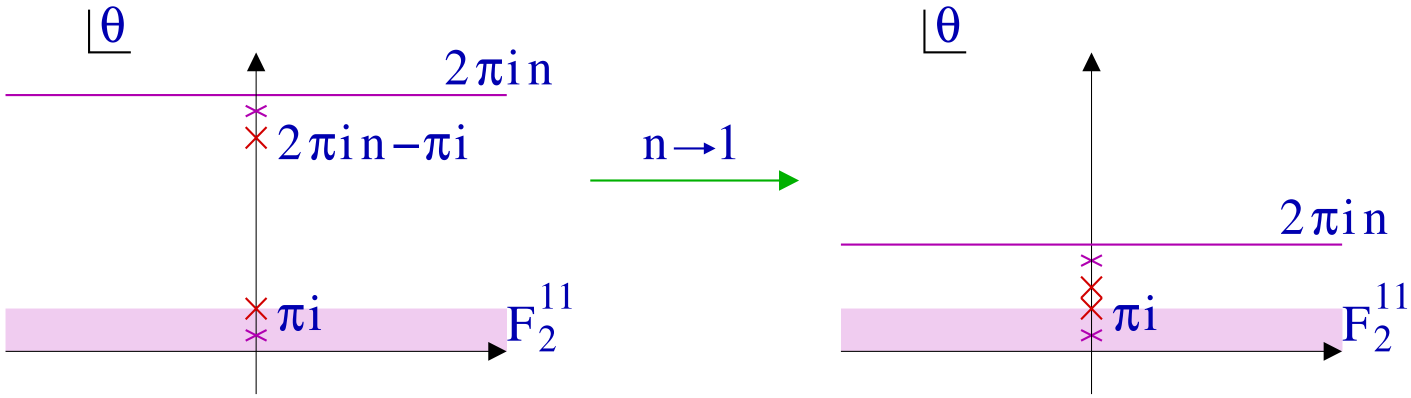
Assuming $F_2^{11}(0) = 0$ and $F_2^{11}(\theta) = 0$ at $|\theta| \to \infty$:

$$\tilde{f}(0, n) = \frac{1}{2} - \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Im}(S(-\theta)) \coth\left(\frac{\theta}{2}\right) |F_2^{11}(\theta)|^2 d\theta$$

Multi-particle and bound-state case (diagonal scattering)

$$|\ldots,\theta_{\mu_i},\theta_{\mu_{i+1}},\ldots\rangle = S_{\mu_i\mu_{i+1}}|\ldots,\theta_{\mu_{i+1}},\theta_{\mu_i},\ldots\rangle\ ,\quad \mu = (\text{type, sheet})$$

- For every particle type, there is a kinematic residue $\Rightarrow$ contribution at $n=1$
- Possible bound states give additional poles on the physical sheet, on the imaginary line of θ, but they never collide $\Rightarrow$ no contribution at $n=1$.



Conclusions

We have derived the first correction to saturation of the entanglement entropy in any IQFT with diagonal scattering, and observed that it is very universal.

- The generalisation to non-diagonal scattering gives the same entropy formula
- The constant U that characterises the saturation itself can be evaluated in the Ising model, and possibly conjectures can be found in interacting models following ideas for evaluating one-point functions by Bazhanov, Lukyanov, Zamolodchikov.
- The evaluation of the higher-particle corrections to the entanglement entropy should be possible
- It would be interesting to understand: 1) if the “link” picture holds, 2) what happens for massless integrable models, 3) if what replaces our formula (or if it still holds) in non-integrable models