

Integrability out of equilibrium: current, fluctuations, correlations

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My works on this subject

J. Phys. A 45 (2012) 362001 with Denis Bernard

J. Phys. A 46 (2013) 372001 with Denis Bernard

arXiv:1212.1077, to appear in Les Houches Lecture Notes

arXiv:1302.3125 with Denis Bernard, accepted in Ann. Inst. H. Poincaré

arXiv:1305.0518 with Yixiong Chen

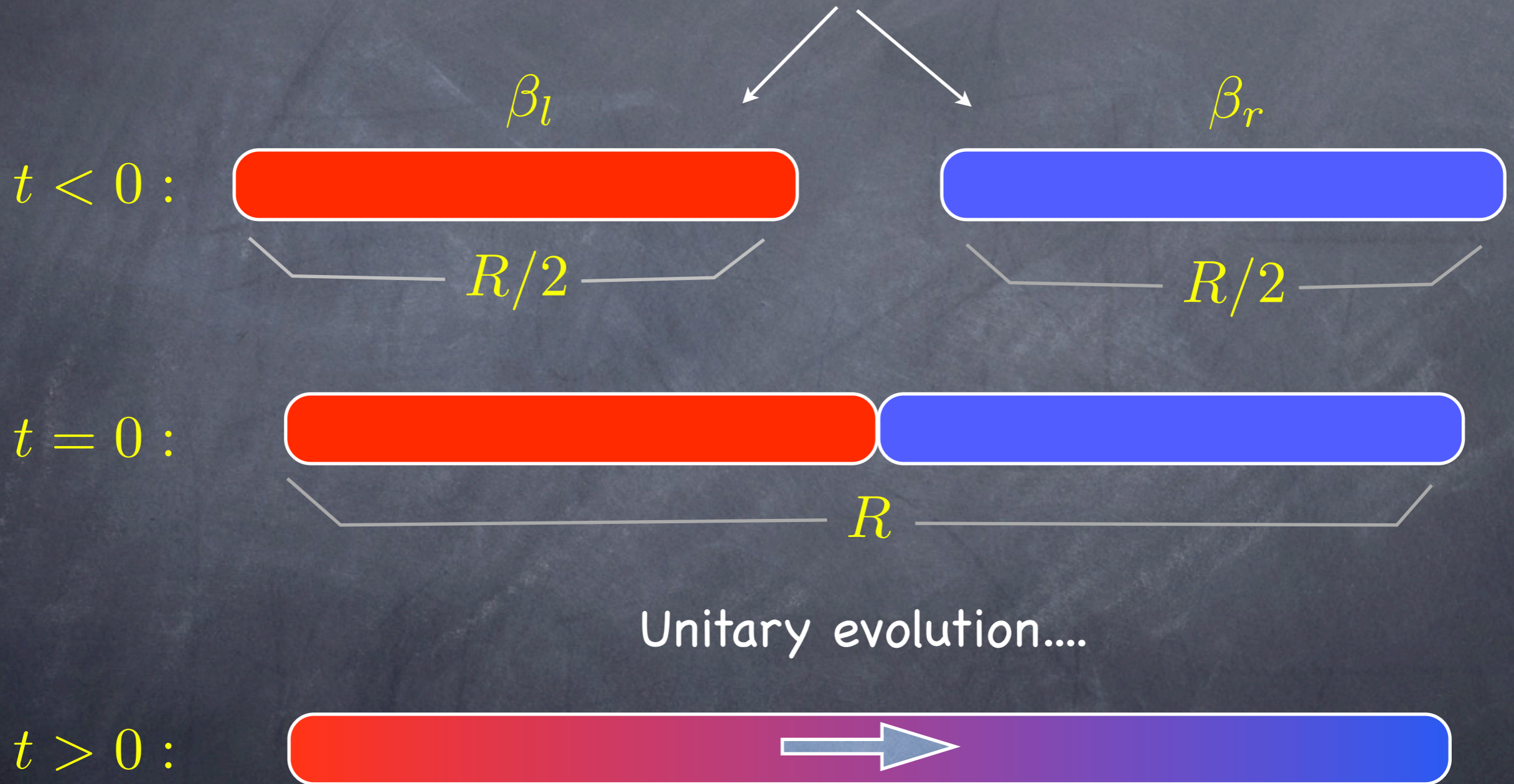
arXiv:1305.4984 with A. De Luca, J. Viti and D. Bernard

arXiv:1306.3192 with Marianne Hoogeveen and Denis Bernard

work in progress with O. A. Castro Alvaredo, Y. Chen, M. Hoogeveen

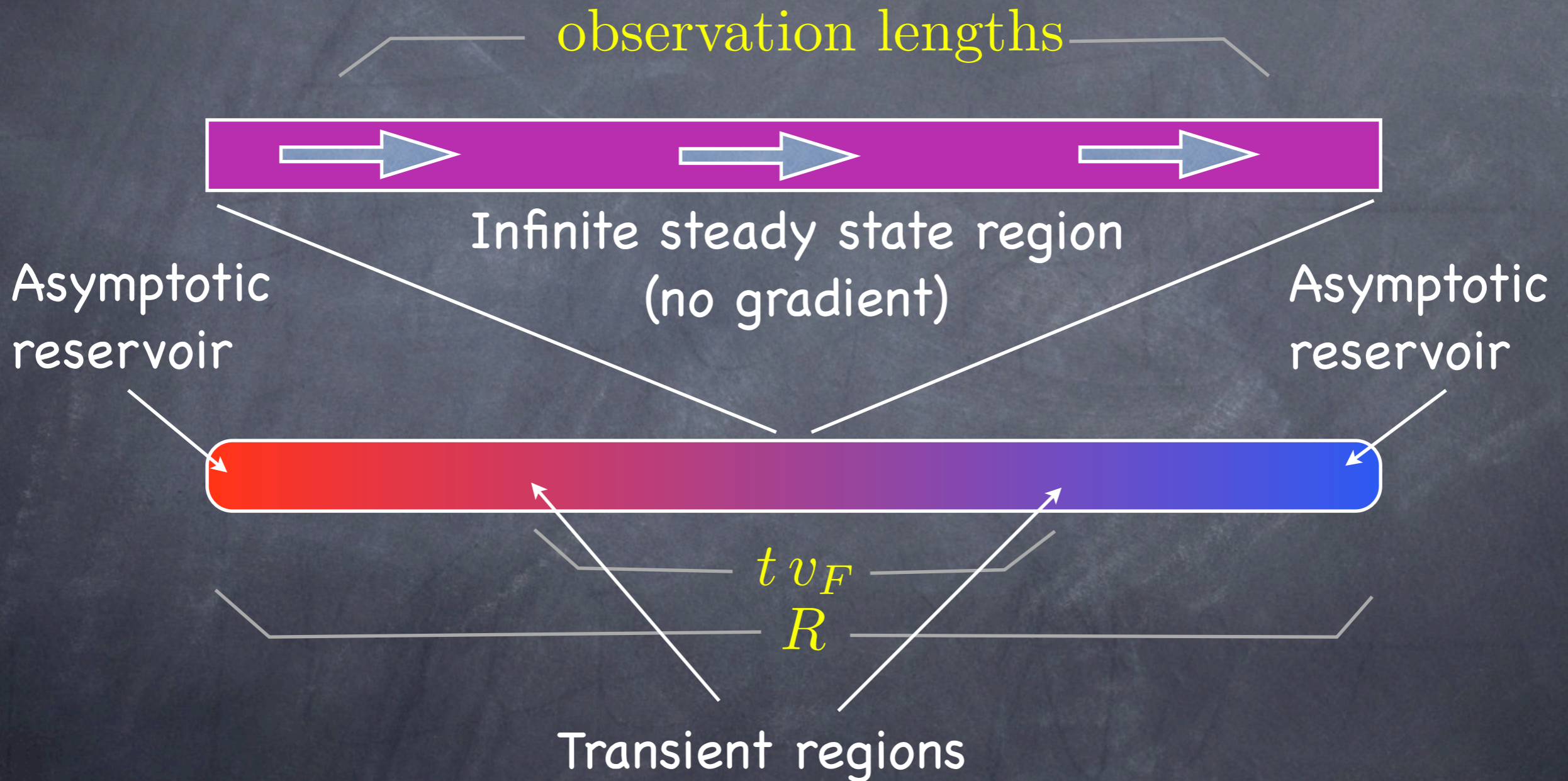
Physical situation

Independent quantum chains
in different thermal states



Steady-state limit

$$R \gg t v_F \gg \text{observation lengths}$$

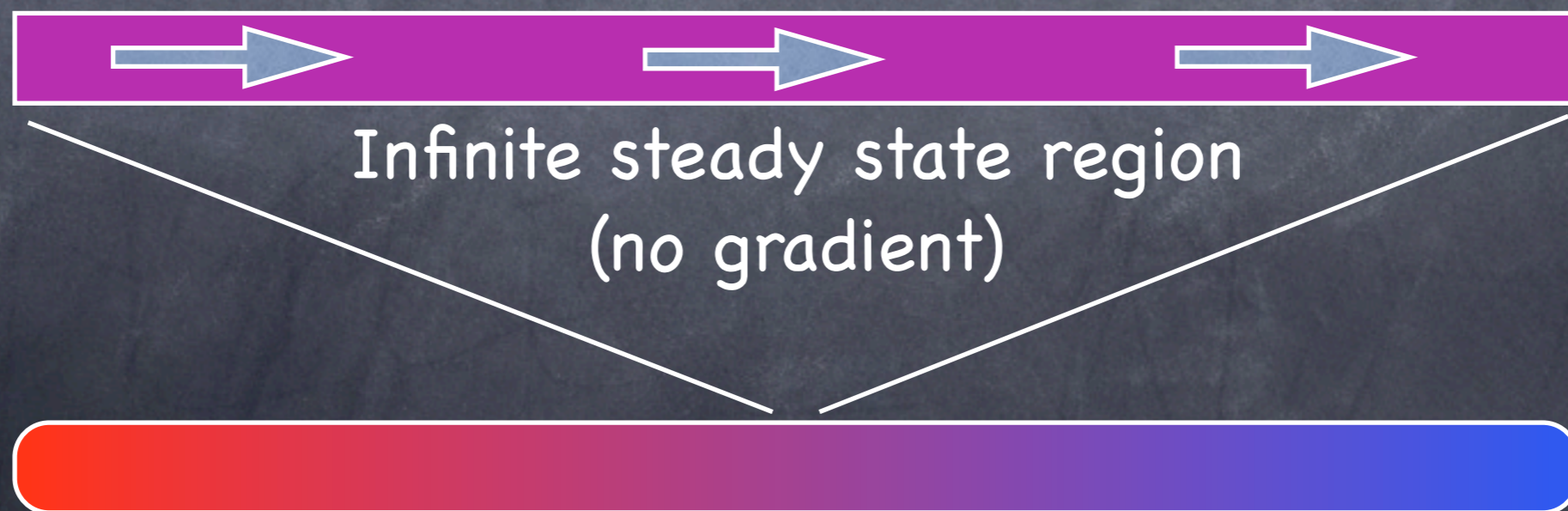


Anything happens?

Generic non-integrable models: Fourier's law says that no gradient \Rightarrow no current; thermal equilibrium in steady state region. **Nothing.**

Integrable models: conservation laws allow a flow of energy to exist; non-equilibrium steady state. **Something interesting!**

Any critical point: collective behaviours allow a flow of energy; non-equilibrium steady state. **Something interesting!** [D. Bernard & BD, 2012]



How / what to calculate

Initially: $\rho_0 = e^{-\beta_l H^l - \beta_r H^r}$

Evolution Hamiltonian: $H = H^l + H^r + H_{\text{contact}}$

Steady state limit:

$$\langle \dots \rangle_{\text{ness}} = \lim_{t_0 \rightarrow -\infty} \lim_{R \rightarrow \infty} \frac{\text{Tr} (e^{iHt_0} \rho_0 e^{-iHt_0} \dots)}{\text{Tr} (\rho_0)}$$

↓
Observables supported on a finite region

Current: $J(\beta_l, \beta_r) = \langle \mathcal{J} \rangle_{\text{ness}}, \quad \mathcal{J} = \frac{d}{dt} \frac{H^r - H^l}{2} = i[H, \frac{H^r - H^l}{2}]$

Correlations: $\langle \mathcal{O}(x_1, t_1) \mathcal{O}(x_2, t_2) \rangle_{\text{ness}}$

Fluctuations of energy transfer:

$$Q = \frac{1}{2} (H^r - H^l)$$



$\Omega_t(q)$ = probability of q at time t

$$= \sum_{q_0} \text{Tr} (P_{q_0+q} e^{-iHt} P_{q_0} \rho_0 P_{q_0} e^{iHt} P_{q_0+q})$$

$F(z)$ = generating function of scaled cumulants

$$= \sum_n \frac{z^n}{n!} \lim_{t \rightarrow \infty} \frac{1}{t} \langle q^n \rangle_{\Omega_t}^{\text{cumulant}} = \sum_n \frac{z^n}{n!} C_n$$

Fluctuations of energy transfer:

It turns out that:

$$F(z) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \left[\langle e^{zQ(t)} e^{-zQ} \rangle_{\text{ness}} \right], \quad Q(t) = e^{iHt} Q e^{-iHt}$$

charge transfer in free fermion models; indirect measurement protocol:
[Levitov & Lesovik 1993; Klich 2002; Schonhamer 2007; D. Bernard & BD 2012]

energy transfer in CFT:
[D. Bernard & BD 2012]

Surprisingly, for energy transfer in integrable models:

$$F(z) = \int_0^z dz' J(\beta_l - z', \beta_r + z')$$

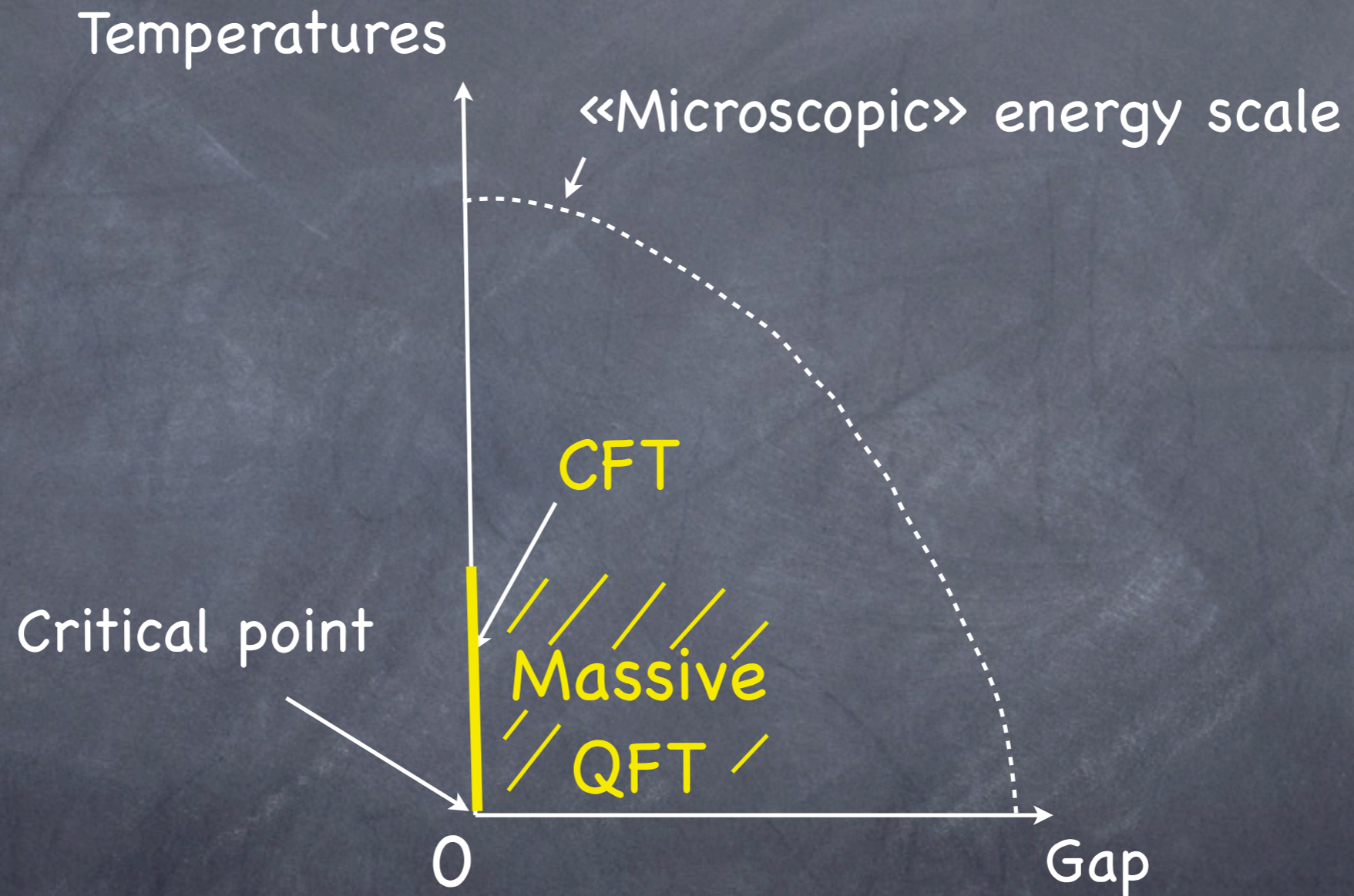
[D. Bernard & BD 2013]

holds whenever there is «pure transmission» $SQ = -QS$
where S is the scattering matrix. [D. Bernard & BD 2013]

implies the standard fluctuation relations of Jarzynski,
Wojcik based on Gallavotti; see Esposito, Harbola,
Mukamel (RMP 2009).

$$F(z) = F(\beta_l - \beta_r - z)$$

Scaling limit: (relativistic) QFT



Formal description of steady state

in massive QFT

[D. Bernard & BD 2012]

[BD 2012]

$$\langle \dots \rangle_{\text{ness}} = \text{Tr} (e^{-W} \dots) / \text{Tr} (e^{-W})$$

$$W = \beta_l \int_0^\infty d\theta E_\theta n_\theta + \beta_r \int_{-\infty}^0 d\theta E_\theta n_\theta$$

Total energy of right-moving
asymptotic particles

Total energy of left-moving
asymptotic particles

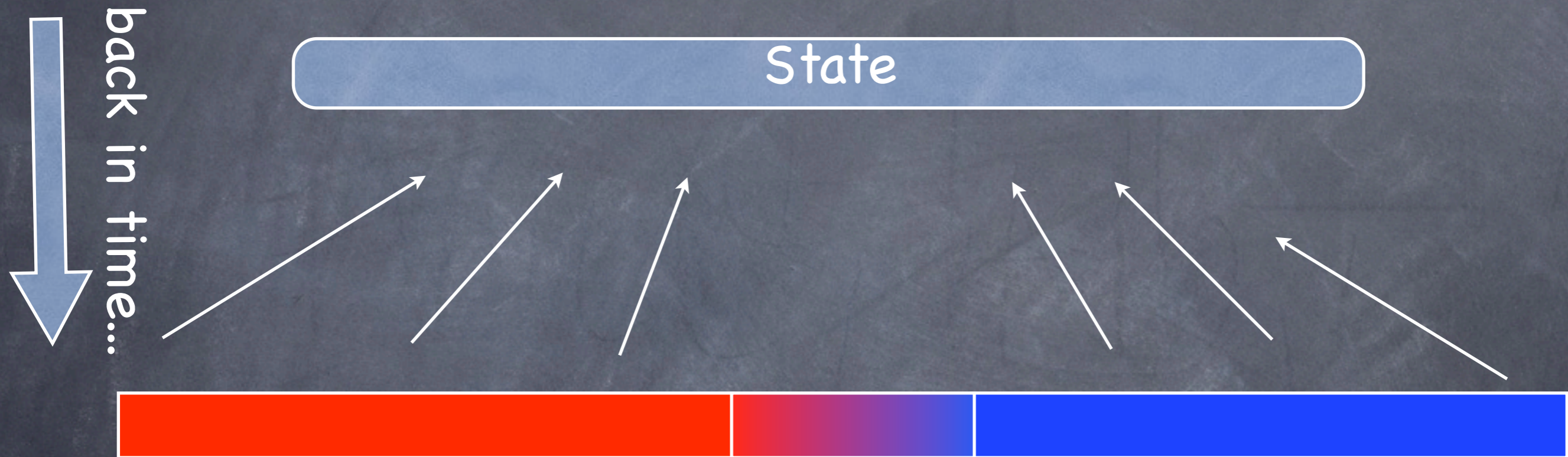
*In massive QFT: states are described by asymptotically free particles characterized by their rapidities

Formal description of steady state

in massive QFT

[D. Bernard & BD 2012]

[BD 2012]



agreement with free fermion calculations (XY model)
[W. H. Aschbacher & C.-A. Pillet, 2003]

Formal description of steady state

in CFT

[D. Bernard & BD 2012, 2013]

$$\langle \dots \rangle_{\text{ness}} = \text{Tr} (e^{-W} \dots) / \text{Tr} (e^{-W})$$

$$W = \beta_l \frac{L_0}{2\pi R} + \beta_r \frac{\bar{L}_0}{2\pi R} \quad (R \rightarrow \infty)$$

Total energy of right-movers

Total energy of left-movers

*In CFT: densities separate into right-movers and left-movers

energy density: $h(x) = h_+(x) + h_-(x)$

momentum density: $p(x) = h_+(x) - h_-(x)$

Current / fluctuations in CFT

[D. Bernard & BD 2012, 2013]

$$J(\beta_l, \beta_r) = \frac{\pi c}{12} (\beta_l^{-2} - \beta_r^{-2}) = \frac{\pi c k_B^2}{12 \hbar} (T_l^2 - T_r^2)$$

central charge of Virasoro algebra

$$h_+(x) \propto -\frac{c}{24} + \sum_{n \in \mathbb{Z}} L_n e^{-\frac{2\pi i n x}{R}}$$

Virasoro

- confirmed by numerics

[C Karrasch, R. Ilan & J. E. Moore, 2012]

- agreement with Luttinger liquid results

[M. Mintchev & P. Sorba, 2012]

[D. B. Gutman, Yu. Gefen & A. D. Mirlin, 2010]

Current / fluctuations in CFT

[D. Bernard & BD 2012, 2013]

[BD, M. Hoogeveen & D. Bernard 2013]

$$F(z) = f(z; \beta_l) + f(-z; \beta_r) \quad f(z; \beta) = \frac{c\pi}{12} \left(\frac{1}{\beta - z} - \frac{1}{\beta} \right)$$

Recall: $F(z)$ = generating function of scaled cumulants

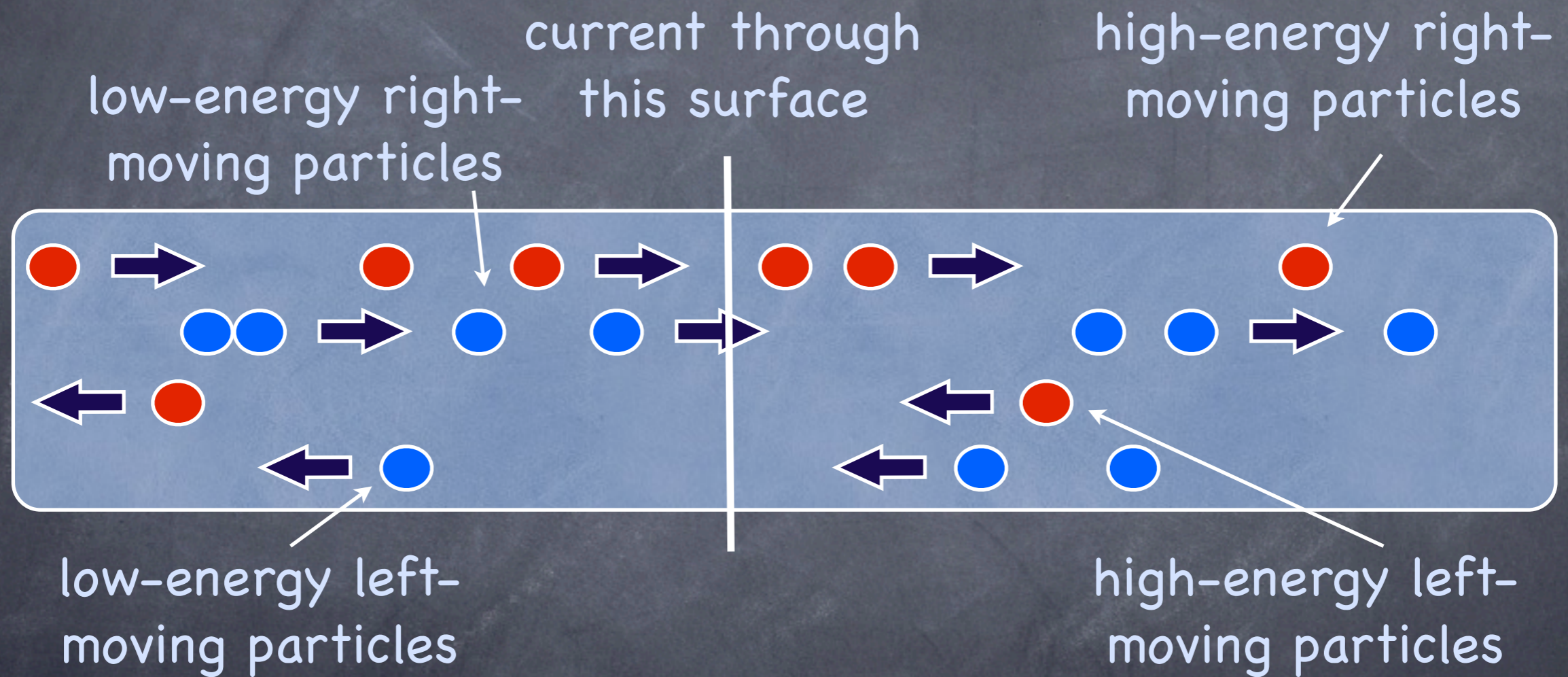
Independent Poisson process for every energy value: independent energy packets, travelling in both directions

$$F(z) = \int dq \omega(q) (e^{zq} - 1), \quad \omega(q) = \frac{c\pi}{12} \cdot \begin{cases} e^{-\beta_l q} & (q > 0) \\ e^{\beta_r q} & (q < 0) \end{cases}$$

intensity

cumulant generating function for a single Poisson process

Poisson process interpretation



Current / fluctuations Ising model

[W. H. Aschbacher & C.-A. Pillet, 2003]

[A. De Luca, J. Viti, D. Bernard & BD 2013]

$$W = \int d\theta w(\theta) a^\dagger(\theta) a(\theta), \quad w(\theta) = m \cosh \theta \cdot \begin{cases} \beta_l & (\theta > 0) \\ \beta_r & (\theta < 0) \end{cases}$$

The current \mathcal{J} is quadratic in the creation and annihilation operators, so a rather simple free-fermion calculation gives

$$J = \frac{\text{Tr} (e^{-W} \mathcal{J})}{\text{Tr} (e^{-W})} = \frac{m^2}{4\pi} \int d\theta \frac{\sinh 2\theta}{1 + e^{w(\theta)}}$$

=> **Landauer form:** $p dp$ times thermal distributions on left / right

Fluctuations: Independent Poisson processes,

$$\omega(q) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2\pi n^2} \cdot \begin{cases} e^{-\beta_l q} & (q > nm) \\ e^{\beta_r q} & (q < -nm) \end{cases}$$

using $F(z) = \int_0^z dz' J(\beta_l - z', \beta_r + z') = \int dq \omega(q) (e^{zq} - 1)$

Current in massive integrable QFT

[O. A. Castro Alvaredo, Y. Chen, BD & M. Hoogeveen in preparation]

We extend the thermodynamic Bethe ansatz [Al. B. Zamolodchikov, 1990] to the non-equilibrium steady state density matrix:

$$J = \frac{\text{Tr} (e^{-W} p(0))}{\text{Tr} (e^{-W})} \longrightarrow \text{current operator} = \text{momentum density}$$

In order to calculate the trace, we make the system finite and periodic, replacing the operator W with an approximate finite-length version W^L . Following Zamolodchikov, a state $|v\rangle$ is described by Bethe roots with energies and momenta that approximate the relativistic ones. So we set:

$$W^L |v\rangle = \sum_k w_k |v\rangle, \quad w_k = e_k \cdot \begin{cases} \beta_l & (p_k > 0) \\ \beta_r & (p_k < 0) \end{cases}$$

$|v\rangle$: energies e_k , momenta p_k

Current in massive integrable QFT

[O. A. Castro Alvaredo, Y. Chen, BD & M. Hoogeveen in preparation]

Using translation invariance and taking the large length limit, we get:

$$J = \lim_{L \rightarrow \infty} L^{-1} \frac{\text{Tr}_L \left[e^{-W^L} P \right]}{\text{Tr}_L \left[e^{-W^L} \right]} \longrightarrow \text{total momentum operator}$$

We define a «free energy» and calculate the current from it:

$$f^a := - \lim_{L \rightarrow \infty} L^{-1} \log \text{Tr}_L \left[e^{-W^L - aP} \right], \quad J = \left. \frac{d}{da} f^a \right|_{a=0}.$$

We can then use the methods of the thermodynamic Bethe ansatz...

[Al. B. Zamolodchikov, 1990]

Current in massive integrable QFT

[O. A. Castro Alvarado, Y. Chen, BD & M. Hoogeveen in preparation]

$$J = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \frac{m \cosh \theta x(\theta)}{1 + e^{\epsilon(\theta)}}$$

$$x(\theta) = m \sinh \theta + \int_{-\infty}^{\infty} \frac{d\gamma}{2\pi} \frac{\varphi(\theta - \gamma) x(\gamma)}{1 + e^{\epsilon(\gamma)}} \quad \text{dressed momentum}$$

$$\epsilon(\theta) = w(\theta) - \int_{-\infty}^{\infty} \frac{d\gamma}{2\pi} \varphi(\theta - \gamma) \log(1 + e^{-\epsilon(\gamma)}) \quad \begin{array}{l} \text{non-equil.} \\ \text{pseudo energy} \end{array}$$

where we recall that

$$w(\theta) = m \cosh \theta \cdot \begin{cases} \beta_l & (\theta > 0) \\ \beta_r & (\theta < 0) \end{cases} \quad \text{driving term with a jump}$$

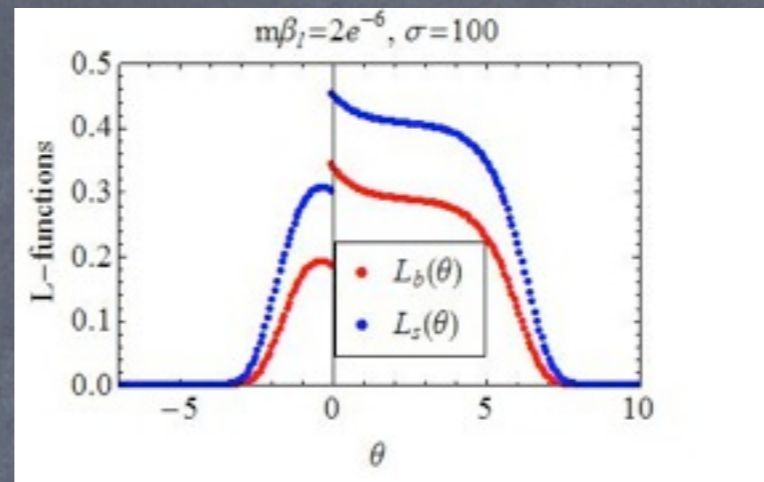
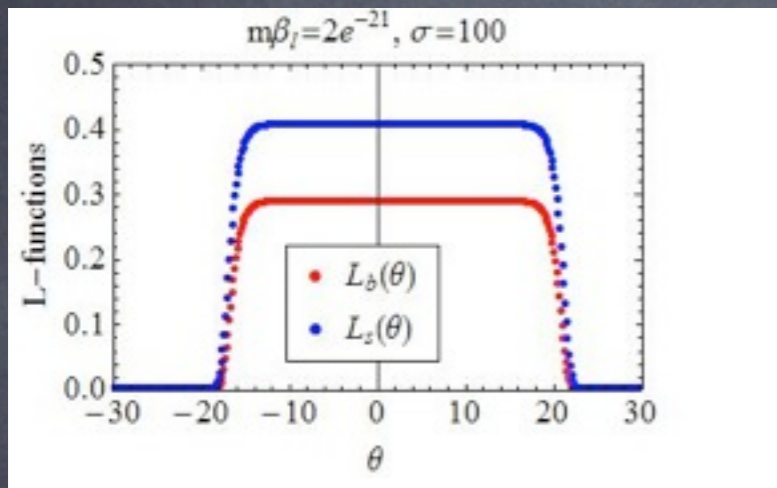
$$\varphi(\theta) = -i \frac{d}{d\theta} S(\theta) \quad \text{differential scattering phase}$$

Current in massive integrable QFT

[O. A. Castro Alvaredo, Y. Chen, BD & M. Hoogeveen in preparation]

L-function has a jump and an asymmetry

$$L(\theta) = \log(1 + e^{-\epsilon(\theta)})$$



Sine-Gordon soliton and breather L-functions at high and medium temperatures

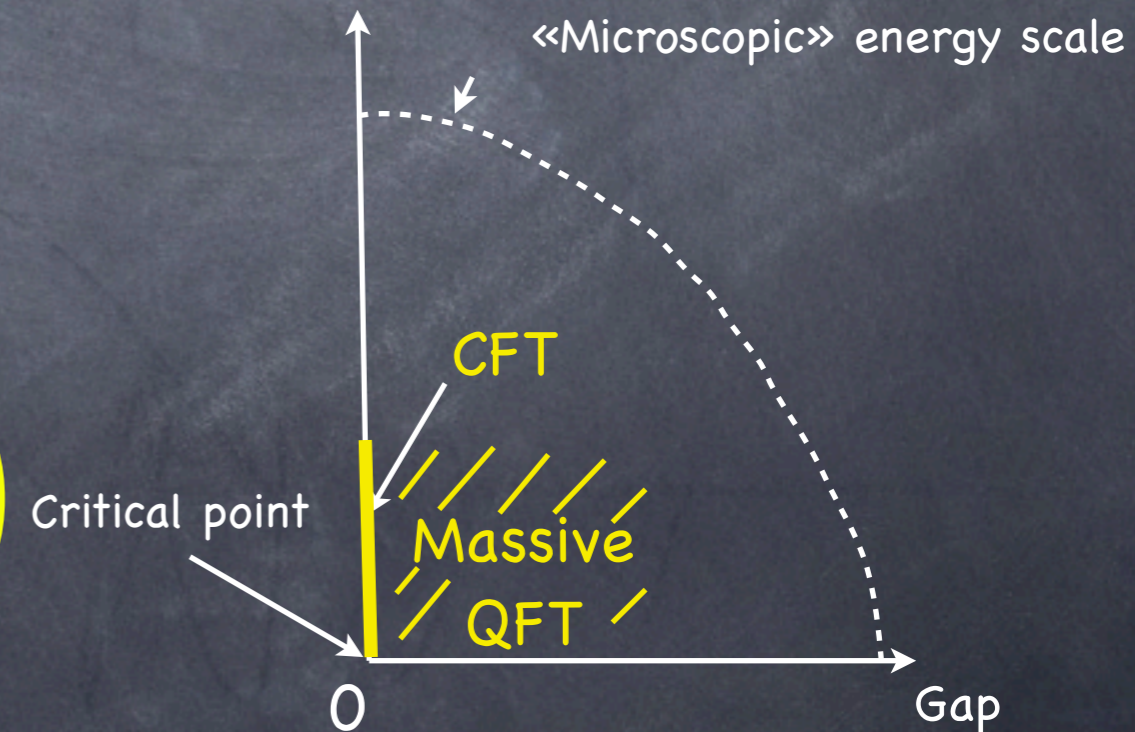
Large temperature / small gap limit: we recover the exact CFT result

$$J(\beta_l, \beta_r) \sim \frac{\pi c}{12} (T_l^2 - T_r^2)$$

$$c = \frac{3}{\pi^2} \int_{\epsilon(0)}^{\infty} d\epsilon \left(\log(1 + e^{-\epsilon}) + \frac{\epsilon}{1 + e^{\epsilon}} \right)$$

An expression for the central charge [Al. B. Zamolodchikov, 1990]

Temperatures

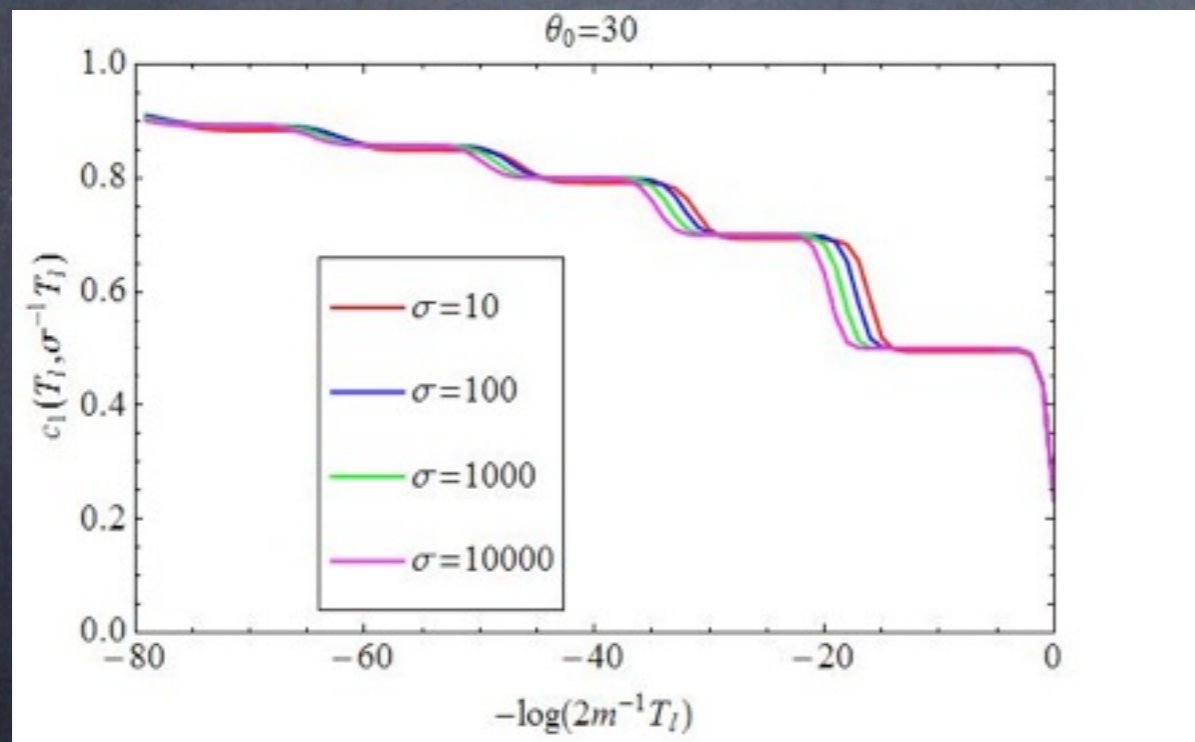


Non-equilibrium c-functions

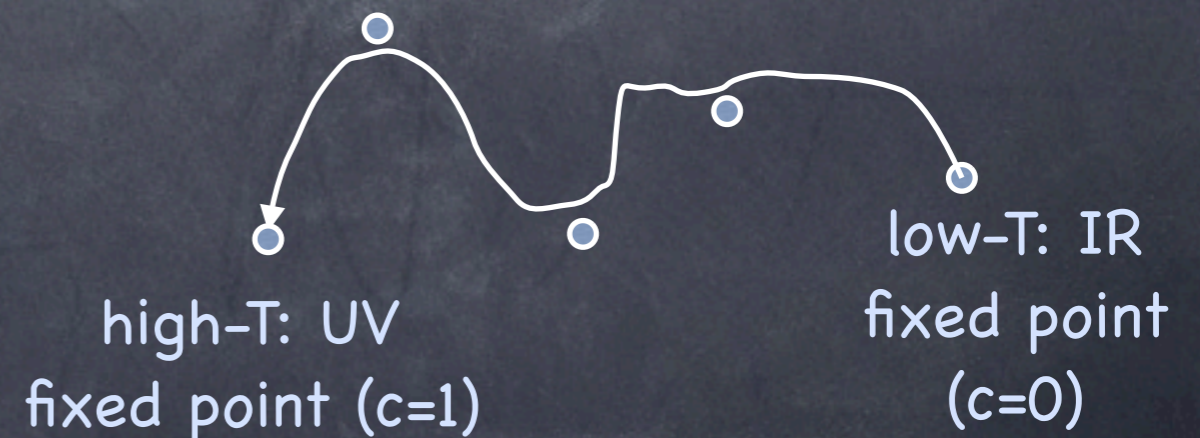
[O. A. Castro Alvarado, Y. Chen, BD & M. Hoogeveen in preparation]

Dividing current by CFT form, get something that equates the central charge at fixed points. In fact, it is monotonic between fixed points: a **c-function** [A. B. Zamolodchikov 1986, Al. B. Zamolodchikov 1990], which measures the number of degrees of freedom along an RG trajectory.

$$c_1(T_l, T_r) = \frac{J(\beta_l, \beta_r)}{(\pi/12)(T_l^2 - T_r^2)}$$



Non-equilibrium c-function
in Al. B. Zamolodchikov's
roaming trajectories model



Non-equilibrium c-functions

[O. A. Castro Alvarado, Y. Chen, BD & M. Hoogeveen in preparation]

It turns out that all cumulants give rise to c-functions:

$$c_n(T_l, T_r) = \frac{12C_n(T_l, T_r)}{\pi n! (T_l^{n+1} + (-1)^n T_r^{n+1})}$$

Explanation: by lowering the mass, keeping the temperatures the same, we allow more excitations to be present, hence there is more current, and more fluctuations

$$\frac{d}{dm} C_n \leq 0.$$

Poisson process interpretation

[O. A. Castro Alvaredo, Y. Chen, BD & M. Hoogeveen in preparation]

We can calculate the Poisson process intensity by Fourier transform of the current; it is an intensity **if and only if it is positive**:

$$\omega(q) = \frac{1}{q} \int \frac{d\lambda}{2\pi} J(\beta_l - i\lambda, \beta_r + i\lambda) e^{-i\lambda q}.$$

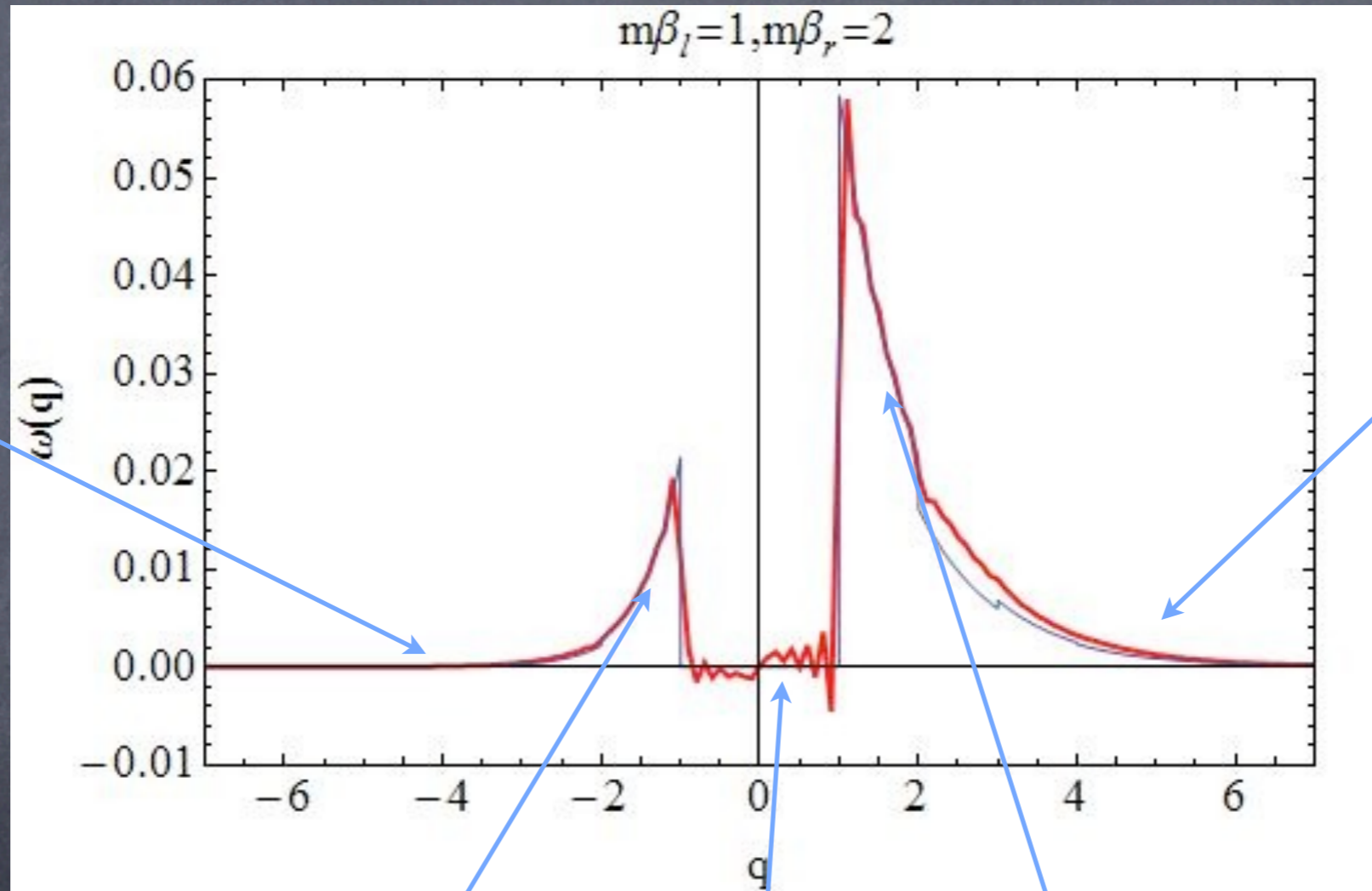
using $F(z) = \int_0^z dz' J(\beta_l - z', \beta_r + z') = \int dq \omega(q) (e^{zq} - 1)$

Poisson processes iff $\omega(q) \geq 0$

Poisson process interpretation

[O. A. Castro Alvarado, Y. Chen, BD & M. Hoogeveen in preparation]

It turns out that indeed, this Fourier transform is positive!



Asymptotically
CFT $c=1$
 $\frac{\pi}{12} e^{\beta_r q}$

Asymptotically
CFT $c=1$
 $\frac{\pi}{12} e^{-\beta_l q}$

Exactly Ising
(one-particle)

Exactly 0

Exactly Ising
(one-particle)

Correlation functions in steady state (Ising model)

[Y. Chen & BD 2013]
extending: [BD 2005]

We use a «spectral decomposition» in the space of operators («Liouville space») in order to obtain a convergent expansion valid at large distances, following an old idea in C^* algebra.

[Gelfand & Naimark 1943, Segal 1947]

Inner product on the space of operators with respect to a density matrix:

$${}^\rho \langle A | B \rangle^\rho = \frac{\text{Tr}(\rho A^\dagger B)}{\text{Tr}(\rho)} \quad A, B \in \text{End}(\mathcal{H})$$

Choice of basis states:

$$|\text{vac}\rangle^\rho := |\mathbf{1}\rangle^\rho$$

creation / annihilation
operators

$$|\theta_1, \dots, \theta_N\rangle_{\epsilon_1, \dots, \epsilon_N}^\rho := Q_{\epsilon_1, \dots, \epsilon_N}^\rho(\theta_1, \dots, \theta_N) |a^{\epsilon_1}(\theta_1) \cdots a^{\epsilon_N}(\theta_N)\rangle^\rho$$

convenient normalization:

$$Q_{\epsilon_1, \dots, \epsilon_N}^\rho(\theta_1, \dots, \theta_N) := \prod_{i=1}^N \left(1 + e^{-\epsilon_i W(\theta_i)}\right).$$

Correlation functions in steady state (Ising model)

[Y. Chen & BD 2013]

The spectral decomposition is:

$$\begin{aligned}
 \frac{\text{Tr} [\rho \sigma(x) \sigma(0)]}{\text{Tr}[\rho]} &= {}^\rho \langle \text{vac} | \sigma(x) \sigma(0) \rangle^\rho \\
 &= {}^\rho \langle \text{vac} | \sigma^\ell(x) \sigma^\ell(0) | \text{vac} \rangle^\rho \quad \text{left-action: } A^\ell |B\rangle^\rho = |AB\rangle^\rho \\
 &= \sum_{\text{states}} {}^\rho \langle \text{vac} | \sigma^\ell(x) | \text{state} \rangle^\rho {}^\rho \langle \text{state} | \sigma^\ell(0) | \text{vac} \rangle^\rho \\
 &\quad \text{Form factor expansion!}
 \end{aligned}$$

We use $\rho = e^{-W}$ with $W = \int d\theta w(\theta) a^\dagger(\theta) a(\theta)$, and we take $\sigma(x)$ to be the Ising spin field (say in ordered phase).

Correlation functions in steady state (Ising model)

[Y. Chen & BD 2013]

Mixed-state form factors are defined as:

$$f_{\epsilon_1, \dots, \epsilon_N}^{\rho}(\theta_1, \dots, \theta_N) := {}^{\rho}\langle \text{vac} | \sigma^{\ell}(0) | \theta_1, \dots, \theta_N \rangle_{\epsilon_1, \dots, \epsilon_N}^{\rho}.$$

Results:

direction of branch of twist field (right or left) expectation value

$$f_{\epsilon_1, \epsilon_2}^{\rho, \pm}(\theta_1, \theta_2) = h_{\epsilon_1}^{\pm}(\theta_1) h_{\epsilon_2}^{\pm}(\theta_2) \left(\tanh \frac{\theta_2 - \theta_1 \pm i(\epsilon_2 - \epsilon_1)\mathbf{0}}{2} \right)^{\epsilon_1 \epsilon_2} \langle \sigma_{\pm} \rangle_{\rho}$$

«leg factors» vacuum form factor

$$h_{\epsilon}^{+}(\theta) = \sqrt{\frac{i}{2\pi}} e^{\epsilon \frac{i\pi}{4}} g(\theta + \epsilon i\mathbf{0})^{\epsilon}, \quad h_{\epsilon}^{-}(\theta) = -i h_{-\epsilon}^{+}(\theta)$$

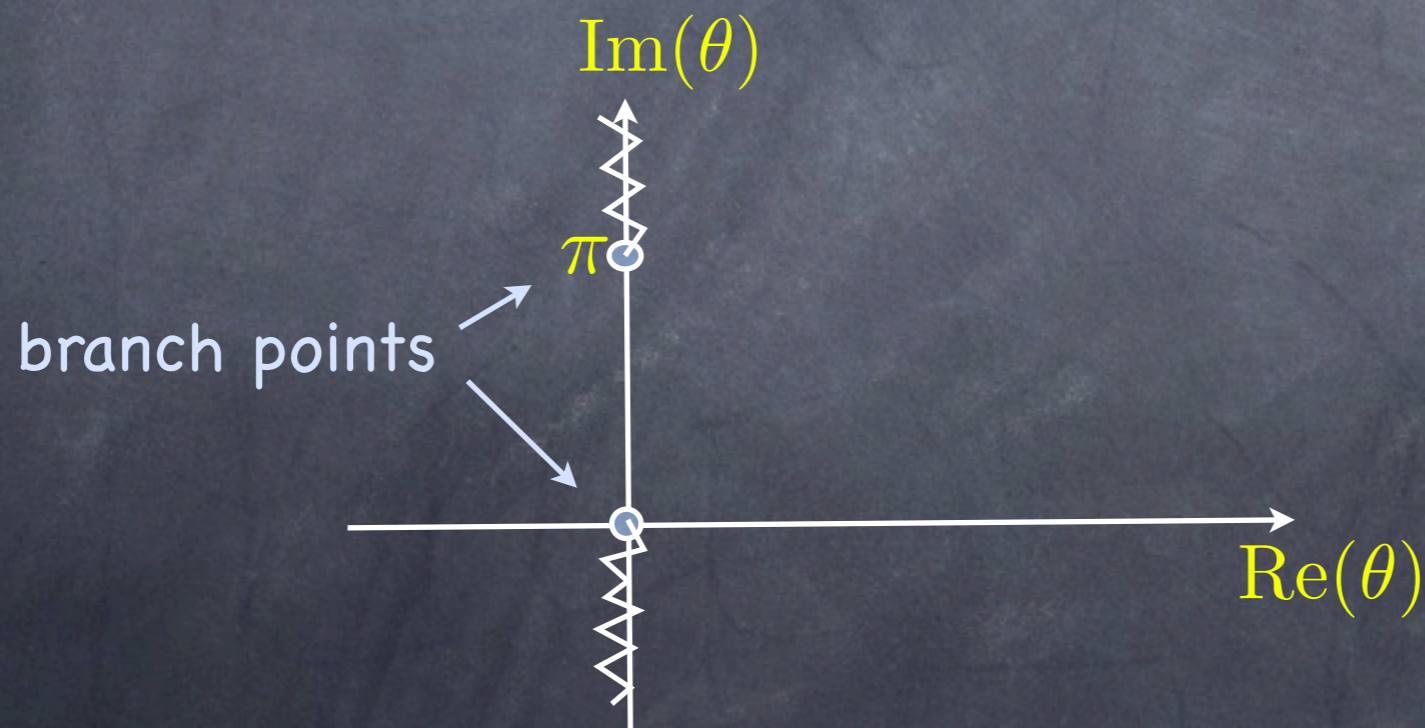
$$g(\theta) = \exp \left[\int \frac{d\theta'}{2\pi i} \log \left(\tanh \frac{\theta - \theta' + i\mathbf{0}}{2} \right) \frac{d}{d\theta'} \log \left(\tanh \frac{w(\theta')}{2} \right) \right]$$

Correlation functions in steady state (Ising model)

[Y. Chen & BD 2013]

Analytic structure of leg factors determines leading asymptotic behaviour of two-point function. This reproduces the correct asymptotic in the GGE occurring after a magnetic-field quench as calculated by Calabrese, Essler and Faggoti (2011–2012).

In the non-equilibrium steady state: there are branch points:



Correlation functions in steady state (Ising model)

[Y. Chen & BD 2013]

We find an oscillatory behaviour at large distances, e.g. in the disordered phase:

$$\frac{e^{-x\mathcal{E}_{\text{ness}}}}{mx} \cos(\nu \log(mx) + B)$$

with a frequency determined by the temperatures:

$$\nu = \frac{1}{\pi} \log \left(\coth \frac{m}{2T_r} \tanh \frac{m}{2T_l} \right)$$

Conclusion and perspectives

- Generalization to presence of impurities
- Current and fluctuations in integrable spin chains
- Non-equilibrium correlation functions / mixed-state form factors in integrable models
- Higher dimensions (in progress – AdS/CFT...)