Conformal field theory from conformal loop ensembles

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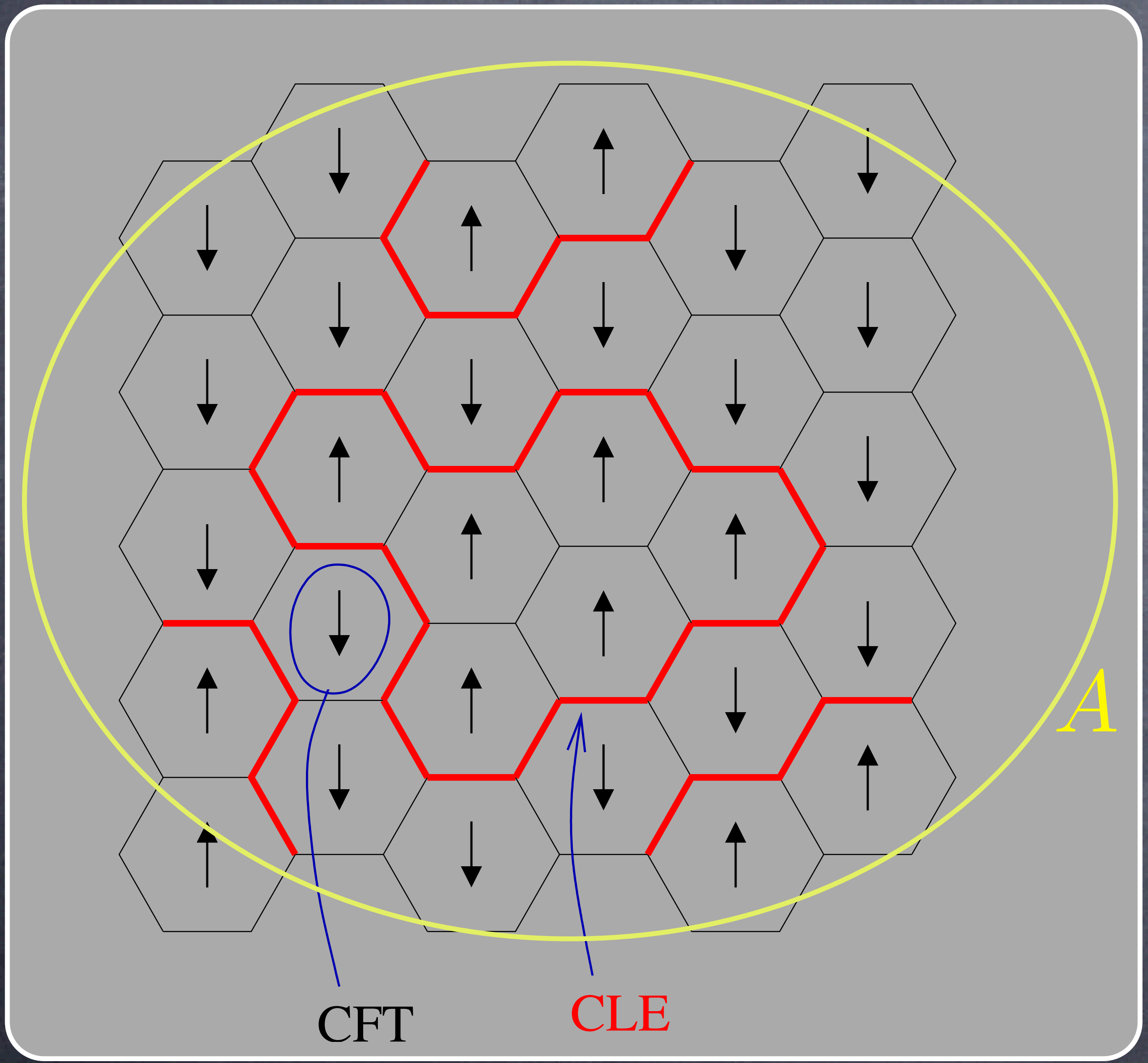
Scaling limit of the Ising statistical model: CFT



$$\mu = e^{-\beta_c \sum_{(jk)} \sigma_j \sigma_k}$$

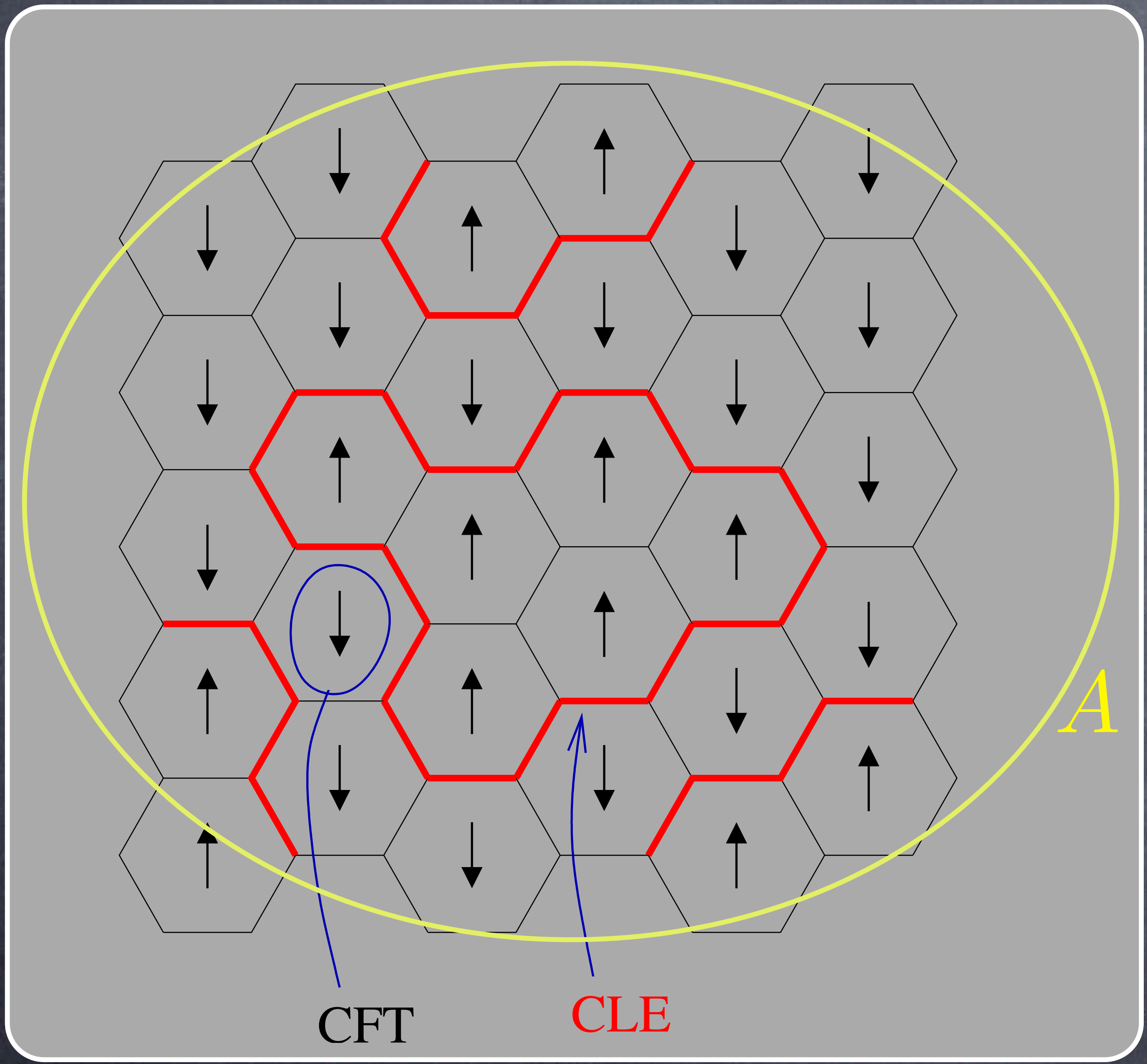
Scaling limit: CFT correlation functions

$$\lim_{r\to\infty} r^{nd} \mathbf{E}_{rA}(\sigma_{z_1 r} \cdots \sigma_{z_n r}) =$$

$$\langle \sigma(z_1) \cdots \sigma(z_n) \rangle_A$$

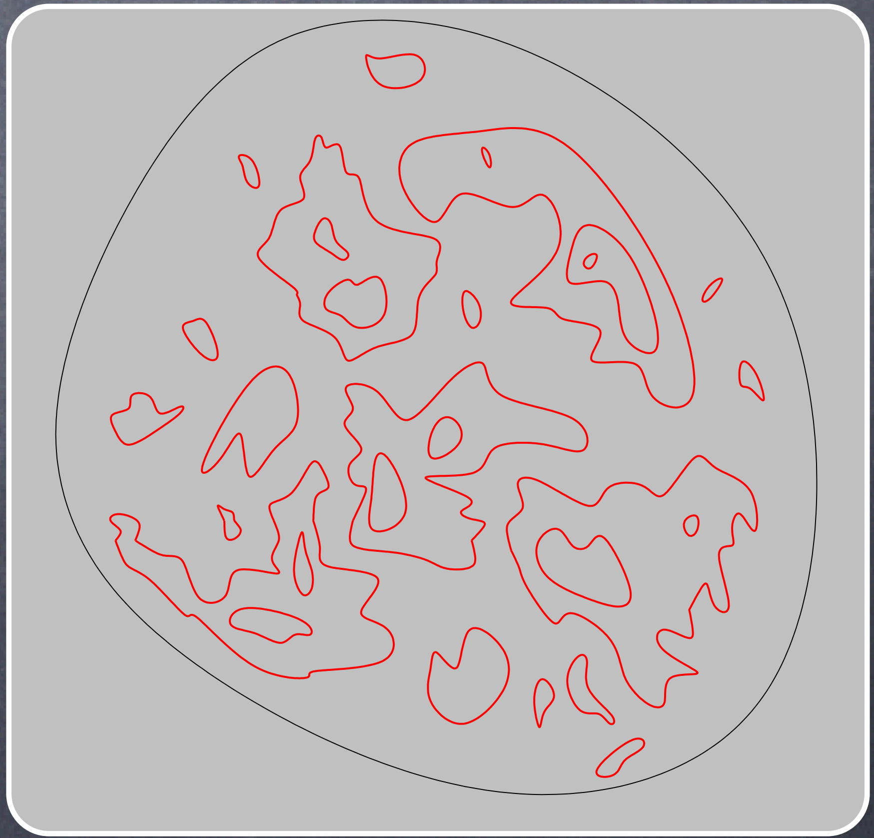
$$(d = 1/8)$$

Scaling limit of the Ising statistical model: CLE



$\mu = x_c^{\text{length}}$

Scaling limit:
conformal loop ensembles



More general: O(n)-models

$$\mu = x_c(n)^{\mathrm{length}} n^{\#\mathrm{loops}}$$

CFT: central charge

$$c = c(n) \in [0,1]$$

CLE: almost-sure fractal dimension of loops

$$\delta = \delta(n) \in [4/3, 7/4]$$

Relation:

$$c = \frac{(7-4\delta)(3\delta-4)}{\delta-1}$$

CFT

- scaling limit of local variables
- powerful algebraic structures giving critical exponents and correlation functions
- conjectures only: no proofs of existence of the scaling limit

CLE

- scaling limit of domain boundaries
- probabilistic description giving some exponents
- direct link with lattice models, proofs of scaling limit for percolation and Ising models

Since CFT and CLE describe the same scaling limits, can we connect them?

CFT $\longleftrightarrow$ CLE

Aim: constructing the CFT algebraic structures from CLE concepts

The Virasoro vertex operator algebra

Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

Identity module: a particular Verma module V

$$L_m \mathbf{1} = 0 \quad (m \geq -1)$$

$$V = \mathrm{span}\big(L_{\mathbf{m}_j}\mathbf{1} : j \geq 0,\ m_k \leq -2\ (k = 1, \ldots, j)\big)$$

$$L_{\mathbf{m}_j} = L_{m_j} \cdots L_{m_1}$$

Vertex operator map

$$Y(\cdot, x) : V \mapsto \mathrm{End}(V)[[x, x^{-1}]]$$

Recursive definition of the vertex operator map

$$Y(\mathbf{1}, x) = 1$$

$$Y(L_m \mathbf{1}, x) = \frac{1}{(-2-m)!} \frac{d^{-2-m}}{dx^{-2-m}} \sum_{n \in \mathbb{Z}} L_n x^{-n-2} \qquad (m \leq -2)$$

$$Y(L_{\mathbf{m}_j} \mathbf{1}, x) = \,:Y(L_{m_j} \mathbf{1}, x) Y(L_{\mathbf{m}_{j-1}} \mathbf{1}, x):$$

involving the normal-ordered product

$$:Y(L_m \mathbf{1}, x) A: \,= Y^{+}(L_m \mathbf{1}, x) A + A Y^{-}(L_m \mathbf{1}, x)$$

$$Y^{+}(v, x) = \text{regular part}$$

$$Y^{-}(v, x) = \text{singular part}$$

Satisfies the main relations of vertex operator algebras

Commutativity

$\forall v, w : \exists k :$

$$(x-y)^k Y(v,x)Y(w,y) = (x-y)^k Y(w,y)Y(v,x)$$

Associativity

$\forall u, w : \exists l \mid \forall v :$

$$(x+y)^l Y(u,x+y)Y(v,y)w = (x+y)^l Y(Y(u,x)v,y)w$$

The VOA can be generated by elements

$$Y(L(m)\mathbf{1}, x)$$

if they satisfy (mainly) commutativity

Relation to CFT correlation functions: inner product

$$(\cdot,\cdot): \quad (\mathbf{1},\mathbf{1}) = 1, \quad L_n^\dagger = L_{-n}$$

$$\langle v_1(x_1)\cdots v_n(x_n)\rangle_{\mathbb{C}} = \iota_{|x_1|>\cdots>|x_n|}(\mathbf{1}, Y(v_1,x_1)\cdots Y(v_n,x_n)\mathbf{1})$$

convergent series in this region:
change to complex variables

Axioms of conformal loop ensembles

- Mesures μ_A for configurations of countably many disjoint, self-avoiding simple loops in simply connected hyperbolic domains A of the plane
- The number of loops of «radius» greater then any given positive number is finite

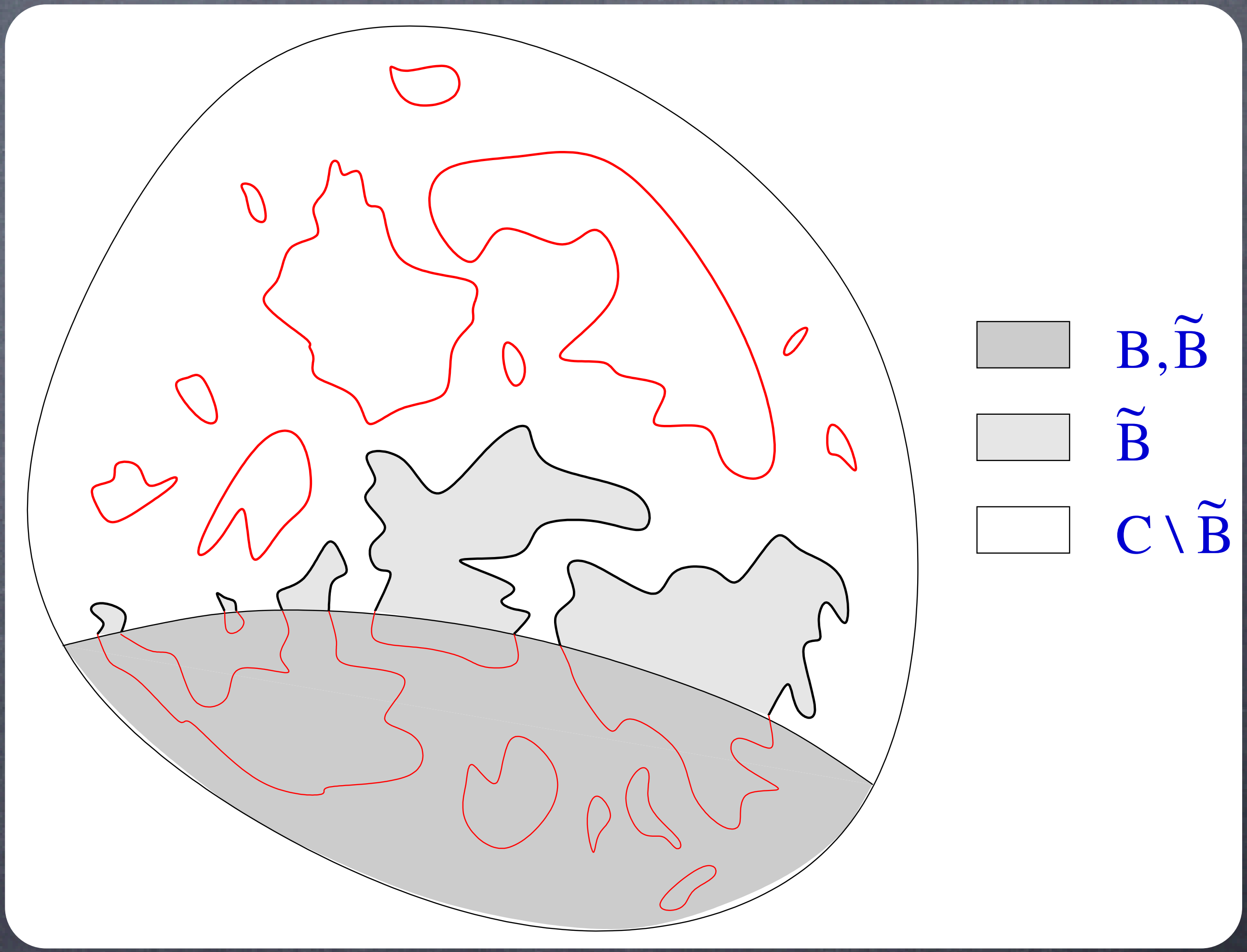
Axioms of conformal loop ensembles

- Conformal invariance

  $$\mu_A = \mu_{g(A)} \circ g$$

- Nesting: inside a loop, the configuration is determined by a CLE bounded by this loop.

- Probabilistic conformal restriction:



CLE and conformal restrictions systems

Random variables and expectation-value functionals

Supported unital algebra

$X \in \mathfrak{X} : \mathrm{Supp}(X) = \text{set of closed subsets of } \hat{\mathbb{C}}$

$\mathfrak{X}^{(A)}$: subalgebra supported in A

Set of linear functionals

$\mathbf{E}[\cdot]_A : \mathfrak{X}^{(A)} \to \mathbb{C}, \quad \mathbf{E}[\mathbf{1}] = 1$

Linear representation of groupoid of conformal maps

$\mathfrak{X}^{(A)} \to \mathfrak{X}^{(g(A))}$

$X \to g \cdot X \qquad\qquad g \cdot (g' \cdot X) = (g \circ g') \cdot X$

Assumed properties

Tubular-neighborhood variables

$$N = \text{tubular neighborhood} : I(N) \in \mathfrak{X}$$

$$N \in \mathrm{Supp}(I(N)), \quad g \cdot I(N) = I(g(N)), \quad \mathbf{E}[I(N)]_A \neq 0,\ A \supset N$$

$$E(N) := \frac{I(N)}{\mathbf{E}[I(N)]_{\hat{\mathbb{C}}}}$$

Weak-local limits: the «configuration-separating» variables

$$E(\alpha) := \lim_{N \to \alpha} E(N) \qquad \alpha = \text{Jordan curve}$$

Assumed properties

Conformal invariance

$$\mathbf{E}[g \cdot X]_{g(A)} = \mathbf{E}[X]_A \quad \forall \quad X \in \mathfrak{X}^{(A)},\ g \text{ conformal on } A$$

Restriction

$$\frac{\mathbf{E}[E(\alpha)X]_A}{\mathbf{E}[E(\alpha)]_A} = \mathbf{E}[X]_{A\setminus\alpha}$$

X : product of factors supported on components of $A \setminus \alpha$

Assumed properties

Smoothness

$$g \mapsto \mathbf{E}[g \cdot X]_{g \cdot A}, \quad g \mapsto \mathbf{E}[X]_{g \cdot A} \quad \text{are smooth at } g = \mathrm{id}$$

where

$A = \cap_i A_i$: in general multiply connected

g : conformal on neighborhood of ∂A_i for all $i \in J$

$g \cdot A_i$: domain bounded by $g(\partial A_i)$

$g \cdot A = \cap_{i \in J} (g \cdot A_i) \cap \cap_{i \notin J} A_i$

Assumed properties

Local covering

$$\lim_{\epsilon \to 0} \nabla \mathbf{E}[X]_{A \setminus K_\epsilon} = \nabla \mathbf{E}[X]_A$$

derivative under smooth conformal deformations

if

$$\lim_{\epsilon \to 0} K_\epsilon = \{z\} \quad \text{(in the sense of smallest covering disks)}$$

X supported in $A \setminus \{z\}$

Theorem 1: relative partition function

Relative partition function (simply connected Jordan domains)

$$Z(\partial C, \partial D) := \frac{1}{\mathbf{E}[E(\partial C)]_{\hat{\mathbb{C}}\setminus\overline{D}}}$$

Theorem:

There exists a complex number $c \in \mathbb{C}$ such that

$$\Delta[h_{-2,w}] \log Z(\partial C, \partial D) = \frac{c}{12}\{s, w\}, \quad s : C \to \mathbb{D} \text{ conformally}$$

Schwartzian derivative

$$w \in D, \quad \overline{D} \subset C$$

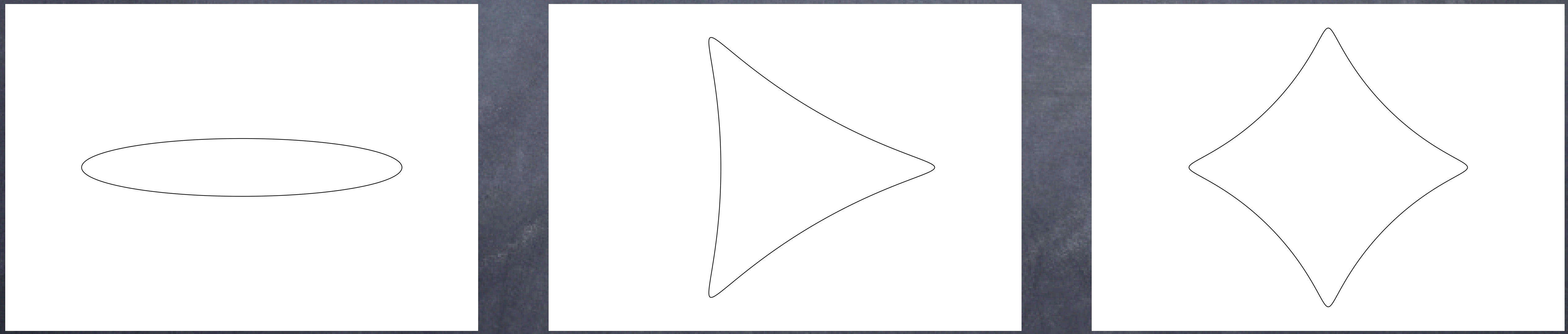
$$\Delta[h_{-2,w}] f(g) = \frac{\partial}{\partial \eta} f(\left\{ z \mapsto z + \frac{\eta}{w - z} \right\})$$

Theorem 2: (partial) VOA structure

Hypotrochoids:

$$C_k(w,\epsilon,\theta,b) := \left\{ w + \epsilon e^{i\theta}(be^{i\alpha} + b^{1-k}e^{(1-k)i\alpha}) : \alpha \in [0,2\pi) \right\}$$

$$k = 2,3,4,\ldots, \quad w \in \mathbb{C}, \quad b > (k-1)^{1/k}$$



Proposition:

Weakly locally,

$$E(C_k(w,\epsilon,\theta,b)) \sim$$

$$\mathbf{1} + o((u\bar{u})^0) + \sum_{m=1}^{\infty} \frac{u^{km}}{m!}\left(T_{k,m}(w) + o((u\bar{u})^0)\right) + c.c.$$

$$u = \epsilon e^{i\theta}$$

Equivalently, we may define

$$T_{k,m}(w) := \lim_{\epsilon\to 0} \frac{m!}{2\pi\epsilon^{km}} \int_0^{2\pi} d\theta\, e^{-kmi\theta} E(C_k(w,\epsilon,\theta,b))$$

Some Virasoro descendents

$$v_{k,m} := \sum_{\lambda\in\Phi(m)} B_\lambda (k-1)^{m-|\lambda|} L_{-k\lambda_{|\lambda|}} \cdots L_{-k\lambda_1}\mathbf{1}$$

$$|\lambda| = j$$

$$\Phi(m) = \{(\lambda_1,\ldots,\lambda_j) : j \geq 1, \sum_i \lambda_i = \lambda,\ \lambda_i \geq 1\}$$

$$B_{(\lambda_1,\ldots,\lambda_j)} = \delta_{\lambda_j,1} B_{(\lambda_1,\ldots,\lambda_{j-1})} + \sum_i (\lambda_i - 1) B_{(\lambda_1,\ldots,\lambda_i - 1,\ldots,\lambda_j)}$$

For instance,

$$v_{k,1} = L_{-k}\mathbf{1}$$

$$v_{k,2} = (L_{-k}^2 + (k-1)L_{-2k})\mathbf{1}$$

$$v_{k,3} = (L_{-k}^3 + 3(k-1)L_{-2k}L_{-k} + 2(k-1)(2k-1)L_{-3k})\mathbf{1}$$

Theorem:

The variables $T_{k,m}$ have the Vertex operator algebra substructure of the descendents $v_{k,m}$, with the identification $T_{k,m} \equiv v_{k,m}$

That is: Laurent expansions of expectation values are reproduced by products of vertex opertors,

$$\Big[\mathbf{E}[T_{k_1,m_1}(w_1)\cdots T_{k_p,m_p}(w_p)X]_A\Big]_{\text{expansion }|w_1|>\cdots|w_p|} =$$

$$Y_R(v_{k_1,m_1},w_1)\cdots Y_R(v_{k_p,m_p},w_p)\mathbf{E}[X]_A$$

with the domain-dependent Virasoro representation R

$$L_\ell \overset{R}{\mapsto} Z(\partial A,v)^{-1}\Delta[h_{\ell,0}]Z(\partial A,v)$$

$$\Delta[h_{\ell,0}]f(g) = \frac{\partial}{\partial\eta} f(\{z\mapsto z-\eta z^{\ell+1}\})$$

v : Jordan curve in A surrounding 0

acting on the function $g\mapsto \mathbf{E}(g\cdot X)_{g\cdot A}$

Conclusion, outlook

- Part of the Virasoro vertex operator algebra recovered.
- Full VOA? What variables are involved?
- Other symmetries? What replaces conformal transformations?
- Proving the axioms of Conformal Restricition Systems from Conformal Loop Ensembles?