Solving Painlevé connection problems using two-dimensional integrable quantum field theory

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Plan of the talk

- Definition of twist fields in QFT
- Definition of the model we will consider: the free Dirac fermion on the Poincaré disk
- How twist fields in this model are related to Painlevé VI
- The connection problems we are interested in
- Constructions of the quantum fields and solutions to the connection problems

Twist fields in quantum field theory

For every global symmetry of a (local) quantum field theory, there exists an associated

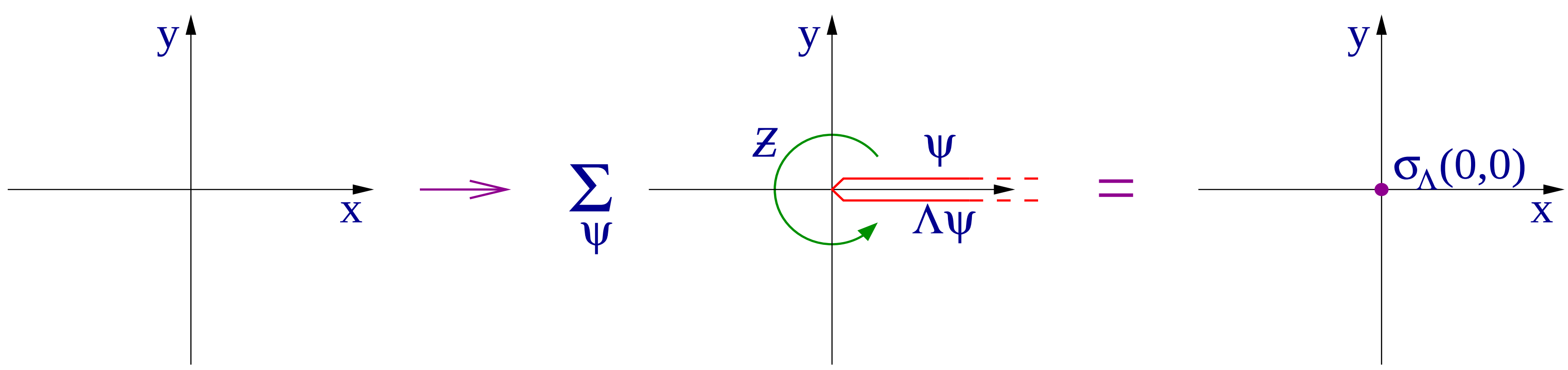
local twist field

Partition function:

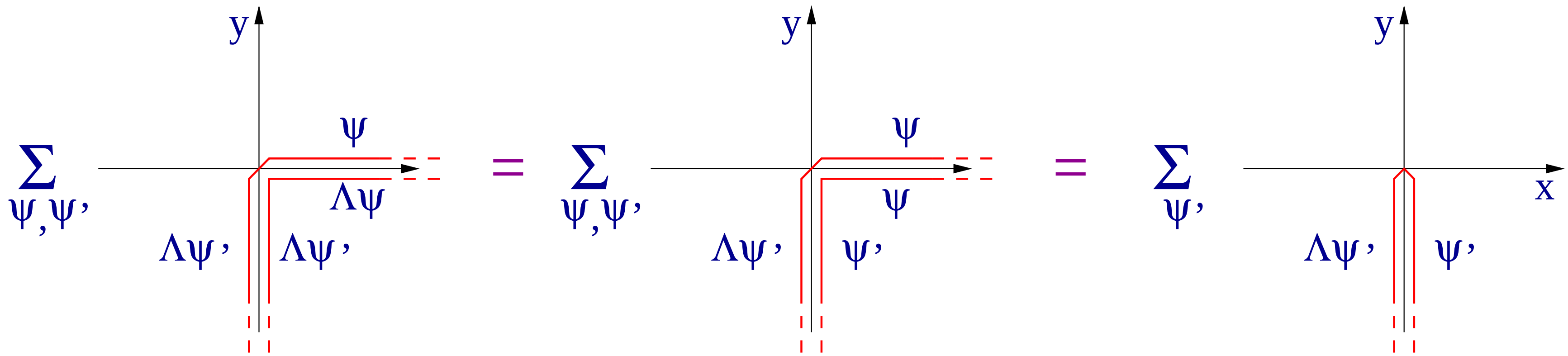
$$Z = \int [d\Psi^\dagger d\Psi] e^{-\mathcal{A}[\Psi^\dagger,\Psi]} \,, \qquad \mathcal{A}[\Lambda\Psi^\dagger,\Lambda\Psi] = \mathcal{A}[\Psi^\dagger,\Psi]$$

Insertion of twist field: universal covering of punctured plane, or plane with a cut

$$Z_{\sigma_\Lambda} = \int_{\Psi(\mathcal{Z}p)=\Lambda\Psi(p)} [d\Psi^\dagger d\Psi] e^{-\mathcal{A}[\Psi^\dagger,\Psi]}$$



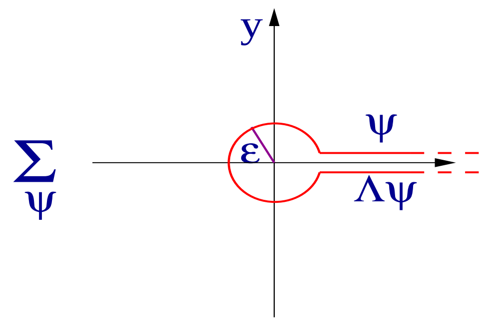
- The result is independent of the shape of the cut:



- Multipoint insertion are defined similarly $\Rightarrow\ Z_{\sigma_{\Lambda_1}(p_1),\sigma_{\Lambda_2}(p_2),\ldots}$

- Correlation functions are regularised ratios:

$$\langle \sigma_{\Lambda_1}(p_1)\sigma_{\Lambda_2}(p_2)\cdots\rangle = \lim_{\epsilon\to 0}\epsilon^{d_1+d_2+\cdots}\frac{Z^{\epsilon_1,\epsilon_2,\ldots}_{\sigma_{\Lambda_1}(p_1),\sigma_{\Lambda_2}(p_2),\ldots}}{Z}$$



- Twist fields are local fields

Example: free fermion theory on the Poincaré disk

Free Dirac fermion **of mass** m on the Poincaré disk **of Gaussian curvature** $-1/R^2$ (maximally symmetric space):

$$\mathcal{A} = \int_{\mathrm{x}^2+\mathrm{y}^2<1} d\mathrm{x}d\mathrm{y}\ \bar{\Psi}\left(\gamma^{\mathrm{x}}\partial_{\mathrm{x}} + \gamma^{\mathrm{y}}\partial_{\mathrm{y}} + \frac{2mR}{1-(\mathrm{x}^2+\mathrm{y}^2)^2}\right)\Psi$$

with

$$\Psi = \begin{pmatrix} \Psi_R \\ \Psi_L \end{pmatrix}, \quad \bar{\Psi} = \Psi^\dagger\gamma^{\mathrm{y}}, \quad \gamma^{\mathrm{x}} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \gamma^{\mathrm{y}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The Dirac fermion has internal $U(1)$ symmetry

$$\Lambda_\alpha : \Psi \mapsto e^{2\pi i\alpha}\Psi, \quad \Psi^\dagger \mapsto e^{-2\pi i\alpha}\Psi^\dagger$$

$$\Rightarrow \sigma_\alpha(x)$$

(we will take $0 < \alpha < 1$)

More precise definitions: correlation functions

Path integral ideas lead to constraints on correlation functions, which completely define them

With p in the universal covering of $\mathbb{D}\backslash\{(0,0),(a_{\rm x},a_{\rm y})\}$, consider for instance the spinor

$$F(p)=\langle\sigma_\alpha(0,0)\tilde\sigma_{\alpha'}(a_{\rm x},a_{\rm y})\Psi(p)\rangle$$

- Equations of motion (where $({\rm x},{\rm y})\in\mathbb{D}$ corresponds to p)

$$\left(\gamma^{\rm x}\partial_{\rm x}+\gamma^{\rm y}\partial_{\rm y}+\frac{2mR}{1-({\rm x}^2+{\rm y}^2)^2}\right)F(p)=0$$

- Asymptotic behaviors

$$|F(p)|=O\left((1-{\rm x}^2-{\rm y}^2)^{1/2+\mu}\right)\quad\text{as}\quad {\rm x}^2+{\rm y}^2\to1$$

$$|F(p)|=O\left(|{\rm x}-a_{\rm x}+i({\rm y}-a_{\rm y})|^{\alpha'-1}\right)\quad\text{as}\quad ({\rm x},{\rm y})\to(a_{\rm x},a_{\rm y})$$

$$|F(p)|=O\left(|{\rm x}+i{\rm y}|^{\alpha}\right)\quad\text{as}\quad ({\rm x},{\rm y})\to(0,0)$$

- Monodromy properties:

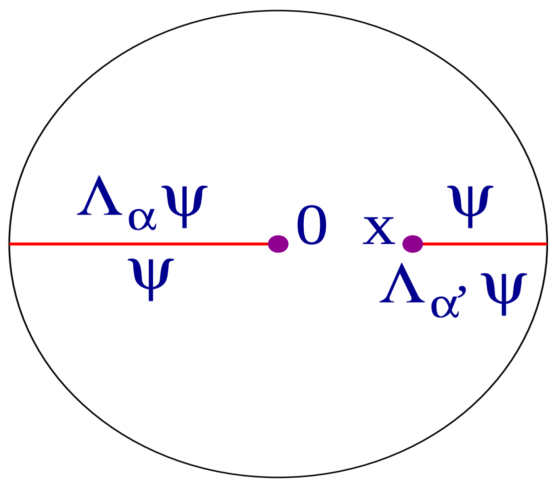
$$F(\mathcal{Z}_{(0,0)}p)=e^{2\pi i\alpha}F(p)\ ,\quad F(\mathcal{Z}_{(a_{\rm x},a_{\rm y})}p)=e^{2\pi i\alpha'}F(p)$$

For multiple twist field insertions, this leads to determinant representation

Path integral ideas also lead to an expression for the two-point function of twist fields as a regularised ratio of determinants (for instance, with $0 < \mathrm{x} < 1$)

$$\langle \sigma_\alpha(0,0)\sigma_{\alpha'}(\mathrm{x},0)\rangle = \lim_{\epsilon_1\to 0,\,\epsilon_2\to 0} \epsilon_1^{d_\alpha}\epsilon_2^{d_{\alpha'}} \frac{\det_{\mathcal{F}_{\alpha,\alpha'}}\left(\gamma^{\mathrm{x}}\partial_{\mathrm{x}} + \gamma^{\mathrm{y}}\partial_{\mathrm{y}} + \frac{2mR}{1-(\mathrm{x}^2+\mathrm{y}^2)^2}\right)}{\det_{\mathcal{F}_{0,0}}\left(\gamma^{\mathrm{x}}\partial_{\mathrm{x}} + \gamma^{\mathrm{y}}\partial_{\mathrm{y}} + \frac{2mR}{1-(\mathrm{x}^2+\mathrm{y}^2)^2}\right)}$$

where $\mathcal{F}_{\alpha,\alpha'}$ is the space of spinor-valued functions on $\mathbb{D}\backslash([-1,0]\cup[\mathrm{x},1])$ which vanish on the boundary of $\mathbb{D}$ and which have the appropriate monodromy properties around the point $(0,0)$ and around the point $(\mathrm{x},0)$.



The eigenvalue problem leads to an **isomonodromic deformation problem**, since changing the positions of twist fields does not change the monodromy they induce. A linear system is obtained by looking at a certain space of solutions to the eigenvalue problem with fixed monodromies, then by considering the action of space-time symmetries on this space and of derivatives with respect to the positions of singular points. Compatibility leads to **Painlevé equations**.

Painlevé equations

It was shown by Palmer, Beatty and Tracy (1993) that the two-point function (in its functional determinant representation) is related to Miwa-Jimbo tau-function of Painlevé VI:

$$\langle \sigma_\alpha(x_1)\sigma_{\alpha'}(x_2)\rangle = \tau(s)$$

$$\frac{d}{ds}\ln\tau(s) = f\left(w, \frac{dw}{ds}, s\right) \qquad\qquad s = \tanh^2\left(\frac{d(x_1,x_2)}{2R}\right)$$

(geodesic distance) $d(x_1,x_2) = 2R\,\mathrm{arctanh}\left(\frac{|z_1-z_2|}{|1-z_1\bar{z}_2|}\right)\,, \quad z_j = \mathrm{x}_j + i\mathrm{y}_j$

$f(w,w',s)$ is a rational function of $w,\ w',\ s$ and of $mR,\ \alpha,\ \alpha'$, and $w=w(s)$ satisfies **Painlevé VI** differential equation

$$w'' - \frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-s}\right)(w')^2 + \left(\frac{1}{s}+\frac{1}{s-1}+\frac{1}{w-s}\right)w'$$

$$= \frac{w(w-1)(w-s)}{s^2(1-s)^2}\left(\frac{(1-4\mu^2)s(s-1)}{2(w-s)^2} - \frac{(\tilde{\lambda}-1)^2 s}{2w^2} + \frac{\gamma(s-1)}{(w-1)^2} + \frac{\lambda^2}{2}\right)$$

with parameters $\mu = mR,\ \lambda = \alpha-\alpha',\ \tilde{\lambda} = \alpha+\alpha',\ \gamma = 0$.

Remarks

- This is a generalisation of much older results, which can be summarised as follows:
  - In the case of the flat geometry ($R \to \infty$), one obtains a description in terms of the Painlevé V equation (Sato, Miwa, Jimbo 1979, 1980)
  - In the case of a theory with only $\mathbb{Z}_2$ symmetry (the Majorana fermion – Ising model) in flat geometry, one obtains a description in terms of the Painlevé III equation (Wu, McCoy, Tracy, Barouch 1976)

  The Dirac theory on the Poincaré disk is the most general case where I am aware of an analysis of Painlevé transcendents from QFT

- Other methods exist for relating Painlevé equations to two-point functions:
  - In the case of flat geometry, it is possible to express the two-point function as a Fredholm determinant, and to derive from this the description in terms of Painlevé equations (Its, Izergin, Korepin, Slavnov 1990; Bernard, Leclair 1997)
  - The occurence of Painlevé equations in free fermion theories was later understood in the context of the Majorana theory as a consequence of certain non-local conserved charges (Fonseca, Zamolodchikov 2003; B.D. Ph.D. thesis Rutgers University 2004)

Asymptotics, exponents from QFT, and Jimbo's formula

Concentrate on singular points $s = 0$ and $s = 1$ only. If one assumes power-law behaviors

$$w \sim Bs^{\kappa_0} \quad \text{as} \quad s \to 0\,, \quad 1 - w \sim A(1-s)^{\kappa_1} \quad \text{as} \quad s \to 1$$

then PVI does not fix the exponents involved.

But correlation functions describe a special transcendent:

- Short distance: $c = 1$ CFT (free massless boson)

$$d_\alpha = \alpha^2 \quad \Rightarrow \quad \tau(s) \sim \langle \sigma_{\alpha+\alpha'} \rangle s^{\alpha\alpha'} \quad \text{as} \quad s \to 0$$
$$\Rightarrow \quad \kappa_0 = \alpha + \alpha' \quad (0 < \alpha + \alpha' < 1)$$

- Large geodesic distance: cluster property of correlation functions

$$\tau(s) \sim \langle \sigma_\alpha \rangle \langle \sigma_{\alpha'} \rangle \quad \text{as} \quad s \to 1 \Rightarrow \kappa_1 = 1 + 2\mu \quad (\mu > 1/2)$$

This should fix the transcendent, and in particular Jimbo's formula (1982) gives

$$B = \mu \frac{\Gamma(\alpha)\Gamma(\alpha')\Gamma(1-\alpha-\alpha')^2\Gamma(\alpha+\alpha'+\mu)}{\Gamma(1-\alpha)\Gamma(1-\alpha')\Gamma(\alpha+\alpha')^2\Gamma(1-\alpha-\alpha'+\mu)}$$

although these values of κ_0, κ_1 were excluded, being a special "degenerate" case.

Connection problem: the value of A and the relative normalisation of the tau-function

- From conformal perturbation theory, one can calculate B
- From form factor expansion

$$A = \frac{\sin(\pi\alpha)\sin(\pi\alpha')\Gamma(\mu+\alpha)\Gamma(1+\mu-\alpha)\Gamma(\mu+\alpha')\Gamma(1+\mu-\alpha')}{\pi^2\Gamma(1+2\mu)^2}$$

(note: Jimbo's formula for A is singular at our values of κ_0, κ_1)

- From vacuum expectation values of twist fields

$$\lim_{s\to 0} \frac{\tau(1-s)s^{\alpha\alpha'}}{\tau(s)} = \frac{\langle\sigma_\alpha\rangle\langle\sigma_{\alpha'}\rangle}{\langle\sigma_{\alpha+\alpha'}\rangle}$$

with

$$\langle\sigma_\alpha\rangle = \prod_{n=1}^{\infty} \left(\frac{1-\frac{\alpha^2}{(\mu+n)^2}}{1-\frac{\alpha^2}{n^2}} \right)^n$$

We will calculate these quantities using a method that is related to Baxter's method of corner transfer matrix in integrable lattice model, obtaining and evaluating trace formulas for the quantities of interest

Constructing correlation functions: quantization schemes and Hilbert spaces

The poincaré disk is maximally symmetric: $SU(1,1)$ space-time symmetry

With

$$\mathcal{S} = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} , \quad \det \mathcal{S} = 1$$

there is an action on fields that preserves $\mathcal{A}$:

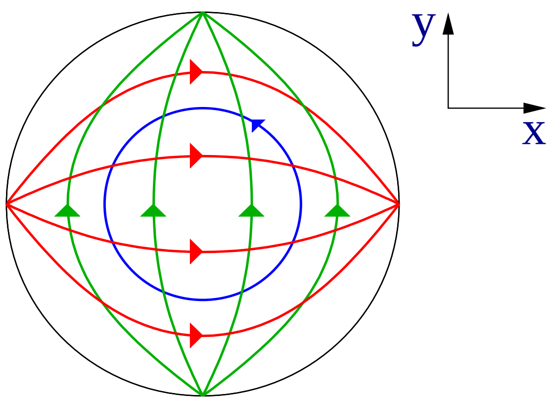
$$\mathcal{S} : \begin{cases} z \mapsto \dfrac{az + \bar{b}}{bz + \bar{a}} , \quad \bar{z} \mapsto \dfrac{\bar{a}\bar{z} + b}{\bar{b}\bar{z} + a} \\ \Psi_R \mapsto (bz + \bar{a})\Psi_R , \quad \Psi_L \mapsto (\bar{b}\bar{z} + a)\Psi_L \end{cases} \qquad (z = \mathrm{x} + i\mathrm{y})$$

It is convenient to consider three $SU(1,1)$ subgroups:

$$\mathcal{X}_\eta = \begin{pmatrix} \cosh(\eta) & \sinh(\eta) \\ \sinh(\eta) & \cosh(\eta) \end{pmatrix}, \quad \eta \in \mathbb{R}$$

$$\mathcal{Y}_\eta = \begin{pmatrix} i\sinh(\eta) & i\cosh(\eta) \\ i\cosh(\eta) & i\sinh(\eta) \end{pmatrix}, \quad \eta \in \mathbb{R}$$

$$\mathcal{R}_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \theta \in [0, 2\pi[$$



Three useful quantization schemes:

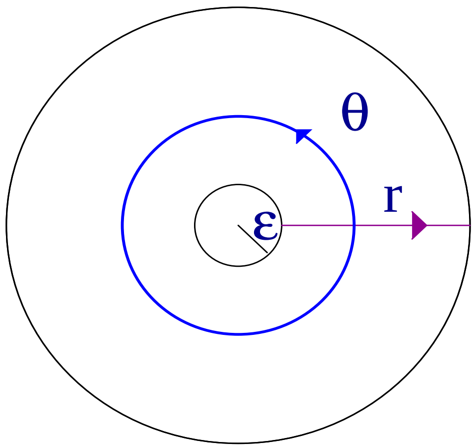
I. Hamiltonian (time translation generator) is generator for $\mathcal{Y}$; space is effectively compact; this gives a good scheme for **large distance expansion** $s \longrightarrow 1$

II. Momentum (space translation generator) is generator for $\mathcal{X}$; space is non-compact; **isometry generator is unitary**, which gives tools for evaluating matrix elements

III. Hamiltonian is generator for $\mathcal{R}$; time is compact and periodic; twist fields have simple realisations allowing **explicit evaluations from trace formulas**

Angular quantization scheme: trace formulas

Hamiltonian H_A generates compact subgroup $\mathcal{R}$:



$$\frac{\partial}{\partial\theta}\mathcal{O} = [H_A, \mathcal{O}]$$

$$H_A = \int_\epsilon^1 \frac{dr}{r} \; : \Psi^\dagger \gamma^y \left(\gamma^x \partial_\eta - \frac{\mu}{\sinh\eta}\right) \Psi :$$

$$\{\Psi(\eta), \Psi^\dagger(\eta')\} = \mathbf{1}\delta(\eta - \eta') \; , \quad r = e^\eta$$

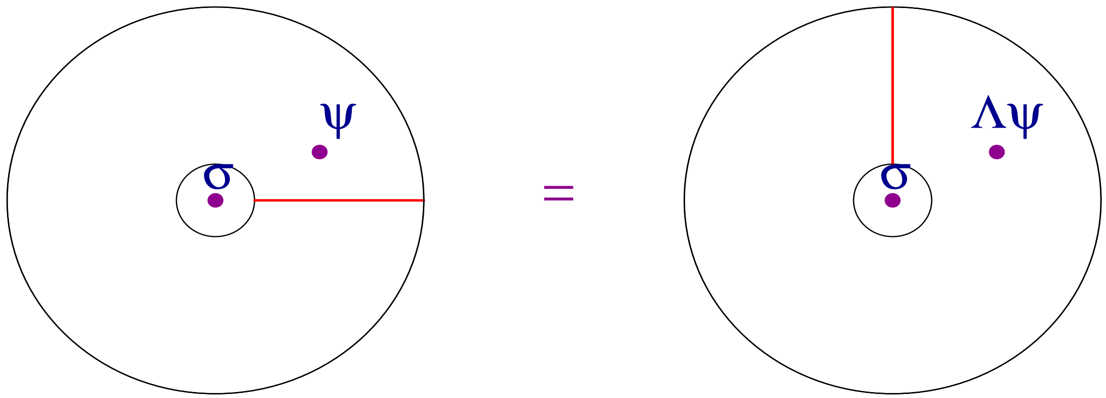
Correlation functions of fermion fields are traces over the Hilbert space $\mathcal{H}_A$ of functions of r on the segment $r \in [\epsilon, 1]$ which form a module for the canonical anti-commutation relations and on which H_A acts and is Hermitian:

$$\langle \cdots \rangle = \lim_{\epsilon \longrightarrow 0} \frac{\mathrm{Tr}_\epsilon \left(e^{-2\pi H_A} \ldots\right)}{\mathrm{Tr}_\epsilon \left(^{-2\pi H_A}\right)}$$

It is easy to diagonalise H_A in terms of partial waves $\mathcal{U}_\nu(\eta) = \begin{pmatrix} u_\nu \\ v_\nu \end{pmatrix}$:

$$\Psi = \sum_\nu c_\nu \mathcal{U}_\nu(\eta) e^{-\nu\theta} \; , \quad \Psi^\dagger = \sum_\nu c_\nu^\dagger \mathcal{U}_\nu^\dagger(\eta) e^{\nu\theta} \; , \quad \{c_\nu, c_{\nu'}^\dagger\} = \delta_{\nu,\nu'}$$

Angular quantization is well adapted to twist fields: they are simply operators producing symmetry transformations on $\mathcal{H}_A$, hence diagonalised by eigenstates of H_A



$$[\sigma_\alpha(0,0)]_A = e^{2\pi i\alpha Q}\ , \quad Q = U(1)\text{-charge}$$

Correlation functions involving fermion fields and one twist fields are regularised traces

$$\langle\sigma_\alpha(0,0)\cdots\rangle = \lim_{\epsilon\to 0}\epsilon^{\alpha^2}\frac{\mathrm{Tr}_\epsilon\left(e^{-2\pi H_A+2\pi i\alpha Q}\ldots\right)}{\mathrm{Tr}_\epsilon\left(e^{-2\pi H_A}\ldots\right)}$$

The proper boundary condition at $r=\epsilon$ for giving the **conformal normalisation** is

$$r=\epsilon\ :\quad \Psi_R=\Psi_L\ ,\quad \Psi_R^\dagger=\Psi_L^\dagger$$

One-point functions can be evaluated

Evaluation of the traces:

- factorisation in independent two-dimensional spaces $\mathcal{H}_{\mathcal{A}}^{(\nu)}$ for each ν: infinite product
- take $\nu < \nu_{max}$ then $\nu_{max} \longrightarrow \infty$ simultaneously on both traces
- in the limit $\epsilon \longrightarrow 0$, what counts is the **density of states** $\partial_\nu \ln S(\nu)$:

$$\begin{pmatrix} u_\nu \\ v_\nu \end{pmatrix} \rightarrow \begin{pmatrix} e^{i\nu} \\ -ie^{-i\nu}\, S(\nu) \end{pmatrix}$$

where

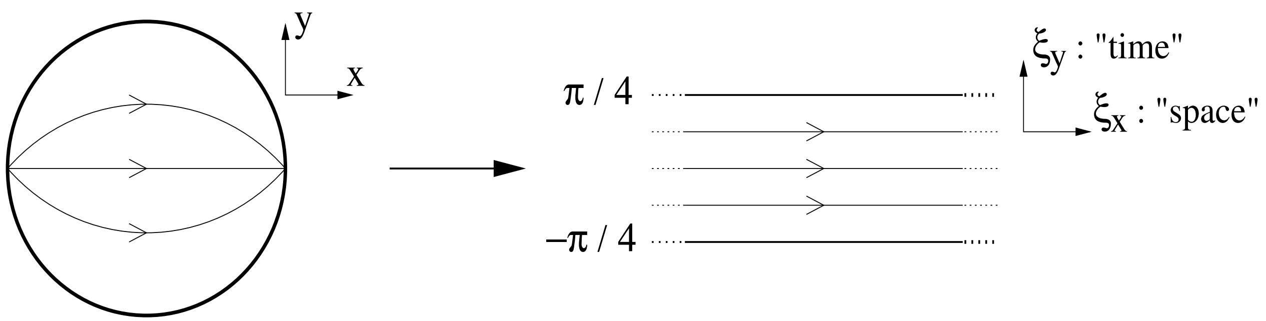
$$S(\nu) = \frac{\Gamma(1/2 + i\nu)\Gamma(1/2 - i\nu + \mu)}{\Gamma(1/2 - i\nu)\Gamma(1/2 + i\nu + \mu)}$$

The result is

$$\langle \sigma_\alpha \rangle = \exp\left[\int_0^\infty \frac{d\nu}{2\pi i} \ln\left(\frac{(1 + e^{-2\pi\nu + 2\pi i\alpha})(1 + e^{-2\pi\nu - 2\pi i\alpha})}{(1 + e^{-2\pi\nu})^2}\right) \partial_\nu \ln S(\nu)\right]$$

Non-stationary quantization scheme: form factors

Momentum operator $P_{\mathcal{X}}$ generates non-compact subgroup $\mathcal{X}$:



$$\mathrm{x} + i\mathrm{y} = \tanh(\xi_\mathrm{x} + i\xi_\mathrm{y}) \,, \quad \mathcal{A} = \int d\xi_\mathrm{x} d\xi_\mathrm{y} \, \bar{\Psi} \left(\gamma^\mathrm{x} \frac{\partial}{\partial \xi_\mathrm{x}} + \gamma^\mathrm{y} \frac{\partial}{\partial \xi_\mathrm{y}} + \frac{2\mu}{\cos 2\xi_\mathrm{y}} \right) \Psi$$

$$-i \frac{\partial}{\partial \xi_\mathrm{x}} \mathcal{O} = [P_{\mathcal{X}}, \mathcal{O}] \,, \quad \{\Psi(\xi_\mathrm{x}), \Psi^\dagger(\xi'_\mathrm{x})\} = \mathbf{1} \delta(\xi_\mathrm{x} - \xi'_\mathrm{x})$$

Correlation functions are “time”-ordered products:

$$\langle \mathcal{O}_1(\xi_{\mathrm{x}\,1}, \xi_{\mathrm{y}\,1}) \mathcal{O}_2(\xi_{\mathrm{x}\,2}, \xi_{\mathrm{y}\,2}) \rangle =$$

$$\begin{cases} \langle \mathrm{vac} | \mathcal{O}_1(\xi_{\mathrm{x}\,1}, \xi_{\mathrm{y}\,1}) \mathcal{O}_2(\xi_{\mathrm{x}\,2}, \xi_{\mathrm{y}\,2}) \cdots | \mathrm{vac} \rangle & \xi_{\mathrm{y}\,1} > \xi_{\mathrm{y}\,2} \\ (-1)^{f_1 f_2} \langle \mathrm{vac} | \mathcal{O}_2(\xi_{\mathrm{x}\,2}, \xi_{\mathrm{y}\,2}) \mathcal{O}_1(\xi_{\mathrm{x}\,1}, \xi_{\mathrm{y}\,1}) \cdots | \mathrm{vac} \rangle & \xi_{\mathrm{y}\,2} > \xi_{\mathrm{y}\,1} \end{cases}$$

Fermion operators are written

$$\Psi(\xi_{\rm x},\xi_{\rm y}) = \int d\omega\ \rho(\omega)\big[i\gamma^{\rm x}\gamma^{\rm y} P^*(\omega,-\xi_{\rm y})\ e^{i\omega\xi_{\rm x}} A_-^\dagger(\omega) + P(\omega,\xi_{\rm y})\ e^{-i\omega\xi_{\rm x}} A_+(\omega)\big]$$

where

- Waves $P(\omega,\xi_{\rm y})e^{-i\omega\xi_{\rm y}}$ fixed from condition that they form module for $su(1,1)$ (as differential operators) with Casimir equal to $\mu^2 - 1/4$ (equations of motion)

- Density contains all singularities in the finite ω-plane

$$\rho(\omega) = \frac{\Gamma\left(\frac{1}{2}+\mu+\frac{i\omega}{2}\right)\Gamma\left(\frac{1}{2}+\mu-\frac{i\omega}{2}\right)}{2\pi\Gamma\left(\frac{1}{2}+\mu\right)^2}$$

- Mode operators satisfy normalised canonical algebra

$$\rho(\omega)\ \{A_\epsilon(\omega), A_{\epsilon'}^\dagger(\omega')\} = \delta(\omega-\omega')\delta_{\epsilon,\epsilon'}$$

Hilbert space $\mathcal{H}$ is space of functions that forms a module for canonical anti-commutation relations (chosen with basis that diagonalises momentum operator) with vacuum $|{\rm vac}\rangle$ that has the property

$$\lim_{\xi_{\rm y}\to-\frac{\pi}{4}} \Psi(\xi_{\rm x},\xi_{\rm y})|{\rm vac}\rangle = \lim_{\xi_{\rm y}\to-\frac{\pi}{4}} \Psi^\dagger(\xi_{\rm x},\xi_{\rm y})|{\rm vac}\rangle = 0$$

Fock space over mode algebra; appropriate choice of $P(\omega,\xi_{\rm y})$ gives $A_\epsilon(\omega)|{\rm vac}\rangle = 0$.

The resolution of the identity gives an expansion for two-point functions in terms of form factors

Denote states by

$$|\omega_1, \ldots, \omega_n\rangle_{\epsilon_1, \ldots, \epsilon_n} = A^\dagger_{\epsilon_1}(\omega_1) \cdots A^\dagger_{\epsilon_n}(\omega_n)|\mathrm{vac}\rangle$$

Then,

$$\mathbf{1}_{\mathcal{H}} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\epsilon_1, \ldots, \epsilon_n} \int \left(\prod_{j=1}^{n} d\omega_j \, \rho(\omega_j) \right) |\omega_1, \ldots, \omega_n\rangle_{\epsilon_1, \ldots, \epsilon_n} \; {}_{\epsilon_n, \ldots, \epsilon_1}\langle \omega_n, \ldots, \omega_1|$$

which gives

$$\langle \mathrm{vac}|\sigma_\alpha(x_1)\sigma_{\alpha'}(x_2)|\mathrm{vac}\rangle = \langle \sigma_\alpha \rangle \langle \sigma_{\alpha'} \rangle \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\epsilon_1, \ldots, \epsilon_n} \int \left(\prod_{j=1}^{n} d\omega_j \, \rho(\omega_j) \right) \times$$
$$\times F_\alpha(\omega_1, \ldots, \omega_n)_{\epsilon_1, \ldots, \epsilon_n} (F_{-\alpha'}(\omega_n, \ldots, \omega_1)_{\epsilon_n, \ldots, \epsilon_1})^* e^{-i(\omega_1 + \cdots + \omega_n)\frac{d(x_1, x_2)}{2R}}$$

where **form factors** are

$$F_\alpha(\omega_1, \ldots, \omega_n)_{\epsilon_1, \ldots, \epsilon_n} = \frac{\langle \mathrm{vac}|\sigma_\alpha(0,0)|\omega_1, \ldots, \omega_n\rangle_{\epsilon_1, \ldots, \epsilon_n}}{\langle \mathrm{vac}|\sigma_\alpha|\mathrm{vac}\rangle}$$

The embedding $\mathcal{H} \hookrightarrow \mathcal{H}_A \otimes \mathcal{H}_A$ allows the evaluation of form factors via trace formulas

- States in $\mathcal{H}$ are associated to operators in $\mathcal{H}_A$:

$$|\omega_1, \ldots, \omega_n\rangle_{\epsilon_1,\ldots,\epsilon_n} \equiv a_{\epsilon_1}(\omega_1)\cdots a_{\epsilon_n}(\omega_n)$$

- Operators acting on $\mathcal{H}$ are identified with left-action on $\mathrm{End}(\mathcal{H}_A)$

$$\mathcal{O}|u\rangle \in \mathcal{H} \;\equiv\; \pi_A(\mathcal{O})U \in \mathrm{End}(\mathcal{H}_A) \quad \text{if } |u\rangle \equiv U$$

- The inner product on $\mathcal{H}$ is associated with traces in $\mathcal{H}_A$:

$$\langle u|v\rangle \equiv \frac{\mathrm{Tr}\left(e^{-2\pi H_A}U^\dagger V\right)}{\mathrm{Tr}\left(e^{-2\pi H_A}\right)} \quad \text{if } |u\rangle \equiv U,\; |v\rangle \equiv V.$$

- Hence form factors are

$$F_\alpha(\omega_1,\ldots,\omega_n)_{\epsilon_1,\ldots,\epsilon_n} = \frac{\mathrm{Tr}\left(e^{-2\pi H_A + 2\pi i\alpha Q} a_{\epsilon_1}(\omega_1)\cdots a_{\epsilon_n}(\omega_n)\right)}{\mathrm{Tr}\left(e^{-2\pi H_A+2\pi i \alpha Q}\right)}$$

- Two sets of conditions define the operators $a_\epsilon(\omega)$:

$$\{a_\epsilon(\omega), \Psi(\eta \to -\infty, \theta)\} = \{a_\epsilon(\omega), \Psi^\dagger(\eta \to -\infty, \theta)\} = 0$$

and

$$\langle \mathrm{vac}|\Psi(\xi_{\mathrm{x}}, \xi_{\mathrm{y}})|\omega\rangle_+ = \frac{\mathrm{Tr}\left(e^{-2\pi H_A}\pi_A(\Psi(\xi_{\mathrm{x}}, \xi_{\mathrm{y}}))a_+(\omega)\right)}{\mathrm{Tr}\left(e^{-2\pi H_A}\right)} = e^{-i\xi_{\mathrm{x}}}P(\omega, \xi_{\mathrm{y}})$$

- They can be calculated explicitly:

$$a_+(\omega) = \int_{-\infty}^{\infty} d\nu\, g(\nu;\omega)\, c_\nu^\dagger\,, \quad a_-(\omega) = \int_{-\infty}^{\infty} d\nu\, g(\nu;\omega)\, c_{-\nu}$$

$$g(\nu;\omega) = \sqrt{\pi}2^{-\mu}e^{i\frac{\pi}{2}\left(\mu+\frac{1}{2}-i\frac{\omega}{2}\right)}\frac{e^{-\pi\nu}\Gamma\left(\frac{1}{2}+\mu+i\nu\right)}{\Gamma\left(1+\mu\right)\Gamma\left(\frac{1}{2}+i\nu\right)} \times$$

$$\times F\left(\mu+\frac{1}{2}+i\nu, \mu+\frac{1}{2}-i\frac{\omega}{2}; 1+2\mu; 2-i0\right)$$

$$F(a, b; c; 2-i0) = \lim_{\varepsilon \to 0^+} F(a, b; c; 2-i\varepsilon)$$

where $F(a, b; c; z)$ is Gauss's hypergeometric function on its principal branch.

Integrals involved can be evaluated by contour deformation: sum of residues of poles

We had:

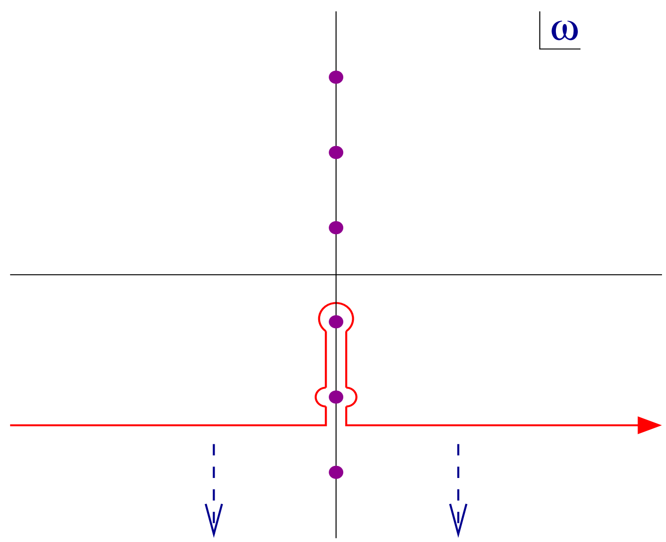
$$\langle \text{vac}|\sigma_\alpha(x_1)\sigma_{\alpha'}(x_2)|\text{vac}\rangle = \langle\sigma_\alpha\rangle\langle\sigma_{\alpha'}\rangle \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\epsilon_1,\ldots,\epsilon_n}\int\left(\prod_{j=1}^{n} d\omega_j\,\rho(\omega_j)\right) \times$$

$$\times F_\alpha(\omega_1,\ldots,\omega_n)_{\epsilon_1,\ldots,\epsilon_n}\left(F_{-\alpha'}(\omega_n,\ldots,\omega_1)_{\epsilon_n,\ldots,\epsilon_1}\right)^* e^{-i(\omega_1+\cdots+\omega_n)\frac{d(x_1,x_2)}{2R}}$$

It turns out that $F_\alpha(\omega_1,\ldots,\omega_n)_{\epsilon_1,\ldots,\epsilon_n}$ **are entire functions of all spectral parameters**

$\Rightarrow$ contour deformation, getting residues at poles of density $\rho(\omega)$:

$$\omega = -i\lambda_k = -i(1+2\mu+2k)\;, \quad k=0,1,2,\ldots$$



Isometric quantization scheme: large-distance expansion

Hamiltonian $H_{\mathcal{Y}}$ generates non-compact subgroup $\mathcal{Y}$.

In isometric quantization, states form a discrete set, parametrized by spectral parameters $k_1, k_2, \ldots$ with energies $\lambda_{k_1} + \lambda_{k_2} + \cdots$. The residue evaluation above is exactly a "form-factor" expansion in isometric quantization.

- All residues can be evaluated in terms of rational and Gamma functions of μ and α
- The exponential of the geodesic distance occurs in the form

$$e^{-(p(1+2\mu)+q)\frac{d(x,y)}{R}}\ , \quad p=0\,, \quad q=0 \quad \text{or} \quad p=1,2,\ldots\ , \quad q=0,1,2,\ldots$$

In particular,

$$\frac{\langle \sigma_\alpha(x)\sigma_{\alpha'}(y)\rangle}{\langle\sigma_\alpha\rangle\langle\sigma_{\alpha'}\rangle} = 1 - 4^{2\mu+1}\frac{(\mu+\alpha)(\mu+\alpha')}{(1+2\mu)^2}Ae^{-(1+2\mu)\frac{d(x,y)}{R}} + \cdots$$

which gives

$$1-w = A(1-s)^{1+2\mu}\sum_{p,q=0}^{\infty} D_{p,q}\,(1-s)^{p(1+2\mu)+q}\ , \quad D_{0,0}=1$$

Conclusions and perspectives

I have described how to solve certain connection problems of Painlevé VI, using its association with correlation functions in 2-dimensional integrable QFT:

- Asymptotic near $s = 0$: conformal perturbation theory
- Asymptotic near $s = 1$: form factor expansion

Questions and future research:

- What about the point $s = \infty$? Is it accessible from QFT?
- The Dirac theory on the sphere can be solved similarly; it leads to PVI with $\mu^2 \mapsto -\mu^2$. Asymptotics? Connection problem?
- Special case $\alpha + \alpha' = 1$ leads to logarithmic behaviors; other operators than σ_α, descendent under fermion algebra, can be considered; all these should correspond to yet other Painlevé VI transcendents.
- More general free fermion theory can be considered: with, for instance, $SU(n)$ invariance, and associated twist fields. What Painlevé equation do they generate? What transcendents? Solving other connection problems?