Conformal field theory from conformal loop ensembles

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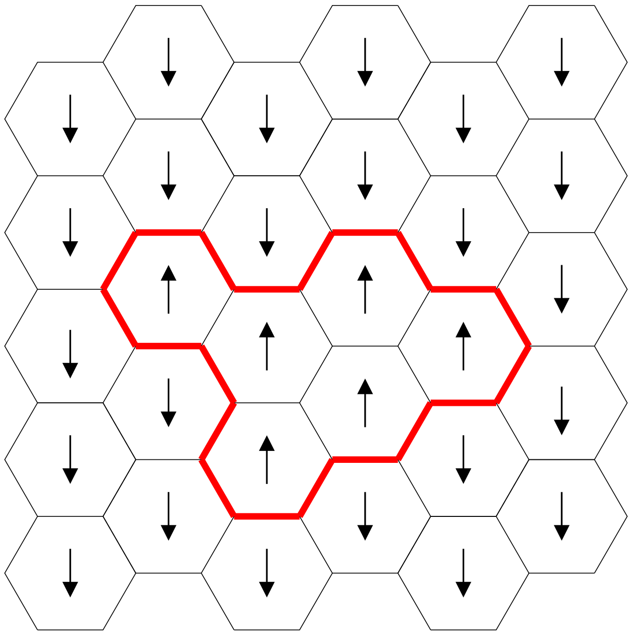
Scaling limits and emergent behaviours

Example: the Ising model

Microscopic model: measure on functions σ from faces of a lattice (ex: hexagonal) to some set (ex: spin $\{\uparrow,\downarrow\} = \{+1,-1\}$), with properties of locality, homogeneity



$$\mu(\sigma) = \exp\left[\beta \sum_{\text{neighbouring faces } j,k} \sigma(j)\sigma(k)\right]$$

Critical point $\beta = \beta_c$: for any β, measure is factorisable into local factors, so fluctuations are usually uncorrelated at large distances. Small temperatures: all spins tend to align (non-zero average local moment). High temperatures: thermal fluctuations break alignment (zero average local moment). Critical point: combination of both effects gives large-distance correlations of fluctuations.

$\Rightarrow$ Universal emergent correlations

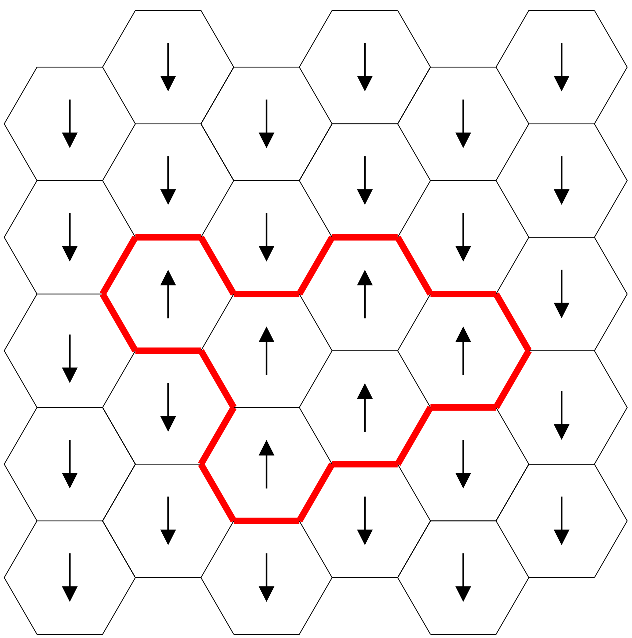
Quantum field theory, a theory for emergent correlations:

The scaling limit of expectations is:

$$\lim_{\varepsilon \to 0} \varepsilon^{-1/4}\, \mathbb{E}^{(\beta=\beta_c-\alpha\varepsilon)}[\sigma(x/\varepsilon)\sigma(y/\varepsilon)] = C^{(\alpha)}(x,y)$$

(here, x, y are in $\mathbb{R}^2$). The coefficient $C^{(\alpha)}(x,y)$ is a correlation function in a QFT

$$C^{(\alpha)}(x,y) = \langle \mathcal{O}(x)\mathcal{O}(y)\rangle^{(\alpha)}$$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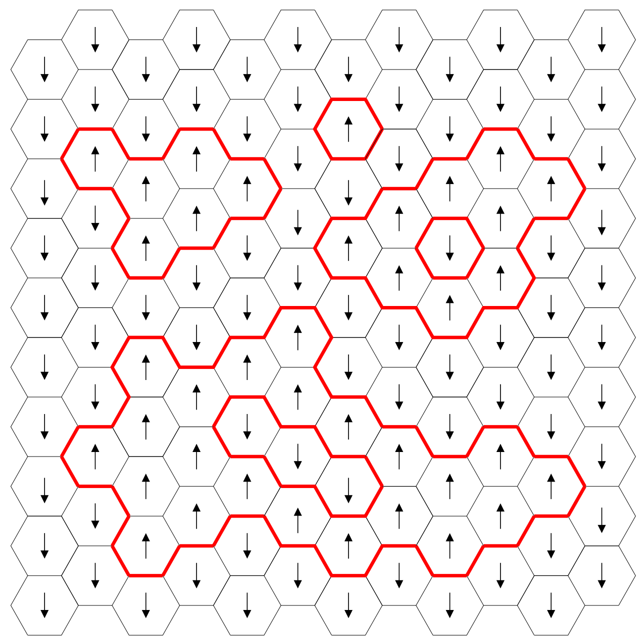

$\longrightarrow \quad \cdots$

The basic ingredients of QFT are

- Local fields $\mathcal{O}(x) \Leftrightarrow$ local variables $1,\ \sigma(k),\ \sigma^2(k),\ \sigma(k)\sigma(\text{neighbour of } k),\ \ldots$
- correlation functions $\langle\cdot\rangle \Leftrightarrow$ expectations of products of local variables $\mathbb{E}[\cdot]$

Questions: 1) What are the emergent objects? 2) What is the measure theory for them? 3) Can we reproduce the QFT local correlations from this theory? 4) Can we prove that it emerges from the microscopic theory?

From quantum fluctuations:

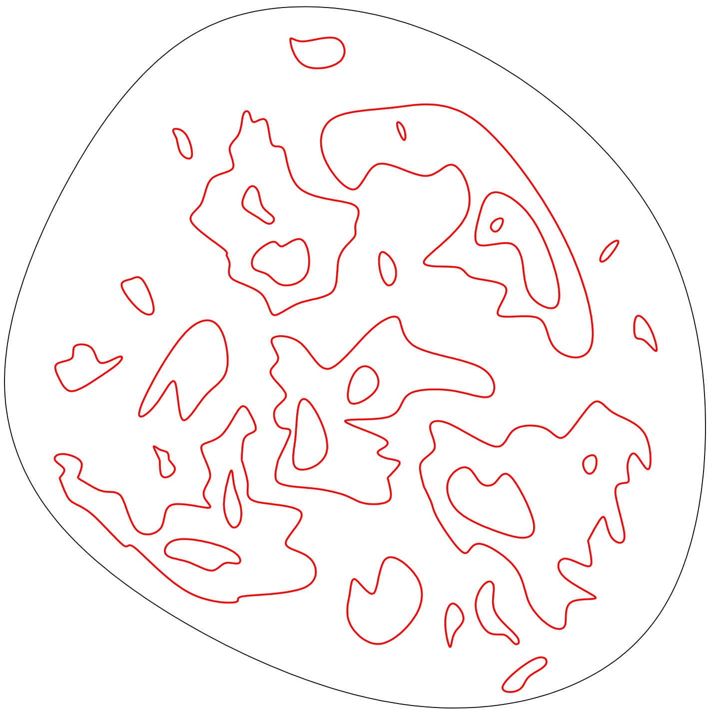
1) relativistic quantum particles; 2) scattering matrix; 3) constructing local field-operators on the Hilbert space (form factor programme in integrable QFT); 4) unknown

From thermal fluctuations:

1) domain walls; 2) conformal loop ensembles (at the critical point) [Sheffield, Werner 2005–]; 3) stress-energy tensor [BD 2009]; 4) proofs in the cases of the critical Ising and percolation models [Smirnov, 2007–].

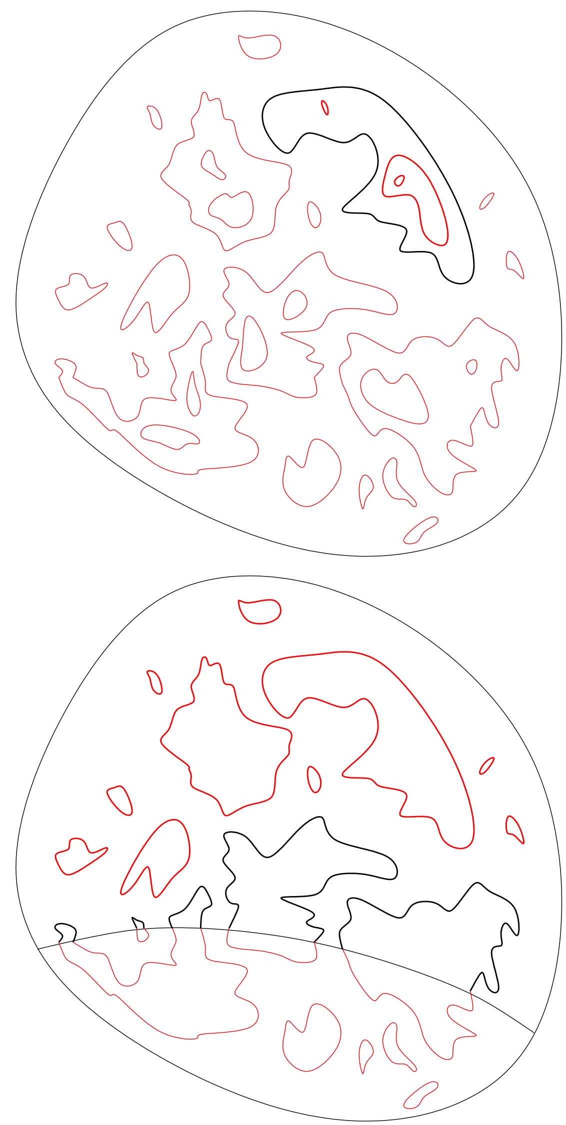
Conformal field theory from conformal loop ensembles

Conformal loop ensembles: Consider the set $\mathcal{S}_D$ whose elements are collections of at most a countable infinity of self-avoiding, disjoint loops lying on a simply connected domain D.



A conformal loop ensemble can be seen as a family of measures μ_D on the sets $\mathcal{S}_D$ for all simply connected domains D, with three defining properties.

1. Conformal invariance. For any conformal transformation $f : D \to D'$, we have $\mu_D = \mu_{D'} \cdot f$.
2. Nesting. The measure μ_D restricted on a loop $\gamma \subset D$ and on all loops outside γ is equal to the CLE measure μ_{D_γ} on the domain $D_\gamma \subset D$ delimited by γ.
3. Conformal restriction. Given a domain $B \subset D$ such that $D \setminus B$ is simply connected, consider $\tilde{B}$, the closure of the set of points of B and points that lie inside loops that intersect B. Then the measure on each component C_i of $D \setminus \tilde{B}$, obtained by restriction on loops that intersect B, is μ_{C_i}.



[Sheffield, Werner 2005–; reviewed in BD 2009]

Conformal field theory: with g conformal on a domain D of $\hat{\mathbb{C}}$, there exists a map $\mathcal{O} \mapsto g \cdot \mathcal{O}$ such that

$$\left\langle \prod_i \mathcal{O}_i(z_i) \right\rangle_D = \left\langle \prod_i (g \cdot \mathcal{O}_i)(g(z_i)) \right\rangle_{g(D)}$$

For primary fields, $(g \cdot \mathcal{O})(g(z)) = (\partial g)^h (\bar{\partial}\bar{g})^{\tilde{h}} \mathcal{O}(g(z))$, with $h, \tilde{h} \in \mathbb{R}^+$. With locality, this implies existence of stress-energy tensor $T(w)$, with conformal Ward identities:

$$\left\langle T(w) \prod_i \mathcal{O}(z_i) \right\rangle_D \sim \sum_i \left(\frac{h_i}{(w-z_i)^2} + \frac{1}{w-z_i} \frac{\partial}{\partial z_i} \right) \left\langle \prod_i \mathcal{O}(z_i) \right\rangle_D$$

T is not a primary field, there is a central charge $c \in \mathbb{R}$:

$$(g \cdot T)(g(w)) = g'(w)^2 T(g(w)) + \frac{c}{12}\{g, w\}, \quad \{g, w\} = \left(\frac{\partial^3 g(w)}{\partial g(w)} - \frac{3}{2} \left(\frac{\partial^2 g(w)}{\partial g(w)} \right)^2 \right)$$

This is the basis for the vertex operator algebra formulation of CFT [Kac, Lepowsky, ..., Cardy, Zamokodchikov, ...; 1980–].

Conformal derivatives: [BD 2009] particular case of Hadamard derivatives, for small conformal transformations (where the tangent space is a space of holomorphic functions). For holomophic functions $h_\eta \to h$ as $\eta \to 0$ uniformly on A,

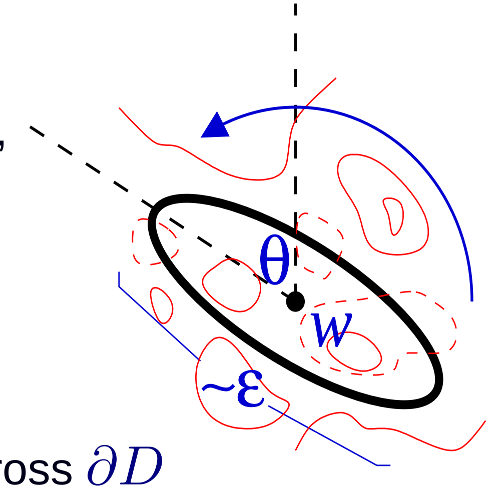
$$\lim_{\eta\to 0} \frac{f((\mathrm{id}+\eta h_\eta)(\Sigma)) - f(\Sigma)}{\eta} = \int_{\partial A} dz\, h(z) \Delta_z^A f(\Sigma) + c.c.$$

We choose $\Delta_z^A f(\Sigma)$ holomorphic in $\hat{\mathbb{C}} \setminus A$. We choose $\Sigma \in$ transformations conformal on $\partial D \cup \{z_i\}$. We define $Z_{\partial D|\partial B} = Z_D Z_{\hat{\mathbb{C}}\setminus\overline{B}} / Z_{D\setminus\overline{B}}$ for any $\overline{B} \subset D$. We choose $A = \hat{\mathbb{C}} \setminus N(w)$ where $N(w)$ is a neighbourhood of w. We find, independently of B,

$$\left\langle T(w) \prod_i \mathcal{O}(z_i) \right\rangle_D =$$

$$Z_{\partial D|\partial B}^{-1}\, \Delta_w^{\hat{\mathbb{C}}\setminus N(w)} \left[Z_{\Sigma(\partial D)|\Sigma(\partial B)} \left\langle \prod_i (\Sigma\cdot\mathcal{O})(\Sigma(z_i)) \right\rangle_{\hat{\mathbb{C}}\setminus\Sigma(\hat{\mathbb{C}}\setminus D)} \right]_{\Sigma=\mathrm{id}}$$

The stress-energy tensor in CLE: [BD 2009]



- $T(w)$: Random variable forbidding loops from crossing a small, spin-2 rotating ellipse (SLE case: [BD, Riva, Cardy 2006])

$$\lim_{\varepsilon\to 0}\frac{8}{\pi\varepsilon^2}\int d\theta\, e^{-2i\theta}\mathbf{1}^{ren}_{E(w,\varepsilon,\theta)}$$

- $Z_{\partial D|\partial B}$: Ratio of probabilities where loops are forbidden to cross ∂D

$$\frac{\mathbb{E}_{\hat{\mathbb{C}}}[\mathbf{1}^{ren}_{\partial D}]}{\mathbb{E}_{\hat{\mathbb{C}}\setminus\overline{B}}[\mathbf{1}^{ren}_{\partial D}]}$$

- I prove conformal Ward identities,

$$\mathbb{E}_D[T(w)\mathbf{1}_X] = Z^{-1}_{\partial D|\partial B}\,\Delta_w^{\hat{\mathbb{C}}\setminus N(w)}\left[Z_{\Sigma(\partial D)|\Sigma(\partial B)}\mathbb{E}_{\hat{\mathbb{C}}\setminus\Sigma(\hat{\mathbb{C}}\setminus D)}(\mathbf{1}_{\Sigma(X)})\right]_{\Sigma=\mathrm{id}}$$

- I prove conformal covariance, $(g\cdot T)(g(w)) =$ the right thing.

- I prove a universality principle: local objects transforming like the stress-energy tensor are the stress-energy tensor.

Other works

with: N. Andrei, J. Cardy, O. Castro Alvaredo, P. Fonseca, A. Gamsa, J. Lepowsky, S. Lukyanov, A. Milas

- **Entanglement entropy in extended quantum systems:** exact universal asymptotics for any 1+1-dimentional massive quantum field theory, branch-point twist fields for massive orbifolds models (co-editor for a special issue of J. Phys. A with P. Calabrese and J. Cardy)
- **Quantum impurities in steady states out of equilibrium:** proof of large-time existence, new calculation techniques (lectures at the Capri spring school 2009)
- **Vertex operator algebras:** general construction results, twisted modules
- **Correlations functions of integrable QFT, zero and finite temperature:** expansions in form factors, conformal perturbation theory, invention of the concept of finite-temperature form factors
- **Massive free fermions on the Poincaré disk:** exact asymptotics of correlation functions, thermodynamics
- **Classical integrability of twist-field correlation functions in massive free-fermion models:** Painlevé equations and connection problem, rôle of finite-temperature form factors in classical inverse scattering problem (invitations by J. Harnad to the Centre de Recherche Mathématiques de Montréal for scientific discussions)