The entanglement entropy in integrable quantum field theory

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Based on:

J. Cardy, O.A. Castro Alvaredo, B.D., *J. Stat. Phys.* 130, 129 (2008)

O.A. Castro Alvaredo, B.D., *J. Phys. A* 41 275203 (2008)

B.D., *Phys. Rev. Lett.* 102 031602 (2009)

O.A. Castro Alvaredo, B.D., *J. Stat. Phys.* 134, 105 (2009)

See the review:

O.A. Castro Alvaredo, B.D., *J. Phys. A* 42 504006 (2009) in special issue “Entanglement entropy in extended quantum systems”, ed. by P. Calabrese, J. Cardy and B.D.

Entanglement in quantum mechanics

- Entanglement: the measurement of a quantum observable immediately affects future measurements of independent observables. Opposite-spin particles from pair production:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\Big)\,, \quad \langle A\rangle = \langle\psi|A|\psi\rangle$$

- Entanglement is the most **fundamental, non-classical** phenomenon of quantum mechanics: neither pure-wave nor pure-particle. It is a useful "**resource**": at the basis of better performances of the (still theoretical) quantum computers.

- Mixed states may describe similar probabilities but without entanglement:

$$\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|\,, \quad \langle A\rangle = \mathrm{Tr}(\rho A)$$

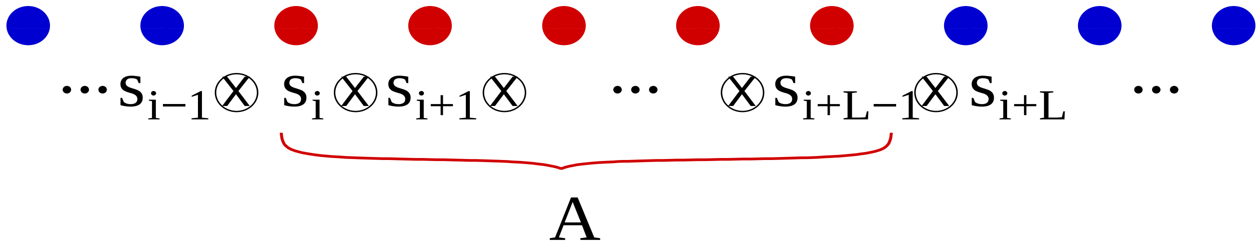
(for pure states, $\rho = |\psi\rangle\langle\psi|$; for finite temperature, $\rho = e^{-H/kT}$). For instance,

$$\rho = \frac{1}{2}\Big(|\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow|\Big)$$

How to measure (or quantify) quantum entanglement?

- There are various propositions for measures of quantum entanglement. Consider the **entanglement entropy**:
  - With the Hilbert space a tensor product $\mathcal{H} = s_1 \otimes s_2 \otimes \cdots \otimes s_N = A \otimes \bar{A}$, and a given state $|\mathrm{gs}\rangle \in \mathcal{H}$, calculate the **reduced density matrix**:

$$\rho_A = \mathrm{Tr}_{\bar{A}}(|\mathrm{gs}\rangle\langle\mathrm{gs}|)$$



  - The entanglement entropy is the resulting **von Neumann entropy**:

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) = -\sum_{\substack{\text{eigenvalues of } \rho_A \\ \lambda \neq 0}} \lambda \log(\lambda)$$

The entanglement entropy

- It is the entropy that is measured in a subsystem A, if its environement $\bar{A}$ is "forgotten". It measures a "number of links" between the subsystem and its environment; the quantity of additional information in the subsystem about its environment.
- It was proposed as a way to understand black hole entropy [Bombelli, Koul, Lee, Sorkin 1986].
- Then it was proposed as a measure of entanglement [Bennet, Bernstein, Popescu, Schumacher 1996].
- Examples:
  - Tensor product state: $|\mathrm{gs}\rangle = |A\rangle \otimes |\bar{A}\rangle \Rightarrow \rho_A = |A\rangle\langle A| \Rightarrow S_A = -1\log(1) = 0$.
  - The state $|\mathrm{gs}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$:
    $\rho_{1^{\mathrm{st}}\ \mathrm{spin}} = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|) \Rightarrow S_{1^{\mathrm{st}}\ \mathrm{spin}} = -2\times\left(\frac{1}{2}\log\left(\frac{1}{2}\right)\right) = \log(2)$

One basic property of entanglement entropy

Entanglement entropy is not "directional": $S_A = S_{\bar{A}}$. Proof:

- Anti-linear maps:
  $f : A \to \bar{A}$ with $f|A\rangle = \langle A|\mathrm{gs}\rangle$,
  $\bar{f} : \bar{A} \to A$ with $\bar{f}|\bar{A}\rangle = \langle \bar{A}|\mathrm{gs}\rangle$.
- Then $\rho_A = \bar{f} f : A \to A$ and $\rho_{\bar{A}} = f\bar{f} : \bar{A} \to \bar{A}$.
- If $\rho_A|A\rangle = \lambda|A\rangle$ then $\bar{f} f|A\rangle = \lambda|A\rangle$, hence $(f\bar{f})f|A\rangle = \lambda f|A\rangle$, whence $\rho_{\bar{A}} f|A\rangle = \lambda f|A\rangle$.
- Hence ρ_A and $\rho_{\bar{A}}$ have the same set of non-zero eigenvalues (with the same degeneracies).

Scaling limit

- Say $|\text{gs}\rangle$ is a ground state of some local spin-chain Hamiltonian, and that the chain is infinitely long.
- An important property of $|\text{gs}\rangle$ is the **correlation length** ξ:

$$\langle \text{gs}|\sigma_i \sigma_j|\text{gs}\rangle - \langle \text{gs}|\sigma_i|\text{gs}\rangle\langle \text{gs}|\sigma_j|\text{gs}\rangle \sim e^{-|i-j|/\xi} \text{ as } |i-j| \to \infty$$

- Suppose there are parameters in the Hamiltonian such that for certain values, $\xi \to \infty$. This is a **quantum critical point**.
- We may adjust these parameters in such a way that the length L of A stays in proportion to ξ: $L/\xi = mr$. This is the **scaling limit**.
- The resulting entanglement entropy **diverges** in that limit: $S_A \propto \log(\xi) + f(mr)$. But the differences $f(mr_1) - f(mr_2)$ are **universal**, and are described by **quantum field theory**. r is the dimensionful length of A; m is the smallest mass of the spectrum.

First universal quantity: short- and large-distance entanglement entropy

Choosing appropriately $\varepsilon \propto 1/(m\xi)$, a non-universal cutoff with dimenions of length:

- Short distance: $0 \ll L \ll \xi$, logarithmic behavior [Holzhey, Larsen, Wilczek 1994; Calabrese, Cardy 2004]

$$S_A \sim \frac{c}{3} \log\left(\frac{r}{\varepsilon}\right) = \frac{c}{3} \log(L) + const.$$

- Large distance: $0 \ll \xi \ll L$, saturation

$$S_A \sim -\frac{c}{3} \log(m\varepsilon) + U = \frac{c}{3} \log(\xi) + U + const.$$

where c is the central charge of the corresponding critical point. In terms of lattice quantities:

$$U = \lim_{x\to\infty} \left(S_A|_{L=\infty,\ \xi=x} - S_A|_{\xi=\infty,\ L=x} \right)$$

Partition functions on multi-sheeted Riemann surfaces

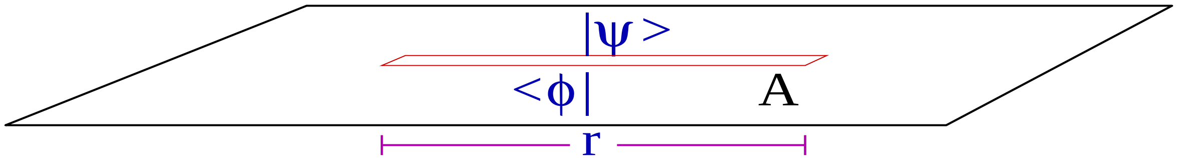
[Callan, Wilczek 1994; Holzhey, Larsen, Wilczek 1994]

- We can use the “replica trick” for evaluating the entanglement entropy:

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) = -\lim_{n\to 1}\frac{d}{dn}\mathrm{Tr}_A(\rho_A^n)$$

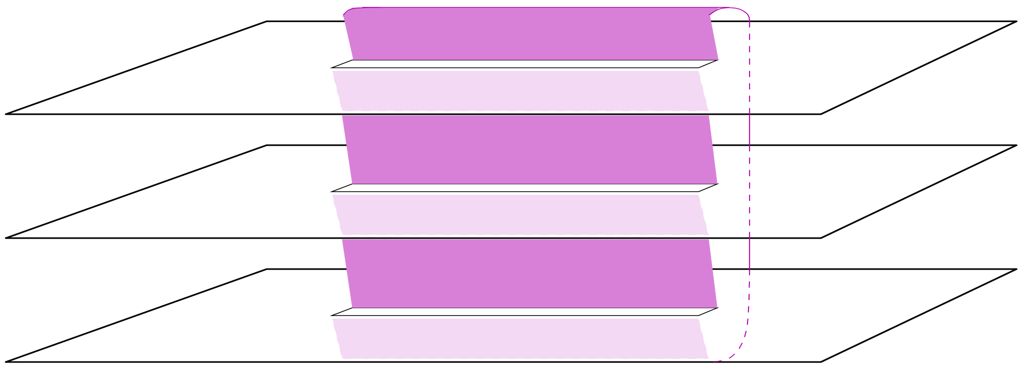
- For integer numbers n of replicas, in the scaling limit, this is a partition function on a Riemann surface:

$${}_A\langle\phi|\rho_A|\psi\rangle_A \quad \sim$$



$$\mathrm{Tr}_A(\rho_A^n) \quad \sim \quad Z_n = \int [d\varphi]_{\mathcal{M}_n} \exp\left[-\int_{\mathcal{M}_n} d^2x\, \mathcal{L}[\varphi](x)\right]$$

$$\mathcal{M}_n \quad :$$



Branch-point twist fields

[Cardy, Castro Alvaredo, Doyon 2007]

- Consider many copies of the QFT model on the usual $\mathbb{R}^2$:

$$\mathcal{L}^{(n)}[\varphi_1, \ldots, \varphi_n](x) = \mathcal{L}[\varphi_1](x) + \ldots + \mathcal{L}[\varphi_n](x)$$

- There is an obvious symmetry under cyclic exchange of the copies:

$$\mathcal{L}^{(n)}[\sigma\varphi_1, \ldots, \sigma\varphi_n] = \mathcal{L}^{(n)}[\varphi_1, \ldots, \varphi_n] \,, \quad \text{with} \quad \sigma\varphi_i = \varphi_{i+1 \bmod n}$$

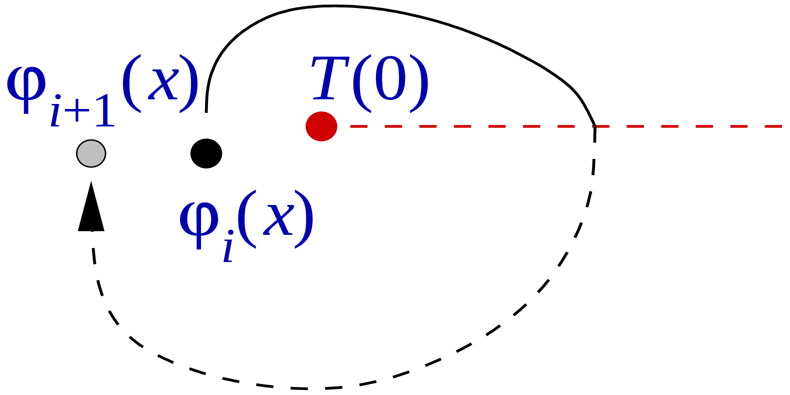
- The associated **twist fields** $\mathcal{T}$, when inside correlation functions, gives

$$\langle \mathcal{T}(0) \cdots \rangle_{\mathcal{L}^{(n)}} \propto \int_{C_0} [d\varphi_1 \cdots d\varphi_n]_{\mathbb{R}^2} \exp\left[- \int_{\mathbb{R}^2} d^2x \, \mathcal{L}^{(n)}[\varphi_1, \ldots, \varphi_n](x) \right] \cdots$$

with branching conditions on the line $\mathrm{x} \in (0, \infty)$ given by

$$C_0 \;:\; \varphi_i(\mathrm{x}, 0^+) = \varphi_{i+1}(\mathrm{x}, 0^-) \quad (\mathrm{x} > 0)$$

- Graphically:



- In operator terms: equal-time exchange relations,

$$\varphi_i(\mathrm{x})\mathcal{T}(0) = \begin{cases} \mathcal{T}(0)\varphi_i(\mathrm{x}) & (\mathrm{x}<0) \\ \mathcal{T}(0)\varphi_{i+1}(\mathrm{x}) & (\mathrm{x}>0) \end{cases}$$

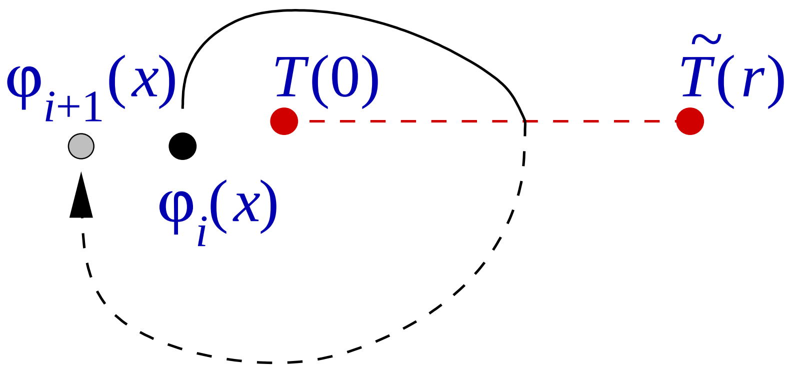
- Locality: commutation with Hamiltonian density $h(\mathrm{x})$,

$$[\mathcal{T}(0), h(\mathrm{x})] = 0 \quad (\mathrm{x} \neq 0)$$

- Another twist field $\tilde{\mathcal{T}}$ is associated to the inverse symmetry σ^{-1}, and we have

$$\begin{aligned}\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} &\propto \int_{C_{0,r}} [d\varphi_1 \cdots d\varphi_n]_{\mathbb{R}^2} \exp\left[-\int_{\mathbb{R}^2} d^2x\, \mathcal{L}^{(n)}[\varphi_1,\ldots,\varphi_n](x)\right] \\ &= Z_n\end{aligned}$$

$C_{0,r}$:



Short- and large-distance entanglement entropy revisited

Hence we have

$$Z_n/Z_1^n = D_n \varepsilon^{2d_n} \langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} \ , \quad S_A = -\lim_{n\to 1} \frac{d}{dn} Z_n$$

where D_n is a normalisation constant, and d_n is the scaling dimension of $\mathcal{T}$ [Calabrese, Cardy 2004]:

$$d_n = \frac{c}{12}\left(n - \frac{1}{n}\right)$$

- Short distance: $0 \ll L \ll \xi$, logarithmic behavior

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} \sim r^{-2d_n} \ \Rightarrow \ S_A \sim \frac{c}{3}\log\left(\frac{r}{\varepsilon}\right)$$

- Large distance: $0 \ll \xi \ll L$, saturation

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} \sim \langle \mathcal{T}\rangle^2_{\mathcal{L}^{(n)}} \ \Rightarrow \ U = -\lim_{n\to 1}\frac{d}{dn}\left(m^{-2d_n}\langle \mathcal{T}\rangle^2_{\mathcal{L}^{(n)}}\right)$$

Evaluation of U

$$U = -\lim_{n\to 1} \frac{d}{dn}\left(m^{-2d_n}\langle\mathcal{T}\rangle^2_{\mathcal{L}^{(n)}}\right)$$

- Idea of Al. Zamolodchikov (unpublished), for twist fields in general.

  In **angular quantization**, $\mathrm{x} + i\mathrm{y} = e^{\eta + i\theta}$, η the "space" and θ the "time":

  twist fields = unitary operator $\mathcal{U}_\sigma$ associated to transformation σ

$$\varphi_i(\eta)\mathcal{T} = \mathcal{T}\varphi_{i+1}(\eta) \quad \Rightarrow \quad \mathcal{T} \propto \mathcal{U}_\sigma$$

- $\mathcal{U}_\sigma$ can be diagonalized simultaneously with angular-quantization hamiltonian $K^{(n)}$:

$$\langle\mathcal{T}(0)\cdots\rangle_{\mathcal{L}^{(n)}} = \mathrm{Tr}_{\text{ang},\mathcal{L}^{(n)}}\left[e^{2\pi i K^{(n)}}\mathcal{U}_\sigma\cdots\right]$$

- Regularization necessary, performed explicitly in free-fermion models; Ising model

  [Cardy, Castro Alvaredo, Doyon 2007], [A. Zamoloschikov, Lukyanov 1997]:

$$U_{\text{Ising}} = \frac{1}{6}\log 2 - \int_0^\infty \frac{dt}{2t}\left(\frac{t\cosh t}{\sinh^3 t} - \frac{1}{\sinh^2 t} - \frac{e^{-2t}}{3}\right) = -0.131984\ldots$$

Second universal quantity: the next correction term

We found [Cardy, Castro Alvaredo, Doyon 2007], [Castro Alvaredo, Doyon 2008], [Doyon 2008]

$$S_A \sim -\frac{c}{3}\log(m_1\varepsilon) + U - \frac{1}{8}\sum_{\alpha=1}^{\ell} K_0(2rm_\alpha) + O\left(e^{-3rm_1}\right)$$

where ℓ is the number of particles in the spectrum of the QFT, and m_α are the masses of the particles, with $m_1 \leq m_\alpha \; \forall\alpha$.

- This next correction term depends only on the particle spectrum, but not on their interaction (i.e. not on the way they scatter off each other).
- In generic QFT, the largest mass is less than twice the smallest mass. Hence, the entanglement entropy provides "clean" information about "half" of the spectrum.

Form factors and two-point function

- In the n-replica model $\mathcal{L}^{(n)}$, there are n times as many particle types, which we will denote by $\mu = (\alpha, j)$ with $j = 1, \ldots, n$ the replica label.
- The two-point function of branch-point twist fields can be decomposed into the in-basis, giving a **large-distance expansion**:

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} = \langle \text{vac}|\mathcal{T}(0)\tilde{\mathcal{T}}(r)|\text{vac}\rangle =$$

$$\sum_{k=0}^{\infty} \sum_{\substack{\alpha_1,\ldots,\alpha_k \\ j_1,\ldots,j_k}} \int \frac{d\theta_1 \cdots d\theta_k}{(2\pi)^k} |F_{\mu_1,\ldots,\mu_k}(\theta_1, \ldots, \theta_k)|^2 e^{-r\sum_{i=1}^{k} m_{\alpha_i} \cosh\theta_i}$$

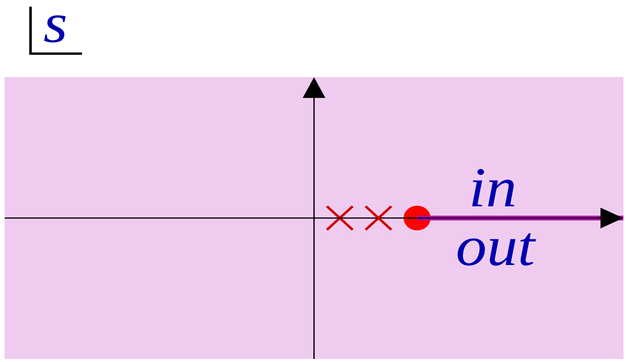
where the **form factors** are:

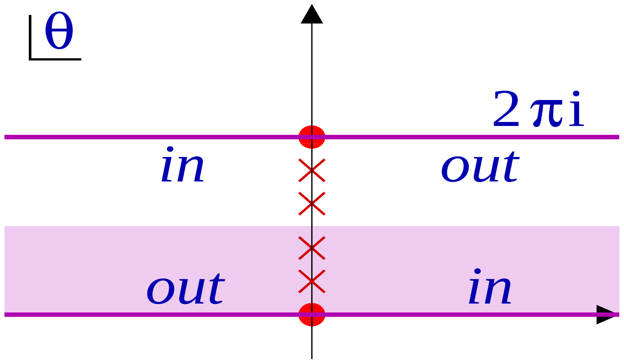
$$F_{\mu_1,\ldots,\mu_k}(\theta_1, \ldots, \theta_k) = \langle \text{vac}|\mathcal{T}(0)|\theta_1, \ldots, \theta_k\rangle^{in}_{\mu_1,\ldots,\mu_k}$$

Analytic properties of two-particle form factors

Consider $F_{\mu_1,\mu_2}(\theta_1,\theta_2) = F_{\mu_1,\mu_2}(\theta_1-\theta_2)$ (by relativistic invariance) as an analytic function of $\theta \equiv \theta_1-\theta_2$.

- Such form factors for **usual (non-twist) fields** have a well-known analytic structure: using Mandelstam's s-variable $s = m_{\alpha_1}^2 + m_{\alpha_2}^2 + 2m_{\alpha_1}m_{\alpha_2}\cosh(\theta)$, there is a branch cut from $s=(m_{\alpha_1}+m_{\alpha_2})^2$ to ∞, just above which we are describing the physical form factor with an in-state, and just below which it is the form factor with an out-state instead. Between 0 and $(m_{\alpha_1}+m_{\alpha_2})^2$, there may be poles due to bound states, and there are no other singularities on the physical sheet.



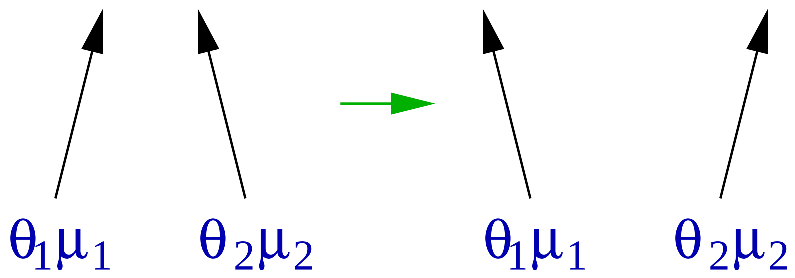


- Form factors for **branch-point twist-fields** have **modified analytic properties**.

Change of sign of θ (as usual)

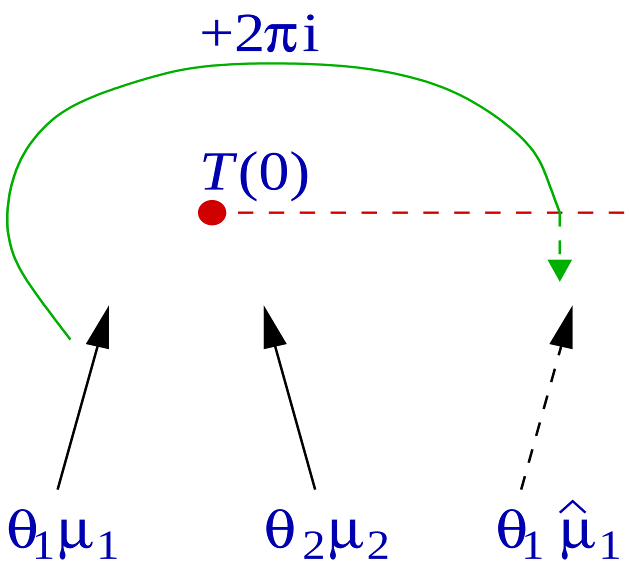
For $\theta_1 < \theta_2$:

$$
\begin{aligned}
F_{\mu_1,\mu_2}(\theta_1 - \theta_2) &= \langle \mathrm{vac}|\mathcal{T}(0)|\theta_1, \theta_2\rangle^{out}_{\mu_1,\mu_2} \\
&\overset{j_1 \neq j_2}{=} \langle \mathrm{vac}|\mathcal{T}(0)|\theta_2, \theta_1\rangle^{in}_{\mu_2,\mu_1} = F_{\mu_2,\mu_1}(\theta_2 - \theta_1)
\end{aligned}
$$



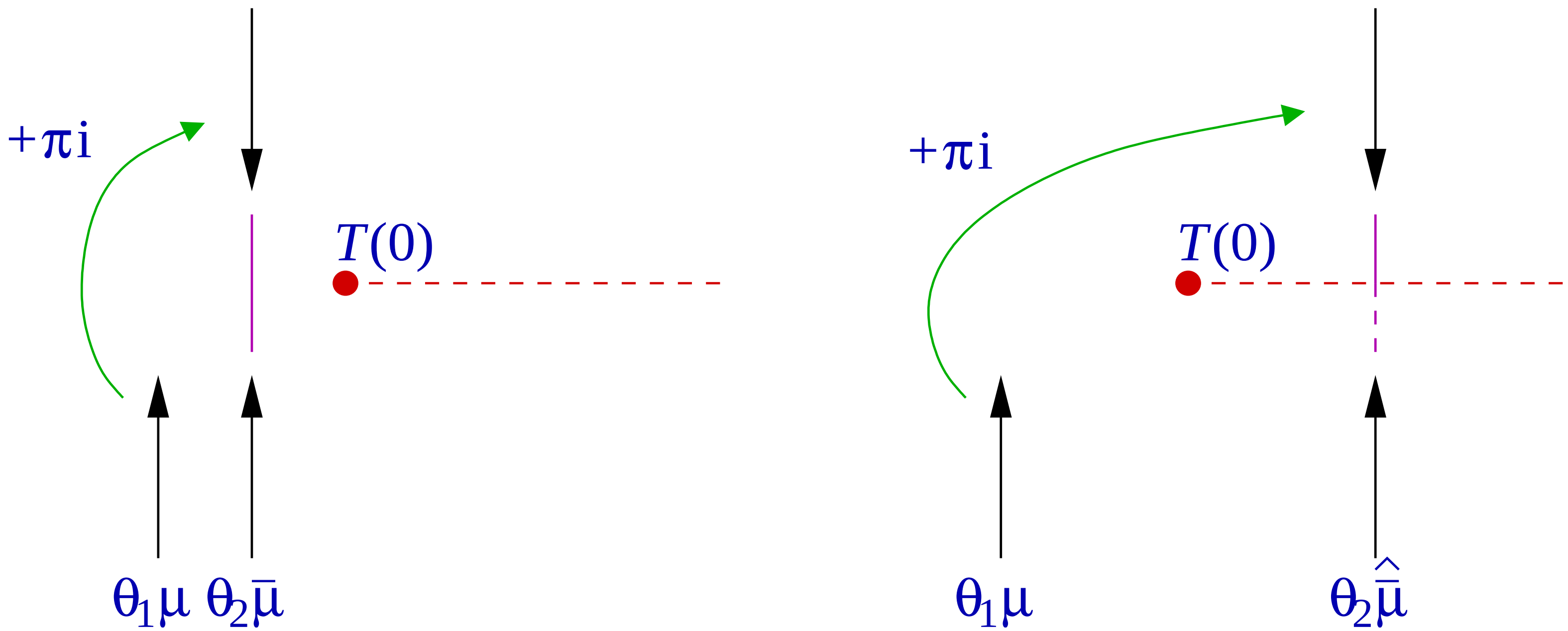
Quasi-periodicity relation (different)

$$F_{\mu_1,\mu_2}(\theta + 2\pi i) = F_{\mu_2,\mu_1}(-\theta)\ , \quad \mu = (\alpha, j + 1 \bmod n)$$



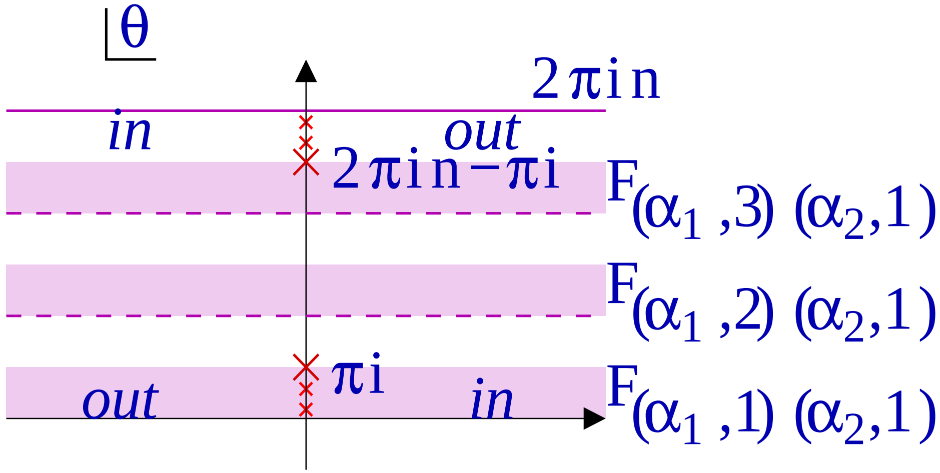
The kinematic residue equation (new)

$$-iF_{\mu_1,\mu_2}(\theta+\pi i) \sim \langle \mathcal{T} \rangle \frac{\delta_{\alpha_1,\bar{\alpha}_2}(\delta_{j_1,j_2}-\delta_{j_1+1,j_2})}{\theta}\ , \quad \bar{\alpha}_2 = \text{anti-particle of } \alpha_2$$



The structure of the two-particle form factors

Putting all that together, only $F_{(\alpha_1,1),(\alpha_2,1)}(\theta)$ matters, thanks to the relation $F_{(\alpha_1,j_1),(\alpha_2,j_2)}(\theta) = F_{(\alpha_1,1),(\alpha_2,1)}(\theta + 2\pi i(j_1 - j_2))$ for $0 \leq j_1 - j_2 \leq n-1$. It has the following analytic structure:



Correction term to the entanglement entropy

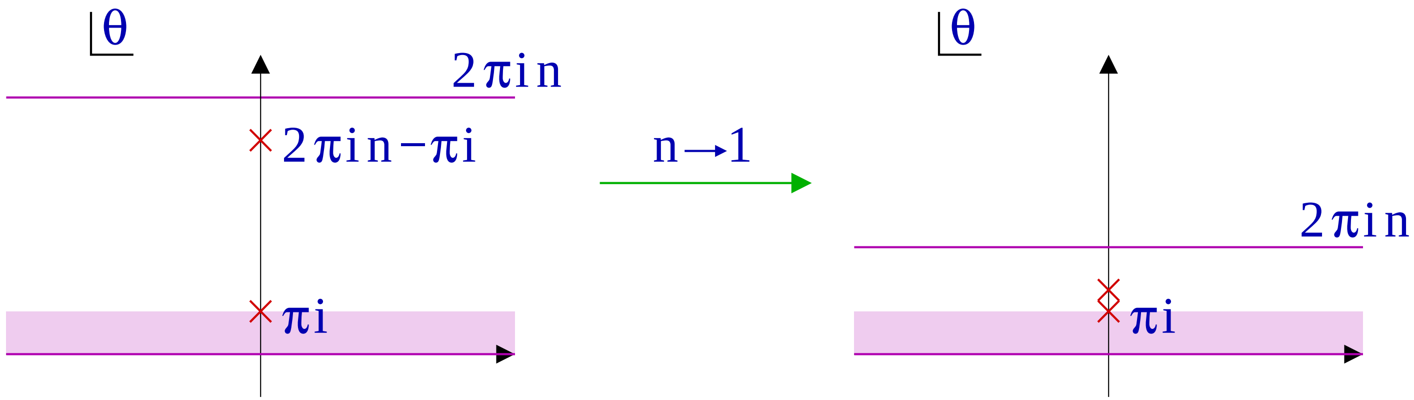
- The two-particle contribution to the entanglement entropy is

$$\frac{d}{dn}\left(\langle\mathcal{T}\rangle\frac{n}{8\pi^2}\sum_{\alpha,\beta=1}^{\ell}\int_{-\infty}^{\infty}d\theta_1 d\theta_2 f_{\alpha,\beta}(\theta_1-\theta_2,n)e^{-r(m_\alpha\cosh\theta_1+m_\beta\cosh\theta_2)}\right)_{n=1}$$

$$\langle\mathcal{T}\rangle f_{\alpha,\beta}(\theta,n)=\sum_{j=0}^{n-1}|F_{(\alpha,1),(\beta,1)}(\theta+2\pi ij)|^2$$

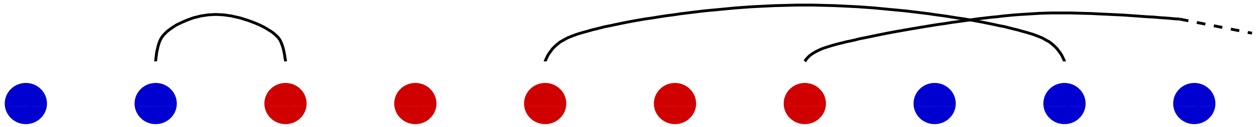
- The form factors themselves vanish like $n-1$ as $n\to 1$, because the branch-point twist field becomes the **identity field**.

- The only contribution to the entanglement entropy comes from the collision of kinematic poles at $\theta=0$, giving $\left(\frac{d}{dn}f_{\alpha,\beta}(\theta,n)\right)_{n=1}=\frac{\pi^2}{2}\delta(\theta)\delta_{\alpha,\bar{\beta}}$:



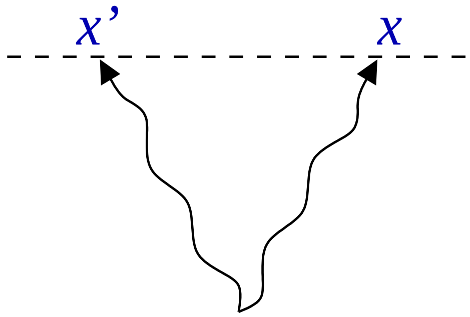
Heuristic: entanglement density and pair creations

- Entanglement entropy should "count" the connections between A and $\bar{A}$, for A of large enough extent:



$$S_A \sim \int_A dx \int_{\bar{A}} dx' s(x - x') \Rightarrow s(x) \sim -\frac{1}{2} \frac{d^2 S_{[0,x]}}{dx^2} \quad (1)$$

- The entanglement density $s(x - x')$ should receive contributions whenever the quantum fluctuation at x is somehow correlated with that at x'.
- At large distances $x - x' \gg m^{-1}$, the main contributions should be due to particles coming from a common virtual pair created far in the past.



- The particles have to survive a time t, and the probability for this is ruled by quantum uncertainty principles, $\propto e^{-Et}$, E the total energy, independently from the interaction.

General two-particle twist-fields form factors

Diagonal scattering without bound states, integral representation for scattering matrix:

$$S(\theta) = \exp\left[\int_0^\infty \frac{dt}{t} g(t) \sinh\left(\frac{t\theta}{i\pi}\right)\right]$$

The general "minimal" solution is

$$F_{j,k}^{\text{min}}(\theta) = \exp\left[\int_0^\infty \frac{dt}{t\sinh(nt)} g(t) \sin^2\left(\frac{itn}{2}\left(1 + \frac{i\theta - 2\pi(j-k))}{\pi}\right)\right)\right]$$

and the full solution is

$$F_{j,k}(\theta) = \frac{\langle \mathcal{T} \rangle \sin\left(\frac{\pi}{n}\right)}{2n \sinh\left(\frac{i\pi(2(j-k)-1)+\theta}{2n}\right) \sinh\left(\frac{i\pi(2(k-j)-1)-\theta}{2n}\right)} \frac{F_{j,k}^{\text{min}}(\theta, n)}{F_{j,k}^{\text{min}}(i\pi, n)}$$

How to evaluate higher-particle twist-fields form factors

- In **models of free fermionic particles**, form factors are given by determinants / pfaffians:

$$\mathcal{T} = :\exp\int d\theta d\theta' \left[a^\dagger(\theta)a^\dagger(\theta')F(\theta,\theta') + a^\dagger(\theta)a(\theta')G(\theta,\theta') + a(\theta)a(\theta')H(\theta,\theta')\right]:$$

- In **interacting integrable models**, one way is to use Lukyanov's angular-quantization method [Lukyanov, 1995],

$$\langle \text{vac}|\mathcal{T}(0)|\theta_1,\ldots,\theta_k\rangle^{in}_{1,\ldots,1} = \frac{\text{Tr}_{\text{ang},\mathcal{L}^{(n)}}\left[e^{2\pi i K^{(n)}}\mathcal{U}_\sigma\, Z_1(\theta_1)\cdots Z_1(\theta_n)\right]}{\text{Tr}_{\text{ang},\mathcal{L}^{(n)}}\left[e^{2\pi i n K^{(n)}}\right]} = \frac{\text{Tr}_{\text{ang},\mathcal{L}}\left[e^{2\pi i n K} Z(\theta_1)\cdots Z(\theta_n)\right]}{\text{Tr}_{\text{ang},\mathcal{L}}\left[e^{2\pi i n K}\right]}$$

Lukyanov observed that:

$K = \int d\nu k(\nu) b_\nu b_{-\nu}$ (bilinear in free bosons),

$Z(\theta) = \sum_j : e^{\int d\nu z_{\nu,j}(\theta) b_\nu}:$ (linear combination of vertex operators).

Calculations: $\langle Z(\theta)Z(\theta')\rangle_{\text{Tr}} = \exp\left[\int d\nu d\nu' z_\nu(\theta) z_{\nu'}(\theta')\langle b_\nu b_{\nu'}\rangle_{\text{Tr}}\right]$, etc.

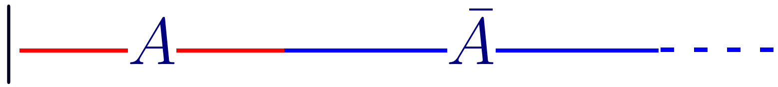
Large-n behaviour of form factors?

[Castro Alvaredo, Doyon 2008]

$\propto n$	for renormalizable models
$\propto n \log n$	for marginally renormalizable models

Third universal quantity: boundary entropy [Castro Alvaredo, Doyon 2008]

System: **half-line** composed of two **connected regions** A (finite) and B (infinite).



$$S_A^{\text{boundary}} \sim \begin{cases} \frac{c}{6}\log(2r/\varepsilon) + V & \varepsilon \ll r \ll m^{-1}, \text{boundary length scale if any} \\ -\frac{c}{6}\log(m\varepsilon) + \frac{U}{2} & r \gg m^{-1} \end{cases}$$

- We found

$$V = s - \log\sqrt{f}$$

where s is the **boundary entropy** of Affleck and Ludwig (1991) and f is the fraction of the massive ground state degeneracy that is broken by the boundary condition.

1. $V = S^{\text{boundary}}(r)_{\text{critical}} - \frac{1}{2}S^{\text{bulk}}(2r)_{\text{critical}} - \log\sqrt{f}$ from looking at $S^{\text{boundary}}(r_1, r_2)$
2. $S^{\text{boundary}}(r)_{\text{critical}} - \frac{1}{2}S^{\text{bulk}}(2r)_{\text{critical}} = s$ [Calabrese, Cardy 2004].

- Consequence:

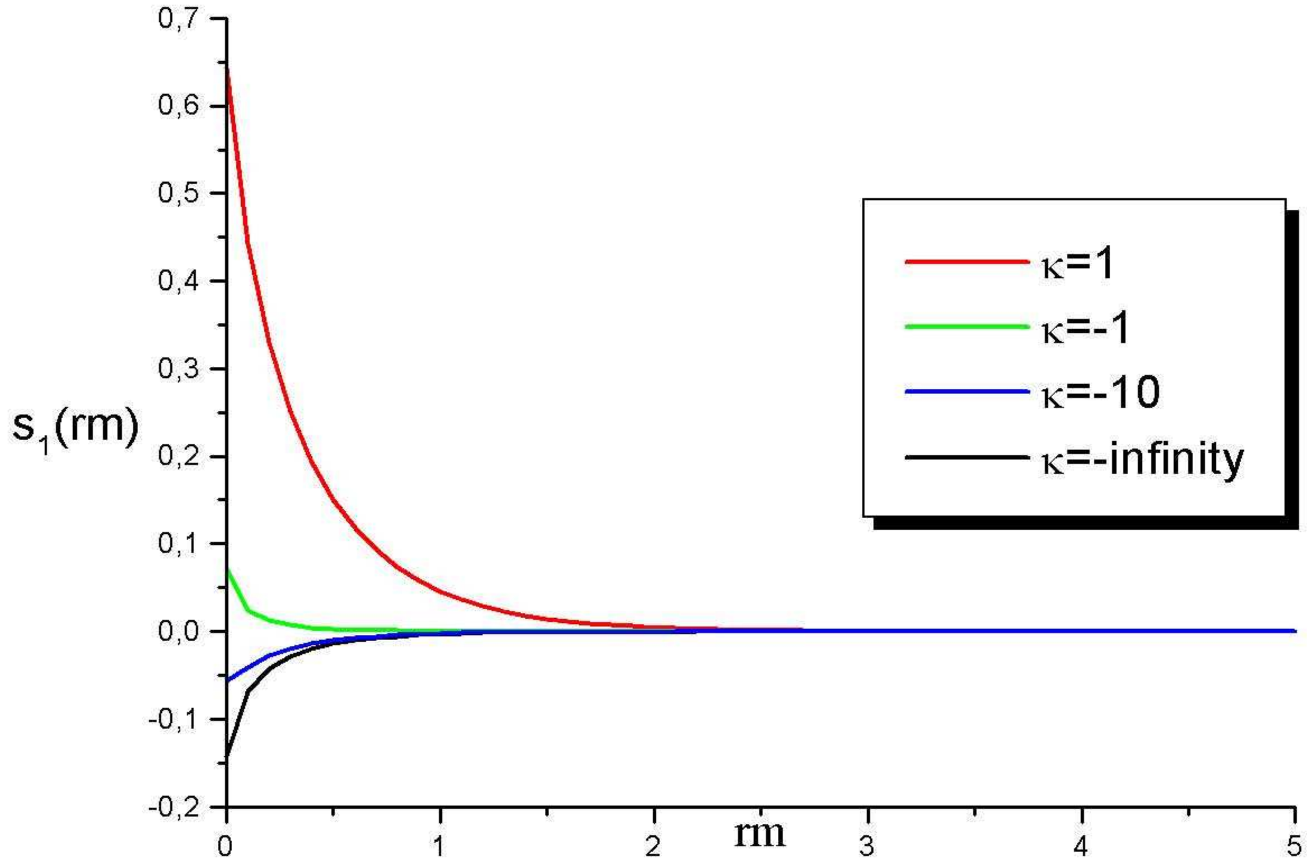
$$\lim_{x\to\infty}\left(S_A|_{L=\infty,\xi=x} - S_A|_{\xi=\infty,L=x/2}\right) = U/2 + \log\sqrt{f} - s.$$

Ising model checks

- Consider Ising quantum chain in transverse magnetic field near to its critical point in the longitudinally-ordered phase, with boundary magnetic field h coupled longitudinally. Use $\kappa = 1 - h^2/(2m)$. Integrable boundary state [Goshal, Zamolodchikov 1994].
- Exact form-factor expression for $V(\kappa)$; 500 terms re-summation of form factors agrees with $1/6 \log(rm) + V(\kappa)$ where

$$V(\kappa) = \begin{cases} \sqrt{2} & \kappa > -\infty \quad \text{(free)} \\ 0 & \kappa = -\infty \quad \text{(fixed)} \end{cases}$$

  This is $V(\kappa) = s - \log \sqrt{f}$ with $f = 1/2$.
- As $n \to 1$, only fully connected terms remain. Analytic continuation from region $n \gg 1$.
- $mr \to 0$ and $\kappa \to -\infty$ simultan.: critical bulk and non-critical boundary condition.
- For $\kappa > -1$ ("critical" value [Goshal, Zamolodchikov 1994]), entropy **not monotonic in rm**: approaches asymptotic value from above. Breaks "subadditivity".



0,7
0,6
0,5
0,4
0,3
0,2
0,1
0,0
-0,1
-0,2
s1(rm)
0
1
2
3
4
5
rm
κ=1
κ=-1
κ=-10
κ=-infinity

Conclusions

- We have shown how three universal quantities associated to the entanglement entropy of one-dimensional quantum chains can be accessed using the methods of massive integrable QFT:
  - the difference between $L \gg \xi \gg 0$ and $\xi \gg L \gg 0$ (the universal constant U),
  - the first correction to saturation at $L \gg \xi \gg 0$ (in terms of the mass spectrum),
  - the difference between $L \gg \xi \gg 0$ and $\xi \gg L \gg 0$ in boundary case (in terms of Affleck and Ludwig's boundary entropy).

  All these relations are valid beyond integrability, in any near-critical quantum chain (i.e. two-dimensional QFT).

- Open problems in massive integrable QFT: other universal corrections to saturation from higher-particle form factors; the entanglement entropy for A a disconnected region from multi-point correlation functions; the entanglement entropy for excited states; etc...