Integrability - solutions to some problems

Benjamin Doyon
King's College London

Fall 2012
London Taught Centre

Exercise 2.1

Let $f \in \mathcal{F}$. Then $[X_F, X_G](f) = X_F(X_G(f)) - X_G(X_F(f)) = \{F, \{G, f\}\} - \{G, \{F, f\}\} = \{\{F, G\}, f\} = X_{\{F,G\}}(f)$. Since this is true for every f, this completes the proof.

Exercise 2.2

Consider

$$dF_i = \sum_j c_{i,j} dp_j + d_{i,j} dq_j = \sum_J C_{iJ} dx_J$$

where we use x_J to represent both p_j and q_j, and $j = 1, \ldots, n$ and $J = 1, \ldots, 2n$. Suppose all dF_i are independent for $i = 1, \ldots, m$ with $m > n$. This means that there is no coefficients a_i such that $\sum_i a_i dF_i = 0$, i.e. such that $\sum_i a_i C_{iJ} = 0$ for all J. We may think of C_{iJ} as m vectors $\vec{C}_i$ with each $2n$ components, and the condition is that these vectors are linearly independent.

Let us denote by ω^{IJ} what gives rise to the Poisson bracket:

$$\{f, g\} = \sum_{IJ} \omega^{IJ} \partial_I f \partial_J g.$$

Then by the involution property $\{F_i, F_j\} = 0$ we have

$$\sum_{IJ} \omega^{IJ} C_{iI} C_{jJ} = 0$$

whence

$$\sum_J C_{jJ} A_i^J = 0, \quad A_i^J = \sum_I \omega^{IJ} C_{iI}.$$

This means that there are m $2n$-dimensional vectors $\vec{A}_i$ such that $\vec{C}_j \cdot \vec{A}_i = 0$ for all j. If all vectors $\vec{A}_i$ are linearly independent, this means that every $2n$-dimensional vector $\vec{C}_j$ must lie in the same $2n - m$-dimensional space. For $m > n$, then the m vectors $\vec{C}_j$ cannot be all independent, which is a contradiction.

Hence the vectors $\vec{A}_i$ cannot be linearly independent. That is, there exist q_i such that $\sum_i q_i \vec{A}_i = 0$. This means

$$\sum_{iI} q_i \omega^{IJ} C_{iI} = 0$$

for all J. That is, $\Omega \vec{v} = 0$ where Ω is the matrix with elements $\Omega_{JI} = \omega^{IJ}$ and $\vec{v} = \sum_i q_i \vec{C}_i$. Since the $\vec{C}_i$ are linearly independent, then $\vec{v} \neq 0$, whence we have found a nonzero eigenvector of Ω with zero eigenvalue. But by explicit calculation, Ω has nonzero determinant, hence all its eigenvalues are nonzero (the Poisson bracket is nondegenerate). Hence this is a contradiction, so that we cannot have $m > n$.

Exercise 2.3

We first inverse in order to find $p_i(F,q)$:

$$p_i(F,q) = \sqrt{F_i - \omega^2 q_i^2}. \tag{0.1}$$

The we integrate to find $S(F,q)$:

$$S(F,q) = \int_0^q \sum_i \sqrt{F_i - \omega^2 q_i^2}\, dq_i \tag{0.2}$$

This is done by the change of variable $q_i = \frac{\sqrt{F_i}}{\omega}\sin\beta_i$ and we obtain

$$S(F,q) = \sum_i \frac{F_i}{\omega}\int d\beta_i \cos^2\beta_i = \sum_i \frac{F_i}{\omega}\left(\frac{\beta_i}{2} + \frac{\sin 2\beta_i}{4}\right) \tag{0.3}$$

so that

$$\Psi_i = \frac{1}{\omega}\left(\frac{\beta_i}{2} + \frac{\sin 2\beta_i}{4}\right) \tag{0.4}$$

In order to calculate the action variables I_j, we integrate over a cycle. We see that we have $q_i = \frac{\sqrt{F_i}}{\omega}\sin\beta_i$ and $p_i = \sqrt{F_i}\cos\beta_i$ so that the β_j describe angles round cycles. Hence an integration over a cycle C_j is an integration on β_j from 0 to 2π. Integrating:

$$I_j = \frac{1}{2\pi}\frac{F_j}{\omega}\int_0^{2\pi} d\beta_j \cos^2\beta_j = \frac{F_j}{2\omega}. \tag{0.5}$$

Then, we have the angle variables

$$\theta_j = 2\omega\Psi_j = \beta_j + \frac{\sin 2\beta_j}{2}.$$

We see that the angles β_j describing the elliptic trajectories are related in a monotonic fashion to the angle variables θ_j.

Exercise 2.4

We calculate

$$\dot{L} = \dot{U}\Lambda U^{-1} + U\dot{\Lambda}U^{-1} - U\Lambda U^{-1}\dot{U}U^{-1} \tag{0.6}$$

and

$$[M,L] = ML - LM = UB\Lambda U^{-1} + \dot{U}\Lambda U^{-1} - U\Lambda BU^{-1} - U\Lambda U^{-1}\dot{U}U^{-1} \tag{0.7}$$

and equating we find what we had to prove.

Exercise 2.5

We have

$$[M,L] = \begin{pmatrix} -\omega^2 q & \omega p \\ \omega p & \omega^2 q \end{pmatrix} \tag{0.8}$$

Equating with $\dot{L}$ we find the correct equations of motion. On the other hand, we see that

$$\frac{1}{4}\mathrm{Tr}L^2 = \frac{1}{2}(p^2 + \omega^2 q^2) \tag{0.9}$$

which is the correct Hamiltonian.

Exercise 2.6

This is done in [1, p 15]

Exercise 2.7

Here, L of [1, p 13] can be used to get coordinates on the invariant submanifold, because in its expansion in the independent algebra elements H_j and E_j we see that the coefficients are I_j and $2I_j\theta_j$, which form a generically nonsingular system of coordinates (and in fact, keeping I_j as constants, this is essentially the system of coordinates given by the angles of the action-angle variables). So the argument presented suggest that the covariant derivatives

$$D_j = \frac{d}{dt_j} - \text{ad}(M_j) \tag{0.10}$$

are commuting. Indeed we see that $\text{ad}(M_j)$ is simply

$$\text{ad}(M_j) = -\sum_{j=1}^{n} \frac{\partial H}{\partial I_j} \text{ad}(E_j) \tag{0.11}$$

which are commuting thanks to $[E_j, E_k] = 0$. Hence, we indeed have a principal bundle over the invariant submanifold characterized by constant I_j, with commuting covariant derivatives $D_j = \frac{d}{dt_j} + \sum_{j=1}^{n} \frac{\partial H}{\partial I_j} \text{ad}(E_j)$, such that the consistent system of equations $D_j L = 0$, where $L = \sum_{j=1}^{n}(I_j H_j + 2I_j\theta_j E_j)$, gives rise to the equations of motion associated to the flows X_{I_j} of the various action variables I_j.

Exercise 3.1

This is just a matter of doing the calculations explicitly. For convenience we write

$$\begin{aligned} U &= \frac{i}{4}\left(\partial_t\phi\,\sigma_z + 2m\sinh u\cos\frac{\phi}{2}\,\sigma_x - 2m\cosh u\sin\frac{\phi}{2}\,\sigma_y\right) \\ V &= \frac{i}{4}\left(\partial_x\phi\,\sigma_z - 2m\cosh u\cos\frac{\phi}{2}\,\sigma_x + 2m\sinh u\sin\frac{\phi}{2}\,\sigma_y\right). \end{aligned} \tag{0.12}$$

Then

$$\begin{aligned} 8i[U,V] &= \left(-\partial_t\phi\,2m\,\cosh u\cos\frac{\phi}{2} - \partial_x\phi\,2m\sinh u\cos\frac{\phi}{2}\right)\sigma_y \\ &\quad + \left(-\partial_t\phi\,2m\,\sinh u\sin\frac{\phi}{2} - \partial_x\phi\,2m\cosh u\sin\frac{\phi}{2}\right)\sigma_x \\ &\quad + \left(-2m^2\sin\phi\right)\sigma_z \\ -4i(\partial_t U - \partial_x V) &= (\partial_t^2 - \partial_x^2)\phi\,\sigma_z \\ &\quad + \left(-\partial_t\phi\,m\,\sinh u\sin\frac{\phi}{2} - \partial_x\phi\,m\cosh u\sin\frac{\phi}{2}\right)\sigma_x \\ &\quad + \left(-\partial_t\phi\,m\,\cosh u\cos\frac{\phi}{2} - \partial_x\phi\,m\sinh u\cos\frac{\phi}{2}\right)\sigma_y \end{aligned} \tag{0.13}$$

so that $\partial_t U - \partial_x V + [U,V] = 0$ is exactly equivalent to the equations of motion.

Exercise 3.2

Exercise 3.3

(assessment question)

Exercise 3.4

Exercise 3.5

Exercise 4.1

Exercise 4.2

We write

$$\begin{aligned}
&R_{a_1,a_2}(\lambda-\mu)T_{a_1}(\lambda)T_{a_2}(\mu)\\
&= R_{a_1,a_2}(\lambda-\mu)L_{N,a_1}(\lambda)\cdots L_{1,a_1}(\lambda)\; L_{N,a_2}(\mu)\cdots L_{1,a_2}(\mu)\\
&= R_{a_1,a_2}(\lambda-\mu)L_{N,a_1}(\lambda)L_{N,a_2}(\mu)\, L_{N-1,a_1}(\lambda)\cdots L_{1,a_1}(\lambda)\; L_{N-1,a_2}(\mu)\cdots L_{1,a_2}(\mu)\\
&= \cdots\\
&= R_{a_1,a_2}(\lambda-\mu)L_{N,a_1}(\lambda)L_{N,a_2}(\mu)\cdots L_{1,a_1}(\lambda)L_{1,a_2}(\mu)\\
&= L_{N,a_2}(\mu)L_{N,a_1}(\lambda)R_{a_1,a_2}(\lambda-\mu)\cdots L_{1,a_1}(\lambda)L_{1,a_2}(\mu)\\
&= \cdots\\
&= L_{N,a_2}(\mu)L_{N,a_1}(\lambda)\cdots L_{1,a_2}(\mu)L_{1,a_1}(\lambda)R_{a_1,a_2}(\lambda-\mu)\\
&= T_{a_2}(\mu)T_{a_1}(\lambda)R_{a_1,a_2}(\lambda-\mu) \qquad (0.14)
\end{aligned}$$

Exercise 4.3

We use the fact that $L_{n,a}(i/2) = iP_{n,a}$. Hence,

$$\begin{aligned}
F(i/2) &= \mathrm{Tr}_a(T_a(i/2))\\
&= \mathrm{Tr}_a(L_{N,a}(i/2)\cdots L_{1,a}(i/2))\\
&= i^N\mathrm{Tr}_a(P_{N,a}\cdots P_{1,a})\\
&= i^N U. \qquad (0.15)
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{d}{d\lambda}F(\lambda)\Big|_{\lambda=i/2} &= \mathrm{Tr}_a\left(\frac{d}{d\lambda}L_{N,a}(\lambda)\cdots L_{1,a}(\lambda)\Big|_{\lambda=i/2}\right)\\
&= i^{N-1}\sum_{j=1}^{N}\mathrm{Tr}_a\left(P_{N,a}\cdots\left(\frac{d}{d\lambda}L_{j,a}(\lambda)\Big|_{\lambda=i/2}\right)\cdots P_{1,a}\right)\\
&= i^{N-1}\sum_{j=1}^{N}\mathrm{Tr}_a\left(P_{N,a}\cdots\mathbf{1}_{j,a}\cdots P_{1,a}\right)\\
&= i^{N-1}\sum_{j=1}^{N}\mathrm{Tr}_a\left(P_{N,a}\cdots\widehat{P_{j,a}}\cdots P_{1,a}\right)\\
&=: i^{N-1}\sum_{j=1}^{N}U_j \qquad (0.16)
\end{aligned}$$

where (as usual) the wide hat means that the factor is missing. Then, we observe that

$$\begin{aligned} U_j &= \mathrm{Tr}_a\left(P_{N,a}\cdots\widehat{P_{j,a}}\cdots P_{1,a}\right) \\ &= \mathrm{Tr}_a\left(P_{j,j+1}P_{j,j+1}P_{N,a}\cdots P_{j+1,a}P_{j-1,a}\cdots P_{1,a}\right) \\ &= \mathrm{Tr}_a\left(P_{j,j+1}P_{N,a}\cdots P_{j,j+1}P_{j+1,a}P_{j-1,a}\cdots P_{1,a}\right) \\ &= \mathrm{Tr}_a\left(P_{j,j+1}P_{N,a}\cdots P_{j+1,a}P_{j,a}P_{j-1,a}\cdots P_{1,a}\right) \\ &= P_{j,j+1}U. \end{aligned} \tag{0.17}$$

Hence,

$$\begin{aligned} \left.\frac{dF(\lambda)}{d\lambda}F(\lambda)^{-1}\right|_{\lambda=i/2} &= i^{-1}\sum_{j=1}^{N}U_jU^{-1} \\ &= i^{-1}\sum_{j=1}^{N}P_{j,j+1}UU^{-1} \\ &= i^{-1}\sum_{j=1}^{N}P_{j,j+1}. \end{aligned} \tag{0.18}$$

This is indeed a local quantity (i.e. a sum over a local density). Then, we simply have to use the expression

$$P_{j,j+1} = \frac{1}{2}\left(1+\vec{\sigma}_j\cdot\vec{\sigma}_{j+1}\right) \tag{0.19}$$

to obtain $Q_1 = i^{-1}(N+2H)$.

References

[1] "Introduction to classical integrable systems", O. Babelon, D. Bernard and M. Talon, Cambridge University Press, 2003

[2] "How algebraic Bethe ansatz works for integrable models", L.D. Faddeev, published in Les Houches 1995, Relativistic gravitation and gravitational radiation pp. 149-219, hep-th/9605187, 1996

[3] "Algebraic analysis of solvable lattice models", M. Jimbo and T. Miwa, Conference Board of the Mathematical Sciences 85, American Mathematical Society ,1993

[4] "On the quantum inverse scattering problem", J. M. Maillet and V. Terras, Nucl. Phys. B575, 627-644, hep-th/9911030, 2000

[5] "Quantum inverse scattering method and correlation functions", V.E. Korepin, N.M. Bogoliubov, A.G. Izergin, Cambridge University Press, 1993

[6] "Form factors in completely integrable models of quantum field theory", F. A. Smirnov, World Scientific, 1992