Integrability

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Contents

1 Introduction

Integrability is a wide subject that comprises many deep ideas and that can be applied to very diverse physical systems. This course will give an overview of some of the ideas behind integrability, and of the way they are developed in applications to four areas of theoretical physics: finite dynamical systems, classical field theory, quantum chains, and quantum field theory. In each case, slightly different techniques are involved, but one of the goal of this course is to emphasise the conceptual similarity between them. In particular, I will try to show how the basic concepts in integrable dynamical systems, such as action-angle variables, conserved quantities, and the simplification of the set of phase space trajectories that ensues, are realised in other contexts. I will develop simple, standard examples in order to illustrate the concepts. Naturally, the course will be divided into four parts, corresponding to the four areas of theoretical physics.

Since the course is only 10 hours long, and since I'd like to be able to emphasise the concepts and their similarities in various areas of applications of integrability, instead of the technicalities, it will be helpful if the students can do some reading before each lecture (the appropriate reading will be suggested in due time). It will also be helpful if the students can make sure they are somewhat comfortable with the important concepts in classical Hamiltonian mechanics, complex analysis, quantum mechanics and quantum field theory before the course starts.

It is important to note that the choice of topics and the way they are presented here reflect mainly the strong time constraints and my personal take on the subject of integrability, rather than "the correct" overview of integrability.

Physical systems and integrability

In physics, what is usually understood as a *system* is a topological space of states M (usually a manifold, although often infinite-dimensional), and an evolution map, a bijective map $U_t : M \to M$ parametrised by the time $t \in \mathbb{R}$. Every state is a mathematical object which encodes observable predictions, and the evolution map determines how the predictions change in time. In various physical applications, the space of states may be very different, hence also the evolution equation may look different. In classical dynamical systems, the space of states is the phase space: a symplectic manifold (hence with a non-degenerate Poisson structure). In classical field theory, it is a space of functions (which we will mainly take to be on $\mathbb{R}$). In quantum mechanics and in quantum field theory, it is a Hilbert space (but in both cases with somewhat different structure).

In most, if not all, physical cases, the evolution equation is such that time evolution of any given state is continuous: the map $t \mapsto U_t(p)$, for any given $p \in M$, is a continuous map (at least in a neighbourhood of $t = 0$). However, the map U_t from M to M, for fixed time t, is in general very complicated. In particular, although it may almost everywhere continuous, this continuity is almost nowhere uniform in time. For instance, in many cases, two nearby states map to states that are very far apart, and that become exponentially further apart as t is increased - this is chaos. Integrability is essentially the opposite of chaos: the map U_t is as nice as it can be. It possesses infinitely many invariant submanifolds that foliate M, parametrised by as many continuous parameters as there are "degrees of freedom", and on these submanifolds, states that start nearby stay nearby uniformly in time. This is what is explicitly seen in classical dynamical systems, and it seems to form a common underlying principle at the basis of integrability in general. Let us see how this works in the four physical situations that I talked about.

Acknowledgment. As the course goes along, I will / did have good questions and comments from the students, which undoubtedly will improve these notes. Hence I am grateful in advance to all attending students for their contributions.

2 Dynamical systems

For this part, I follow Chapter 2 of [1] (but many other references exist).

2.1 Liouville intetrability

Consider a classical system. First, this is a symplectic $2n$-dimensional manifold, and for simplicity we will take it to be described by the usual coordinates $p_i \in \mathbb{R}$, $q_i \in \mathbb{R} : i = 1, \dots, n$ with canonical Poisson brackets

$$\{p_i, q_j\} = \delta_{i,j}.$$

Let us denote by $\mathcal{F}$ the real linear space of smooth real functions on our symplectic manifold, which of course also has the structure of a ring. We have, as usual, for any functions $F, G \in \mathcal{F}$,

$$\{F, G\} = \sum_i \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \right).$$

This Poisson bracket is antisymmetric, bilinear, and satisfies the Jacobi identity. Hence, this gives on $\mathcal{F}$ the structure of a Lie algebra. Further, thanks to the Poisson structure, any function on phase space

$F \in \mathcal{F}$ has an associated vector field which we will denote X_F:

$$X_F = \{F, \cdot\} = \sum_i \left(\frac{\partial F}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial}{\partial p_i} \right). \tag{2.1}$$

In particular, the map $F \mapsto X_F$ is a homomorphism of Lie algebras,

$$[X_F, X_G] = X_{\{F,G\}}. \tag{2.2}$$

Exercise 2.1 *Show (2.2).*

Second, the classical system is endowed with equations of motion. These are equations representing the evolution along the vector field associated to a particular function on phase space, the Hamiltonian $H = H(p, q)$. That is, the equations of motion are the statement that for any function F on phase space[1],

$$\dot{F} = X_H(F) = \{H, F\} \tag{2.3}$$

(where the dot means time derivative d/dt). Flows induced by vector fields associated to functions on phase space like this are usually referred to as *Hamiltonian flows*.

The definition of Liouville integrability is as follows.

Definition 2.1 *A dynamical system (of $2n$-dimensional phase space) is Liouville integrable is there exists n independent conserved quantity in involution.*

This definition requires explanation for the various terms involved. A *conserved quantity* is a function $F \in \mathcal{F}$ that is invariant under time evolution,

$$\{H, F\} = 0.$$

We require there to be n of them, F_i, $i = 1, \ldots, n$. These are required to be *independent*. This means that at generic points on the symplectic manifold, the tangent space of the surface defined by $F_i = f_i = const.\ \forall\ i$ exists and is n-dimensional (hence, these relations give rise, locally, to a n-dimensional submanifold); or equivalently, that the total variations dF_i are linearly independent. Finally, the conserved quantities are in *involution* if

$$\{F_i, F_j\} = 0 \quad \forall \quad i, j.$$

An important remark is that *there cannot be more than n independent conserved quantities in involution.*

Exercise 2.2 *Prove the remark just made.*

This remark implies that the Hamiltonian itself, which is obviously conserved and in involution with all of F_i, must be a function of the F_i (hence it can always be taken as one of them). In fact, a Liouville integrable system is not really characterized by its Hamiltonian, but rather by its set of quantities in involution (modulo non-singular transformations amongst them); the choice of a Hamiltonian as a function of these is essentially irrelevant.

[1] We consider here only dynamical systems with explicitly time-independent Hamiltonians.

Before we go to the Liouville theorem, let us discuss already some simple geometric consequences of Liouville integrability. Let us assume that at every point (not just "generically"), dF_i are linearly independent. Then we have an n-dimensional submanifold $\mathtt{M}_f$ defined by fixing the conserved quantities F_i to constants, $\mathtt{M}_f = \{(p,q) : F_i = f_i\}$. Since all conserved quantities are in involution, we can define a map $m_t : \mathtt{M}_f \to \mathtt{M}_f,\ (p,q) \mapsto (p(t), q(t))$ for every $t \in \mathbb{R}^n$ by solving

$$\frac{d}{dt_1} \cdots \frac{d}{dt_n} G = \{F_1, \{\cdots, \{F_n, G\} \cdots\}\}$$

(for any $G \in \mathcal{F}$), and this gives rise to a transitive and locally free action of the abelian group $\mathbb{R}^n$ on $\mathtt{M}_f$,

$$m_t \circ m_{t'} = m_{t+t'}.$$

Let us assume further that $\mathtt{M}_f$ is compact and connected. This means that for any point on the manifold, there will be a discrete subgroup of $\mathbb{R}^n$ that fixes it, which must be isomorphic to $\mathbb{Z}^n$ (because there will be n independent directions in $\mathbb{R}^n$ in which the point is fixed). Hence, the manifold has the quotient structure $\mathbb{R}^n/\mathbb{Z}^n$, which is topologically (and from the viewpoint of the differentiable manifold structure) a (multi-dimensional) torus. Another way of saying the same is that, basically, since the vector fields X_{F_i} commute,

$$[X_{F_i}, X_{F_j}] = X_{\{F_i, F_j\}} = 0,$$

then there exists a connection on the manifold that makes it flat, hence it must be a torus. We'll say a word on flat connections at the end of this section.

Remark 2.1 *Clearly, any trajectory under time evolution produced by the Hamiltonian H lies on one such manifold $\mathtt{M}_f$ (which one depends on the initial conditions). Hence, we have found that the trajectories, in Liouville integrable systems and under certain global conditions, lie on very simple submanifolds. These trajectories cannot be very complex; this is at the basis of the connection between Liouville integrability and "solvability". In fact, we will see, in particular vie the action-angle variables, that trajectories do not exponentially diverge from each other in integrable systems, contrary to chaotic systems.*

Theorem 2.1 (the Liouville theorem) *The solution of the equations of motion of a Liouville integrable system is obtained by "quadrature".*

By quadrature means that we only have to solve some algebraic equations and do some integrals in order to obtain the solution. That this be the case is certainly not immediate from the definition of a dynamical system in general.

Proof. The main argument of the proof is to find a canonical transformation to coordinates (F, Ψ):

$$\{F_i, \Psi_j\} = \delta_{i,j}. \tag{2.4}$$

This means that we can solve by quadrature:

$$\dot{F}_i = \{H, F_i\} = 0, \quad \dot{\Psi}_i = \{H, \Psi_i\} = \frac{\partial H}{\partial F_i} = \Omega_i(F)$$

the latter one because H is only a function of the F_i. This is trivial to solve.

The nontrivial aspect of the quadrature is involved in finding Ψ_j. We have three set of n variables: the F_i, the p_i and the q_i, and one set of n relations amongst them (the expressions of the F_i on phase

space). We will play with variations keeping various sets constant, in a fashion similar to what is done in thermodynamics.

We will assume in the following that the matrix with value

$$\left.\frac{\partial F_k}{\partial p_j}\right|_q \tag{2.5}$$

at position (k, j) is invertible. This seems a bit stronger than the independence of the conserved quantities in the definition of Liouville integrability, but it is the case in all interesting examples, and essentially, one can choose coordinates (p, q) in such a way.

Let us first consider a function S on phase space whose total variation under fixed F_i is given by

$$dS|_F = \sum_i p_i dq_i. \tag{2.6}$$

That is, $\partial S/\partial q_i|_F = p_i$ (on the left hand side, all of F_j and all of q_j, $j \neq i$ are fixed, the latter condition being implicit in our notation). For this to be the total variation of a function of phase space, we must have

$$\left.\frac{\partial p_i}{\partial q_j}\right|_F = \left.\frac{\partial p_j}{\partial q_i}\right|_F. \tag{2.7}$$

Let us show that this holds in Liouville integrable systems. We evaluate $\partial F_k/\partial q_i|_p$ in two different ways. First, using the fact that X_{F_k} is a differential operator and using involution $\{F_k, F_j\} = 0$ (implied summation over repeated indices),

$$\begin{aligned}
-\left.\frac{\partial F_k}{\partial q_i}\right|_p &= \{F_k, p_i\} \\
&= X_{F_k}(p_i) \\
&= X_{F_k}(F_j)\left.\frac{\partial p_i}{\partial F_j}\right|_q + X_{F_k}(q_j)\left.\frac{\partial p_i}{\partial q_j}\right|_F \\
&= \{F_k, q_j\}\left.\frac{\partial p_i}{\partial q_j}\right|_F \\
&= \left.\frac{\partial F_k}{\partial p_j}\right|_q \left.\frac{\partial p_i}{\partial q_j}\right|_F.
\end{aligned}$$

Second, $\partial F_k/\partial q_i|_F = 0$ which implies

$$\left.\frac{\partial F_k}{\partial q_i}\right|_p + \left.\frac{\partial F_k}{\partial p_j}\right|_q \left.\frac{\partial p_j}{\partial q_i}\right|_F = 0$$

Combining these two and using invertibility of (2.5) gives (2.7).

Hence there is a function (at least locally) defined by (2.6). From it, we define the variables Ψ_j:

$$\Psi_j = \left.\frac{\partial S}{\partial F_j}\right|_q. \tag{2.8}$$

This gives canonical coordinates:

$$\begin{aligned}
\{F_i, \Psi_j\} \;=\; X_{F_i}(\Psi_j) &= X_{F_i}(q_k)\, \frac{\partial}{\partial q_k}\bigg|_F \frac{\partial S}{\partial F_j}\bigg|_q + X_{F_i}(F_k)\, \frac{\partial}{\partial F_k}\bigg|_q \frac{\partial S}{\partial F_j}\bigg|_q \\
&= \{F_i, q_k\}\, \frac{\partial}{\partial F_j}\bigg|_q \frac{\partial S}{\partial q_k}\bigg|_F \\
&= \frac{\partial F_i}{\partial p_k}\bigg|_q \frac{\partial p_k}{\partial F_j}\bigg|_q \\
&= \frac{\partial F_i}{\partial F_j}\bigg|_q \\
&= \delta_{i,j}.
\end{aligned}$$

This completes the proof. ∎

Note that the quadrature involves inverting the relation $F_i = F_i(p,q)$ to give $p_i = p_i(F,q)$, evaluating the integral

$$S(F,q) = \int_{q^{(0)}}^{q} \sum_i p_i(F,q) dq_i, \tag{2.9}$$

and finally inverting $\Psi_j = \Psi_j(F,q)$ to give $q = q(F,\Psi)$. The trivial evolution of (F,Ψ) then gives the nontrivial evolution of (p,q).

Remark 2.2 *We see that the principles according to which integrability occurs and the ideas leading to coordinates with simple evolution equations are very nice and relatively simple, but that the explicit solution, in practice, will require some more "dirty" work: some potentially complicated algebraic manipulations with the conserved quantities. In integrability, the problem of solving a given system is often divided into two parts: the "direct" problem whereby we obtain variables with simple dynamics, and the "inverse" problem whereby we try to reconstruct the original variables and their dynamics in terms of the new ones. The inverse problem is the most complicated one.*

2.2 Action-angle variables

We have seen that we have (at least in the compact, connected case) invariant n-dimensional submanifolds $\mathtt{M}_f$, parametrised by as many parameters as there are degrees of freedom (n). Since these are tori, there are n cycles. In particular, there are closed paths that cannot be continuously deformed to points. The integral on the right-hand side of (2.9), starting and ending on a point q along such a closed path, does not, in general, give zero. Hence, the function $S(F,q)$, although locally well defined, is globally multivalued. We can consider, in particular, its variation along the n cycles. Let us denote by c_j for $j = 1, \ldots, n$ the n cycles on $\mathtt{M}_f$, and by

$$I_j = \frac{1}{2\pi} \int_{c_j} dS|_F \tag{2.10}$$

the (normalized) variations of S along these cycles. Clearly, where continuous deformation of a cycle does not change this value. Hence, I_j are not functions of the start / end-point q of the cycle, so they are only functions of F: they are conserved quantities.

The variables conjugated to I_j are the angle variables θ_j, particular linear combinations of the Ψ_i

with coefficients that depend on the I_j, defined by

$$\theta_j = \left.\frac{\partial S}{\partial I_j}\right|_q = \sum_i \frac{\partial F_i}{\partial I_j}\Psi_i. \tag{2.11}$$

These are angle variables:

$$\int_{c_j} d\theta_k|_I = \int_{c_j} \sum_i \left.\frac{\partial}{\partial q_i}\right|_I \left.\frac{\partial S}{\partial I_k}\right|_q dq_i = \frac{\partial}{\partial I_k}\int_{c_j} dS|_I = \delta_{k,j}$$

and form with I_j canonical coordinates:

$$\begin{aligned} \{I_j, \theta_k\} &= \sum_i \frac{\partial F_i}{\partial I_k}\{I_j, \Psi_i\} \\ &= \sum_i \frac{\partial F_i}{\partial I_k}\frac{\partial I_j}{\partial F_i} \\ &= \delta_{j,k}. \end{aligned} \tag{2.12}$$

The action-angle variables described above give a clear picture of the motion on the invariant submanifold, in agreement with Remark 2.1.

Exercise 2.3 *Find the action-angle variables for a set of n harmonic oscillator,*

$$H = \frac{1}{2}\sum_i (p_i^2 + \omega^2 q_i^2).$$

You can take the conserved charges $F_i = (p_i^2 + \omega^2 q_i^2)$.

2.3 Lax pairs and classical r-matrices

We now have the general structure of integrability, but an important question is: given a Hamiltonian, how do we know that it is integrable? That is, how do we find the n independent conserved quantities in involution? Unfortunately, there is no general answer to this question – it is always a difficult task to determine if a given system is integrable (and in more complicated physical situation, this is usually worth one or more scientific papers!). Yet, a wealth of techniques have been developed in the context of integrability essentially in order to try to classify the possible integrable systems. In fact, often one starts with a description that immediately implies the existence of these conserved quantities, and often that contain some part of the direct or inverse problem. These, rather than the solution by quadrature, are the techniques that generalize most naturally to field theories and quantum systems, where a clear concept generalizing Liouville integrability is not immediate. I will here discuss Lax pairs and the associated classical r-matrix, and how conserved quantities in involution naturally emerge from these.

Let L and M be two square matrices (of dimension d, say) whose entries are functions in $\mathcal{F}$. Suppose the following holds:

$$\dot{L} = [M, L] = ML - LM \tag{2.13}$$

where on the right-hand side we have the usual matrix commutator. Then L and M are called a *Lax pair*. It is a simple matter to see that a Lax pair gives d (not necessarily independent) conserved charges. For instance,

$$\frac{d}{dt}\mathrm{Tr}(L^m) = m\mathrm{Tr}(L^{m-1}\dot{L}) = m\mathrm{Tr}(L^{m-1}[M, L]) = \mathrm{Tr}([M, L^m]) = 0 \tag{2.14}$$

for all m. Basically, this can be recast into saying that the d eigenvalues of L are conserved. More directly, suppose L can be diagonalized: $L = U\Lambda U^{-1}$ where Λ is diagonal. Then, with the gauge transformation

$$M = UBU^{-1} + \dot{U}U^{-1} \tag{2.15}$$

we have

$$\dot{\Lambda} = [B, \Lambda]. \tag{2.16}$$

But since Λ is diagonal, the right-hand side has no element on the diagonal (this being true for any B). Hence $\dot{\Lambda} = 0$.

Exercise 2.4 *Show (2.16).*

Exercise 2.5 *Verify that a Lax pair for the harmonic oscillator is*

$$L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix}, \quad M = \begin{pmatrix} 0 & -\omega/2 \\ \omega/2 & 0 \end{pmatrix}$$

with $H = \frac{1}{4}\mathrm{Tr}(L^2)$.

Of course, having conserved quantities does not mean that we have Liouville integrability. We need them to be in involution. This means that we need to specify something with the Poisson bracket itself. This is where the classical r-matrix comes into play. Below, we assume for simplicity that L is diagonalizable (at generic points).

First some notation. Let us denote by $\mathcal{M}$ the space of matrices with entries in $\mathcal{F}$. If $A, B \in \mathcal{M}$, then there are two natural multiplication operations we can do, mixing the matrix multiplication with a product structure on $\mathcal{F}$: the usual multiplication AB (using the ring structure of $\mathcal{F}$), and the operation

$$\{A, B\} := \sum_{\alpha,\beta} \{A_\alpha, B_\beta\} E_\alpha E_\beta$$

(using the Lie algebra structure of $\mathcal{F}$), where E_α form a basis of constant matrices (any basis will do) and $A = \sum_\alpha A_\alpha E_\alpha$, etc. The latter satisfies the usual bilinearity, anti-symmetry and Jacobi identity. Further, let us consider maps $\mathcal{M} \to \mathcal{M} \otimes \mathcal{M}$ into tensor products of matrices given by $A \mapsto A_1 := A \otimes \mathbf{1}$ and $A \mapsto A_2 := \mathbf{1} \otimes A$ (and similarly when there is more factors); the indices $1, 2, \ldots$ are not to be confused with the α index used above. Note that

$$\{A_1 B_1, C_2\} = A_1\{B_1, C_2\} + \{A_1, C_2\}B_1. \tag{2.17}$$

Similarly, for any $r = \sum_{\alpha,\beta} r_{\alpha\beta} E_\alpha \otimes E_\beta \in \mathcal{M} \otimes \mathcal{M}$, let us consider the identity map $r \mapsto r_{12}$ (for agreement with the standard notation) and the permutation map

$$r \mapsto r_{21} := \sum_{\alpha,\beta} r_{\beta\alpha} E_\alpha \otimes E_\beta$$

(with a similar notation for more tensor products).

Theorem 2.2 *The eigenvalues of L are in involution if and only if there exists a $r \in \mathcal{M} \otimes \mathcal{M}$ such that*

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]. \tag{2.18}$$

Proof. We write $L = U\Lambda U^{-1}$ where Λ is diagonal. Then by direct calculation using (2.17), we have

$$\{L_1, L_2\} = U_1 U_2 \{\Lambda_1, \Lambda_2\} U_1^{-1} U_2^{-1} + [r^\star_{12}, L_1] - [r^\star_{21}, L_2] \tag{2.19}$$

where

$$r^\star = U_2 \{U_1, \Lambda_2\} U_1^{-1} U_2^{-1} + \frac{1}{2}\left[\{U_1, U_2\} U_1^{-1} U_2^{-1}, L_2\right] \tag{2.20}$$

Hence (2.18) holds if the eigenvalues are in involution. On the other hand, assume that (2.18) holds and let us denote $s = U_1^{-1} U_2^{-1}(r - r^\star) U_1 U_2$. Then

$$\{\Lambda_1, \Lambda_2\} = [s_{12}, \Lambda_1] - [s_{21}, \Lambda_2]. \tag{2.21}$$

Since Λ_1 and Λ_2 are diagonal, the right-hand side does not contain any element on the diagonal, hence the eigenvalues are in involution. ∎

Exercise 2.6 *Derive (2.19) and (2.20).*

If the matrix r involved in (2.18) is further constrained to be a constant matrix, then the relation (2.18) has a solution if and only if r is such that the Jacobi identity holds, leading to the condition

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] = 0. \tag{2.22}$$

If $r_{12} = -r_{21}$, then this is the *classical Yang-Baxter equation*.

The connection from the r-matrix to the Lax pair is obtained by considering the associated classical flows. Let L and r satisfy (2.18). In order to be specific, let us consider the quantities in involution given by

$$F_j = \mathrm{Tr}\left(L^j\right)$$

(the Hamiltonian, which we assume to be a function of the L-eigenvalues, will in general have the form $H = \sum_j a_j F_j$, up to convergence issues). Then we find, by evaluating $\mathrm{Tr}_1\left(\{L_1^j, L_2\}\right)$ using (2.18), that

$$\{F_j, L\} = [M_j, L], \quad M_j = -j\mathrm{Tr}_1\left(L_1^{j-1} r_{21}\right). \tag{2.23}$$

That is, we can define "times" t_j associated to the F_j and we have the set of Lax-pair flows

$$\frac{dL}{dt_j} = [M_j, L]. \tag{2.24}$$

Clearly, this then holds for any power of L, hence for any function $h(L)$ with converging power series:

$$\frac{d}{dt_j} h(L) = [M_j, h(L)]. \tag{2.25}$$

This may also hold for more general functions of L (e.g. power series with matrix coefficients that commute with all M_j). If matrix elements of L can be used to obtain a set of (generically non-singular) coordinates on the invariant submanifold, then M_j is a matrix-valued function of the matrix elements of L (seeing the conserved quantities as fixed coefficients). Hence $[M_j, h(L)] = g_{j,h}(L)$ is a function of L. If further $g_{j,h}(L)$ is, for any j, h, a function for which (2.25) holds, then we find that

$$D_j = \frac{d}{dt_j} - \mathrm{ad}(M_j) \tag{2.26}$$

are commuting "covariant derivatives". That is, in this case, we have a principal $GL(d)$-bundle over the invariant submanifold $\mathtt{M}_f$, where the action of $U \in GL(d)$ on a fiber is $L \mapsto ULU^{-1}$; and we have put on it a flat connection $-\mathrm{ad}(M_j)$ (in the t_j coordinates on $\mathtt{M}_f$). A similar zero-curvature idea will be crucial in the next subsection where we look at the inverse scattering method for integrable classical field theories.

Exercise 2.7 *Work out explicitly the covariant derivative (2.26) on the $GL(2n)$-bundle for the construction of [1, pp.12,13], where Lax pairs are presented from the action-angle variables. You can take the F_j to be simply equal to the action variables I_j.*

3 Classical field theory: the inverse scattering method

We now consider a classical system with infinitely-many degrees of freedom. We do not consider any arbitrary such system, but very special ones, which can be described by PDEs. In particular, we look at the example of the sine-Gordon model:

$$\partial_t^2\phi - \partial_x^2\phi = -m^2 \sin\phi \tag{3.1}$$

(comparing with [1, Chap13], we have done $\varphi \mapsto \phi/(2\beta)$ and $m \mapsto m/4$). This field equation can be recast into a Hamiltonian flow equation: we have the Poisson structure

$$\{\Pi(x), \phi(x')\} = \delta(x - x'), \tag{3.2}$$

and the Hamiltonian

$$H = \int dx \, \left(\frac{1}{2}\Pi(x)^2 + \frac{1}{2}\big(\partial_x\phi(x)\big)^2 + m^2(1 - \cos\phi) \right) \tag{3.3}$$

(the integral is on the line). The constant term in the potential has been chosen such that the field configuration $\phi = 0$ has finite (in fact, zero) energy. Here, the infinitely-many degrees of freedom could be thought of, naively, as the values of ϕ at different points x in space; in a sense, the variable x replaces the indices used in the previous section ($q_i \mapsto \phi(x)$). But this means that, in principle, ϕ can be a very "wild" function: with discontinuities, non-differentiable, diverging at $\pm\infty$, etc. This causes difficulties in the interpretation of (3.1), and in practice, we often look at nice enough initial condition, and the niceness usually stays under time evolution. Hence, we may restrict the space of functions we are looking at – we will take the same space of ϕ and Π. The phase space is the direct product of two copies of that space of function. For simplicity, we will be considering functions which are smooth (infinitely-many times differentiable) and which decay as $x \to \pm\infty$ at least exponentially fast. The Hamiltonian takes finite values on such field configurations, and in fact, in order to get the equations of motion

$$\partial_t\phi(x) = \pi(x), \quad \partial_t\Pi(x) = \partial_x^2\phi - m^2 \sin\phi \tag{3.4}$$

which lead to (3.1), we need to use integration by parts, hence we do need some asymptotic conditions on the fields[2]. The space of such functions is clearly infinite, so we still have infinitely many degrees of freedom.

How does integrability work for such infinite systems? How do we define integrability? The important notion here is that of *locality*. Locality here can be seen as meaning that the way ϕ evolves in time is determined locally: $\phi(x, t + \delta t)$ is determined, for small δt, from $\phi(x', t)$ for x' in a neighborhood of x. Equation (3.1) is clearly local. There is no universally accepted definition of integrability when there are infinitely-many degrees of freedom, but the presence of locality gives rise to a principle that generalizes that of Liouville integrability:

[2]We could have chosen $\phi \to 2\pi Q$, as $x \to \infty$, exponentially fast, for any $Q \in \mathbb{Z}$, and still have finite-energy field configurations and the correct equations of motion.

Definition 3.1 *A local field theory model is integrable if there exists infinitely many independent local conserved quantities in involution.*

The new idea is that of a local conserved quantity, rather than simply a conserved quantity. A local conserved quantity is a quantity $Q = \int dx\, \rho(x)$ that is the integral over space of a density $\rho(x)$ that depends only on the fields and their derivatives at the point x. The problem with this definition is the "infinitely-many": this is not necessarily sufficient (not all infinities are "the same"). Yet, most of the times, discovering few conserved quantities in involution, say 3 or 4, makes it very likely that the model is integrable.

Remark 3.1 *The fact that the PDE has one space (and one time) dimension is crucial for (at least the usual developments of) integrability. Taking infinitely-many degrees of freedom in a nice, local way as above introduces the concept of dimensionality of space-time: there is a notion of neighborhoods, and of diffeomorphism with $\mathbb{R}^d$ for some d (manifold structure). It turns out that only with one space dimension do the standard developments of integrability actually work. In a sense, having only one space dimension is constraining enough for integrability to have a possibility of existing; for instance, particles can never miss each other, and there is only two ways they can go (right or left)!*

In the sine-Gordon model, the Hamiltonian (3.3) is clearly a local conserved quantity. The "momentum"

$$P = \int dx\, \Pi(x)\partial_x\phi(x) \tag{3.5}$$

is also a local conserved quantity. It can be checked that there are more, for instance (up to typos of mistake in my calculation!):

$$Q_3 = \int dx\, \left(\big(\Pi(x) - \partial_x\phi(x)\big) \sin\phi - \frac{1}{2m}\big(\partial_x\Pi(x) - \partial_x^2\phi(x)\big)\big((\Pi(x) - \partial_x\phi(x))^2 - 2m^2\cos\phi(x)\big) \right). \tag{3.6}$$

Checking that this commutes with H is a bit tedious, but its commutatitivity with P is immediate (since P is equivalent to ∂_x). Hence the quantities H, P and Q_3 all Poisson commute with each other. It does look like the sine-Gordon model is integrable according to Definition (3.1). But clearly we need a better framework to find the local conserved quantities, and then a framework to solve the model. Here, we will use the zero-curvature formulation. There are other methods, some more algebraic, some more analytic, in order to construct local conserved quantities and in order to obtain explicit solutions to (3.1). The zero-curvature formulation has the advantage of putting in a unique framework both the construction of conserved quantities, and the construction of explicit solutions via the inverse scattering method; and the latter method has the advantage of being powerful enough to be applicable for very general initial conditions.

Remark 3.2 *In fact, perhaps a better concept of locality in relativistically invariant models like (3.1) can be borrowed from quantum field theory. A local field $\varphi(x)$ is a field that Poisson commutes with the energy density $h(x')$, i.e. $\{\rho(x), h(x')\} = 0$, whenever $x \neq x'$ (by relativistic invariance: at space-like distances). A local conserved quantity is an integral over space of a density that is itself a local field. The definition of space is crucial, and it's fundamentally defined through the momentum operator. This more general definition becomes particularly important in certain situations in quantum field theory, when one looks at local fields in various locality sectors.*

3.1 Zero-curvature formulation, the monodromy matrix, conserved quantities

In order to find infinitely-many conserved quantities, we should use the idea of Lax pairs. However, there are two elements which we need to take proper care of.

First, the infinitely-many quantities seem to indicate that we should have infinite-dimensional matrices. Effectively this is the case, but in practice this is achieved by using the concept of *spectral parameter*. Basically, we will have a pair of finite-dimensional (below, 2 by 2) matrices $L(\lambda)$ and $M(\lambda)$ that depend on a parameter λ, and that satisfy, for all λ,

$$\frac{d}{dt}L(\lambda) = [M(\lambda), L(\lambda)]. \tag{3.7}$$

Then $\mathrm{Tr}(L(\lambda))$ is a conserved quantity for each λ, and expanding in λ (in an appropriate way) gives a lot of conserved quantities; for instance all coefficients of a Taylor series expansion in λ or λ^{-1}, or more generally Laurent series expansions. In a sense, we are looking at a representation of a loop algebra, so effectively at infinite-dimensional matrices. But the idea of a spectral parameter has its own advantages compared to that of infinite-dimensional matrices; for instance, bringing the concept of analyticity in λ into play can be very powerful.

Second, we need the conserved quantities to be local. This means that not any type of Lax pair will be appropriate. It turns out that the locality principle is conveniently taken care of by translating the Lax-pair condition into a zero-curvature condition for a connection on a vector bundle (the fibers being vectors acted on by the L and M matrices) over space-time (bringing space along with time in this way, as opposition to space being just a set of indices, is what allows us to talk about locality in a natural way).

Consider the following connections on a $SU(2)$ vector bundle over space-time:

$$\begin{aligned} U = U(x,t) &= \frac{i}{4}\begin{pmatrix} \Pi & 2m\sinh(u+i\phi/2) \\ 2m\sinh(u-i\phi/2) & -\Pi \end{pmatrix} \\ V = V(x,t) &= \frac{i}{4}\begin{pmatrix} \partial_x\phi & -2m\cosh(u+i\phi/2) \\ -2m\cosh(u-i\phi/2) & -\partial_x\phi \end{pmatrix}, \end{aligned} \tag{3.8}$$

where the spectral parameter is $\lambda = e^u$.

It is a simple matter to check that the zero curvature condition,

$$[\partial_x - U, \partial_t - V] \equiv \partial_t U - \partial_x V + [U, V] = 0 \tag{3.9}$$

(commutativity of the associated covariant derivatives), is *equivalent* (i.e. implies and is implied by) the sine-Gordon equation (3.1).

Exercise 3.1 *Show the equivalence of (3.9) and (3.1).*

For any (smooth enough) 2 by 2 matrices U and V, we can construct the space-time dependent matrix $T_0(x,t)$ uniquely defined by $T_0(0,0) = \mathbf{1}$, $(\partial_x - U)T_0(x,0) = 0$, $(\partial_t - V)T_0(x,t) = 0$. The first is an initial condition and the second a first-order differential equation, together giving a unique function $T_0(x,0)$. This is then seen as an initial condition for the third equation, giving $T(x,t)$. If we have (3.9), then the

third equation implies that we further have $(\partial_t - V)(\partial_x - U)T_0(x,t) = 0$ which implies, by uniqueness of the solution of $(\partial_t - V)S(x,t) = 0,\ S(x,0) = 0$, that $(\partial_x - U)T_0(x,t) = 0$. In other words, the zero-curvature condition (3.9) is the compatibility condition for

$$(\partial_x - U)T_0(x,t) = 0, \quad (\partial_t - V)T_0(x,t) = 0 \tag{3.10}$$

and the condition $T_0(0,0) = \mathbf{1}$ uniquely fixes the solution.

Now construct the *transport matrix* (for $x \geq y$ say)

$$T(x,y,t) = T_0(x,t)T_0(y,t)^{-1}. \tag{3.11}$$

Then clearly

$$\frac{\partial}{\partial t}T(x,y,t) = V(x,t)T(x,y,t) - T(x,y,t)V(y,t). \tag{3.12}$$

Note that $V(\infty,t) = V(-\infty,t)$ thanks to our asymptotic condition on ϕ. Hence, setting the L matrix to the *monodromy matrix*

$$L(\lambda) = L(\lambda;t) = T(\infty,-\infty,t) \tag{3.13}$$

we find (3.7) with

$$M(\lambda) = V(\infty,t) = -\frac{im}{4}(\lambda + \lambda^{-1})\,\sigma_x. \tag{3.14}$$

This is the Lax pair that we can obtain from the zero-curvature condition, itself a consequence of the sine-Gordon equation (3.1). Note that $L(\lambda;t)$ is a functional of the phase space variables Π, ϕ (seen as functions of space, at the fixed time t) through the solutions to (3.10) (whence it gets its t dependence), and that M, which could also be in general, is in fact just a constant.

It is possible to express more explicitly L in terms of phase space variables, and to get a clear geometrical picture of the above construction, as follows. The solution to the first-order differential equation $(\partial_x - U)T(x,y,t) = 0$, with the initial condition $T(y,y,t) = 0$, is obtained using the path-ordered exponentials:

$$T(x,y,t) = \mathcal{P}\exp\left[\int_y^x ds\, U(s,t)\right] \tag{3.15}$$

where the ordering places $U(s,t)$ with larger values of s to the left of those with smaller values. This explicit solution involves an integration over the connection along the x axis, representing a parallel transport along the x axis. We could define similar objects by integrating along any path in space-time, using the connection U, V. The zero-curvature statement means that multiplying these parallel-transport operators along a close contour gives the identity $\mathbf{1}$. In particular, taking a path that goes from $(-\infty,t)$ to (∞,t) to (∞,t') to $(-\infty,t')$ to $(-\infty,t)$ we obtain

$$e^{(t-t')M}L^{-1}(\lambda;t')e^{(t'-t)M}L(\lambda;t) = \mathbf{1}. \tag{3.16}$$

Re-writing this as $e^{(t'-t)M}L(\lambda;t)e^{(t-t')M} = L(\lambda;t')$ and taking the trace, we indeed find that $\mathrm{Tr}(L(\lambda;t))$ is independent of t.

Remark 3.3 *The construction of L and M leading to (3.13) and (3.14) works as well if U and V do not define a flat connection (constructing $T(x,t)$ as described above). The difference is that, of course, (3.15) does not hold in this case. Hence, in this case, the L matrix will depend not only on Π and ϕ at time t, but also at all other times between 0 and t. It is, then, not a "proper" L matrix, and does not give rise to "proper" conserved quantities.*

Remark 3.4 *As we mentioned, the sine-Gordon model also has finite-energy solutions with the asymptotic conditions $\phi(\infty,t) = 2\pi Q$, $Q \in \mathbb{Z}$ (these are minima of the potential). These solutions correspond to the presence of Q solitonic charges. The above derivation needs some small adjustments (see [1, Chap 13] in this situation.*

The explicit construction of the conserved quantities is somewhat technical but straightforward, and can be performed as follows. First, one may perform a gauge transformation

$$\tilde{T}(x,y,t) = e^{-\frac{i}{4}\phi(x)\sigma_z} T(x,y,t) e^{\frac{i}{4}\phi(y)\sigma_z} \tag{3.17}$$

giving the new L matrix $\tilde{L}(\lambda;t) = \tilde{T}(\infty,-\infty,t)$ and the new connection

$$\tilde{U} = \frac{i}{4}\left((\Pi - \partial_x\phi)\sigma_z + m\lambda(1-\lambda^{-2}e^{-i\phi})\sigma_+ + m\lambda(1-\lambda^{-2}e^{i\phi})\sigma_-\right). \tag{3.18}$$

Clearly $\mathrm{Tr}(\tilde{L}(\lambda)) = \mathrm{Tr}(L(\lambda))$. Second, one diagonalizes the connection, in order to evaluate the trace more easily. It turns out that it is always possible to do so for a connection formed by Pauli matrices (see [1, pp 384-385] for more precision; this is in fact a statement about the $su(2)$ algebra). If the connection is in the form $\tilde{U} = U_z\sigma_z + U_+\sigma_+ + U_-\sigma_-$, then by appropriate gauge transformations, one finds a connection that is purely diagonal (proportional to σ_z), and

$$\tilde{L}(\lambda;t) = \mathcal{P}\exp\left[\int_{-\infty}^{\infty} dx\, \tilde{U}(x,t)\right] = \exp\left[\sigma_z \int_{-\infty}^{\infty} dx\, v(x)\right] \tag{3.19}$$

where v solves the Ricatti equation

$$\partial_x v + v^2 = \partial_x\left(U_z - \frac{1}{2}\partial_x \log U_+\right) + \left(U_z - \frac{1}{2}\partial_x \log U_+\right)^2 + U_-U_+. \tag{3.20}$$

Finally, one simply expands everything in increasing powers of λ^{-1}. The conserved charges are $\mathrm{Tr}(\tilde{L}(\lambda)) = 2\cos\int dx\, v(x)$, or we can take $\int dx\, v(x)$. It turns out that

$$v(x) = \frac{im}{4}\lambda + v_{-1}\lambda^{-1} + v_{-2}\lambda^{-2} + \dots \tag{3.21}$$

and $v_{-1}, v_{-2}, \dots$ can be obtained recursively from (3.20). They are explicitly expressed in terms of the phase-space variables, and it is clear also that $v(x)$ contains ϕ and Π and their derivatives at x only. Hence, we have local conserved charges. The full set is obtained by performing the same steps above with the gauge transformation $\tilde{T}(x,y,t) = e^{\frac{i}{4}\phi(x)\sigma_z} T(x,y,t) e^{-\frac{i}{4}\phi(y)\sigma_z}$ and expanding in increasing powers of λ. Combining v_{-1} and v_1 one finds, in particular, the Hamiltonian and the momentum operators. On the other hand, v_{-3} gives rise to the conserved charge Q_3 written above.

Exercise 3.2 *Evaluate the conserved quantities corresponding to v_3 and v_{-3}. What happens with v_2 and v_{-2}?*

Finally, we must discuss the involution of the local conserved quantities obtained (here we omit the explicit time dependence, as we are only interested in Poisson brackets). Here we have a similar r-matrix idea as that for finite dynamical system, but written in its "exponentiated" form. We want to find $r_{12}(\lambda,\mu)$ such that the *fundamental Sklyanin relation* holds:

$$\{L_1(\lambda), L_2(\mu)\} = [r_{12}(\lambda,\mu), L_1(\lambda)L_2(\mu)]. \tag{3.22}$$

We must naturally have anti-symmetry $r_{12}(\lambda,\mu) = -r_{21}(\mu,\lambda)$. This immediately implies that the trace of $L(\lambda)$ generates Poisson commuting quantities. Note that λ and μ are spectral parameters attached to the different copies 1 and 2; this is in agreement with seeing them as providing an "infinite-dimensional matrix" structure. This relation is particularly useful, because it follows from the simpler *local* relation on the connections, similar to the finite-dimensional L-matrix Poisson bracket relation (2.18) (in the case where r_{12} is anti-symmetric):

$$\{U_1(\lambda;x), U_2(\mu;y)\} = [r_{12}(\lambda,\mu), U_1(\lambda,x) + U_2(\mu,y)]\delta(x-y). \tag{3.23}$$

Not all integrable models have such an ultra-local (delta-function only, no derivatives of delta-functions) Poisson bracket between connections, but many do. A straightforward calculation shows that

$$r_{12}(\lambda,\mu) = \frac{1}{16}\left(\frac{\lambda^2+\mu^2}{\lambda^2-\mu^2}\sigma_z\otimes\sigma_z + \frac{4\lambda\mu}{\lambda^2-\mu^2}(\sigma_+\otimes\sigma_- + \sigma_-\otimes\sigma_+)\right) \tag{3.24}$$

does the job in the sine-Gordon model.

Exercise 3.3 *Show that (3.22) follows from (3.23).*

Exercise 3.4 *Show that with (3.24), the relation (3.23) follows from the canonical Poisson relations (3.2).*

3.2 Solution by inverse scattering

The inverse scattering method is a method by which the phase space variables Π and ϕ are transformed into other variables (like a Fourier transform), whose time evolution is much simpler. The transformation is performed through a scattering problem, and the new variables are the scattering data; this is the direct scattering transform. The solution to the initial problem is obtained by inverting this on the evolved scattering data; this is the inverse scattering transform. The scattering data essentially form the action-angle variables; in particular, they include the conserved charges constructed above.

The scattering problem is that of a $\mathbb{C}^2$ vector Ψ obeying the linear equation

$$(\partial_x - U)\Psi = 0. \tag{3.25}$$

The main idea is to see (3.25) as an equation for a two-component wave scattering off the potential U. Given the potential U (so given the functions ϕ and Π), we have reflection and transmission coefficients. We will show below that these coefficients evolve in time in a simple way if ϕ and Π evolve according to (3.4); this will be done by using the fact that if the asymptotic form of Ψ is annihilated by $\partial_t - V$, then by the zero-curvature condition,

$$(\partial_t - V)\Psi = 0 \tag{3.26}$$

for all x. This is the direct problem. On the other hand, if we are given reflection and transmission coefficients for waves of all possible wave numbers, then this should give us enough information to reconstruct U, as these waves should be enough to fully probe the potential. This is the inverse problem. We will see that we can have incoming waves of all possible wave numbers thanks to the presence of the spectral parameter. This is the basis of the inverse scattering transform.

In more detail: As $x \to \pm\infty$, the linear system becomes

$$\left(\partial_x - \frac{im\sinh u}{2}\,\sigma_1\right)\Psi \overset{x\to\pm\infty}{=} 0. \tag{3.27}$$

Hence, the solutions are of the form

$$\Psi \overset{x\to\pm\infty}{\sim} c_1^{\pm}\begin{pmatrix}1\\1\end{pmatrix}e^{ikx} + c_2^{\pm}\begin{pmatrix}1\\-1\end{pmatrix}e^{-ikx} \tag{3.28}$$

where $k = \frac{m\sinh u}{2}$; these are the wave numbers, and for $\lambda \in \mathbb{R}$, we have all of $k \in \mathbb{R}$. We define *Jost solutions* as follows:

Definition 3.2 *The Jost solutions $\Psi = f_+$ and $\Psi = f_-$ are solutions to (3.25) defined by the following asymptotic behaviors:*

$$f_+ \overset{x\to\infty}{\sim} \begin{pmatrix}1\\1\end{pmatrix}e^{ikx}, \quad f_- \overset{x\to-\infty}{\sim} \begin{pmatrix}1\\-1\end{pmatrix}e^{-ikx}, \tag{3.29}$$

The first represent an incoming wave from $x = -\infty$, the second, incoming from $x = \infty$. Note that these asymptotic conditions are not annihilated by $\partial_t - V$, but we will take care of this in due time.

The scattering data are essentially the constants $c_{1,2}^{\pm}$ (functions of λ) associated to the Jost solutions appearing in the other x asymptotic (the reflection and transmission coefficients):

$$\begin{aligned} f_+ &\overset{x\to-\infty}{\sim} c_1^-\begin{pmatrix}1\\1\end{pmatrix}e^{ikx} + c_2^-\begin{pmatrix}1\\-1\end{pmatrix}e^{-ikx} \\ f_- &\overset{x\to\infty}{\sim} c_1^+\begin{pmatrix}1\\1\end{pmatrix}e^{ikx} + c_2^+\begin{pmatrix}1\\-1\end{pmatrix}e^{-ikx}. \end{aligned} \tag{3.30}$$

These constants are not all independent. Indeed, let ξ, Ψ be to solutions to (3.25) and le us consider the Wronskian

$$W(\xi,\Psi) = \det F, \quad F := (\xi,\Psi) \tag{3.31}$$

This is independent of x thanks to $\mathrm{Tr}(U) = 0$:

$$\partial_x W(\xi,\Psi) = \partial_x \exp \mathrm{Tr}\log F = \mathrm{Tr}(\partial_x F\, F^{-1})W(\xi,\Psi) = \mathrm{Tr}(U)W(\xi,\Psi) = 0. \tag{3.32}$$

Further, one can check that $U^* = \sigma_y U \sigma_y$. Hence, if Ψ is a solution to (3.25), then so is

$$\bar\Psi := \sigma_y \Psi^* \tag{3.33}$$

(this defines a conjugation on the space of solutions). Equating Wronskians at $x = \infty$ and $x = \infty$, we find:

$$\begin{aligned} W(f_+, f_-) &: \quad c_1^- = c_2^+ \\ W(f_+, \bar f_-) &: \quad c_2^- = (c_1^+)^* \\ W(f_+, \bar f_+) &: \quad |c_1^-|^2 + |c_2^-|^2 = 1 \end{aligned}$$

whereby we may define the two scattering coefficients

$$a = a(\lambda) = c_1^-, \quad b = b(\lambda) = -c_2^-, \quad |a|^2 + |b|^2 = 1. \tag{3.34}$$

Note that $a = -W(f_+, f_-)/2$ and that putting things together,

$$\begin{aligned} f_+ &\overset{x\to-\infty}{\sim} a\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{ikx} - b\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-ikx} \\ f_- &\overset{x\to\infty}{\sim} -b^*\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{ikx} + a\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-ikx}. \end{aligned} \tag{3.35}$$

From Definition (3.2), it is natural that both f_+ and f_- can be analytically continued, as functions of λ, towards the positive imaginary direction. This is because the exponentials become decaying in the asymptotic regions specified in Definition 3.2, hence the definition doesn't allow for any non-analytic changes of the other asymptotic behavior, the increasing exponential – it is set to zero. Hence also can $a(\lambda)$, because it is the coefficient of an increasing exponential, and only the coefficient of the decreasing exponential may suffer non-analytic changes (in the presence of an increasing exponential component to the asymptotics). Further, if $a(\lambda) = 0$ somewhere in the upper half plane of λ, then the solution f_+ and f_- are linearly dependent (because the Wronskian is zero), and also this solution is now exponentially decaying in both directions, hence normalizable (this is a bound state of the linear scattering problem). For such zeros λ_n, we have

$$f_-(\lambda_n; x) = c_n f_+(\lambda_n; x), \quad (a(\lambda_n) = 0). \tag{3.36}$$

The analytic function $a(\lambda)$ (which contains the information of its zeros λ_n), the function $b(\lambda)$, and the coefficents c_n are the *scattering data* of the scattering problem. The passage from Π and ϕ to these quantities is the direct scattering transform. It turns out that the knowledge of these quantities is sufficient to reconstruct the potential U, whence the phase space variables Π and ϕ. The function $a(\lambda)$ is called the *Jost function*.

Before we indicate how to perform the inverse scattering transform, let us look at the time evolution of the scattering data, using (3.26). Consider $\Psi = w(t)f_+$ and let us impose (3.26). Since $V \sim -\frac{im\cosh u}{2}\sigma_x$ at $x = \pm\infty$, this gives

$$\dot{w} = -\frac{im\cosh u}{2} w. \tag{3.37}$$

Therefore, looking at $x \to -\infty$, we obtain

$$\frac{d}{dt}(wa) = -\frac{im\cosh u}{2} wa, \quad \frac{d}{dt}(wb) = \frac{im\cosh u}{2} wb. \tag{3.38}$$

This implies

$$\dot{a} = 0, \quad \dot{b} = im\cosh u\, b. \tag{3.39}$$

Clearly the zeros λ_n of a do not change, and a similar calculation as the one above gives

$$\dot{c}_n = -im\cosh u\, c_n. \tag{3.40}$$

Hence, rather trivially, the scattering data evolve in a simple fashion.

Remark 3.5 *Note that the scattering transform doesn't have, in principle, anything to do with integrability by itself. It is just a transform operation, much like the Fourier transform. What is important is that the potential involved U is part, along with V, of a flat connection. This flatness condition is the main statement of integrability. It is this condition that gives a simple time evolution of the scattering data. This is similar to the Fourier transform being useful for solving linear problems, because the Fourier coefficients evolve in time simply.*

Finally, let us indicate how to go from the scattering data to the phase space variables. The point is that from functions of the spectral parameter λ, we must obtain functions of the space variable x. Note that this is pure "scattering transform theory", not really "integrable system theory": we do not use the presence of a flat connection (which we used only for the time evolution).

First, from the scattering data, we construct the following functions:

$$F_j(x) = \lim_{\epsilon\to 0}\left(\int_{-\epsilon^{-1}}^{-\epsilon} + \int_{\epsilon}^{\epsilon^{-1}}\right) d\lambda\, \lambda^j\, e^{ikx}\, r - 2i\pi \sum_n e^{ik_n x}\, \lambda_n^j\, \frac{c_n}{a'(\lambda_n)} \tag{3.41}$$

where

$$r = r(\lambda) = -\frac{b^*(\lambda)}{a(\lambda)} \tag{3.42}$$

(and recall that $k(\lambda) = \frac{m}{4}(\lambda - \lambda^{-1})$, with $k_n = k(\lambda_n)$). This is basically a Fourier transform, with some subtleties because of the integration region of λ and because of the zeroes of $a(\lambda)$ on the upper half plane (the bound states of the linear problem).

Second, from the Jost solutions, we construct the two-components functions $A_\pm(x,y)$ and $B_\pm(x,y)$, crucially which are independent of λ, as follows:

$$\begin{aligned} f_+ &= e^{\frac{i\phi}{4}\sigma_z}\left(e^{ikx}\begin{pmatrix}1\\1\end{pmatrix} + \int_x^\infty \big(A_+(x,y) + \lambda^{-1}B_+(x,y)\big)\, e^{iky}\right) \\ f_- &= e^{\frac{i\phi}{4}\sigma_z}\left(e^{ikx}\begin{pmatrix}1\\1\end{pmatrix} + \int_{-\infty}^x \big(A_-(x,y) + \lambda^{-1}B_-(x,y)\big)\, e^{iky}\right). \end{aligned} \tag{3.43}$$

This is again a type of Fourier transform, and the choice of the integration limits guarantees that we have the right asymptotics. The fact that these equations hold is thanks to the special λ dependence of U. Of course, the kernels $A_\pm$ and $B_\pm$ are determined by (3.25): they satisfy linear first-order differential equations with special boundary conditions. In particular, we can easily recover the potential U from $A_\pm$ and $B_\pm$, in fact even from A_+ and B_+ only.

The Gelfand-Levitan-Marchenko equations are linear integral equations that allow us to determine $A_\pm$ and $B_\pm$ from F_j. This is a rather nontrivial result, the proof of which can be found in [1, Chap 13] (much beyond the scope of these notes!):

$$\begin{aligned} -\frac{8i\pi}{m}\bar{A}_+(x,y) &= F_0(x+y)\begin{pmatrix}1\\1\end{pmatrix} + \int_x^\infty \big(F_0(y+z)A_+(x,z) + F_{-1}(y+z)B_+(x,z)\big)\, dz \\ -\frac{8i\pi}{m}\bar{B}_+(x,y) &= F_{-1}(x+y)\begin{pmatrix}1\\1\end{pmatrix} + \int_x^\infty \big(F_{-1}(y+z)A_+(x,z) + F_{-2}(y+z)B_+(x,z)\big)\, dz \end{aligned}$$

Remark 3.6 *Soliton solutions are finite-energy solutions to the sine-Gordon equation with very nice characteristics. They manifest themselves by relatively sharp jumps in the field ϕ around positions $x(t) = x_0 + vt$ that move with fixed velocities. When such jumps encounter each other, the field configurations become quite complicated, but after some time the jumps re-emerged unchanged, except for finite shifts of their centers x_0 (which can also be understood as time shifts $t \mapsto t + \Delta t$). Soliton solutions are obtained from the inverse scattering method by assuming ...*

Exercise 3.5 *Construct the soliton solutions to the sine-Gordon model using the GLM equations (see [1, Chap 13.6]).*

4 Quantum chains: the quantum inverse scattering method

In the last section, we generalized the concept of integrability from finite dynamical systems to dynamical systems with infinitely-many degrees of freedom, represented by PDEs. In the present section, we make a generalization in a "perpendicular" way: we keep (at least initially) a finite number of degrees of freedom, but we quantize the system. One could think of finitely-many particles in interaction, whose positions can lie in $\mathbb{R}$; here however we consider the space in which the classical degrees of freedom lie to be compact, so that we have a discrete basis for the Hilbert space. More precisely, each particle has spin degrees of freedom only. We further consider the case where the Hilbert space is finite-dimensional, where the total spin of each particle is fixed (here: 1/2). It will be important, however, to discuss the limit where the number of particle goes to infinity, whereby the dimensionality of the Hilbert space also becomes infinite.

For definiteness, consider the Heisenberg quantum chain with N sites. The Hilbert space is a tensor product of N spin-1/2 Hilbert spaces, $\mathcal{H}_N = (\mathbb{C}^2)^{\otimes N}$, and the Hamiltonian has the simple form

$$H = \frac{1}{4}\sum_{j=1}^{N}(\vec{\sigma}_j \cdot \vec{\sigma}_{j+1} - 1) \tag{4.1}$$

For simplicity, we will restrict ourselves to the case where the chain is periodic: $\vec{\sigma}_{N+1} \equiv \vec{\sigma}_1$. Note that the total-spin operator

$$\vec{S} := \frac{1}{2}\sum_{j=1}^{N}\vec{\sigma}_j \tag{4.2}$$

commutes with the Hamiltonian, $[H, \vec{S}] = 0$, reflecting the $sl(2)$-symmetry of the model.

Our first task, as usual, is to define integrability. It may appear straightforward as we have just performed a simple quantization, but it is in fact rather tricky. The problem is that for any finite N, the Hamiltonian (4.1) is just a finite-dimensional Hermitian matrix 2^N by 2^N, hence it can be diagonalized. In the diagonal basis, any other matrix that is diagonal will commute with the Hamiltonian. There are exactly 2^N such other, independent matrices, and they all commute with each other. Hence, we have automatically 2^N commuting conserved quantities in involution. Does that mean that any quantum chain is integrable – in fact, so integrable that instead of the N conserved quantities we could expect for N degrees of freedom, we have 2^N of them? Of course not. Again, there is no universally accepted, simple definition of integrability for quantum spin chains. However, again, the concept of locality can be of use.

Locality in quantum spin chains makes sense only when considering the *thermodynamic limit* $N \to \infty$. The proper way of doing this on the Hilbert spaces $\mathcal{H}_N$ is via the inductive limit, whereby a chain with N sites is seen as a N-site subchain of a chain with $N+1$ sites, etc. In this context, it makes sense to talk about an operator on the chain, for instance $h_1 := \vec{\sigma}_1 \cdot \vec{\sigma}_2$ or the operator $\vec{S}$ above, for any number of sites in the chain. The operator h_1 is supported on the sites 1 and 2, and the operator $\vec{S}$ on the whole chain: the support is where the operator is nontrivial (on sites away from the support, the operator has a tensor factor with is just the identity). For any such operator, we may also translate it to any other site, for instance $h_j := \vec{\sigma}_j \cdot \vec{\sigma}_{j+1}$, supported on the sites j and $j+1$. We may say that the quantum spin chain model is *local* if the Hamiltonian is the sum over sites j of a Hamiltonian density h_j that is supported on few sites situated a bounded distance (in site number) from j (here we use the natural topology on the chain, and the concept of a bounded distance is meaningful only with respect to the thermodynamic limit $N \to \infty$). Clearly, this is the case of the Heisenberg Hamiltonian. Likewise, an operator $\mathcal{O}$ is local if it is supported on a bounded number of sites. In fact, a more general definition of locality for a general

operator $\mathcal{O}$ is that its translates $\mathcal{O}_j$ commute with the Hamiltonian density h_k whenever $|j-k|$ is large enough (again, "large enough" is meaningful in the context of the thermodynamic limit):

$$[\mathcal{O}_j, h_k] = 0, \quad |j-k| \text{ large enough.} \tag{4.3}$$

We can also say that two operators $\mathcal{O}$ and $\mathcal{O}'$ are local with respect to each other if $[\mathcal{O}_j, \mathcal{O}'_k] = 0$ for $|j-k|$ large enough. By a slight abuse of language, and following the definition we took in the case of PDEs, a local conserved quantity will be a sum of the form

$$\sum_{j=1}^{N} \rho_j \tag{4.4}$$

where the density ρ_j is local.

Using these concepts, our notion of integrability is as follows:

Definition 4.1 *A local quantum spin chain model is integrable if there exists infinitely many local conserved quantities that are in involution, and whose densities are local with respect with each other.*

Being in involution, here, of course means that their commutators are zero.

Remark 4.1 *The concept of locality and the thermodynamic limit gives us a notion of dimensionality. Here: the Heisenberg chain, as the name indicate, is one-dimensional. Again, as for PDEs, it appears crucial for the standard tools of integrability that the quantum system be one-dimensional.*

Much like the case of integrable PDEs, there is a framework whereby local conserved quantities can be found, which can be used to get the simply-evolving degrees of freedom, and which can also be used also to solve the inverse problem of expressing the physical degrees of freedom (like the Pauli matrices $\vec{\sigma}_j$) in terms of the simply evolving ones. This is the quantum inverse scattering method, which includes the algebraic Bethe ansatz and the solution to the inverse problem. However, in the context of quantum spin chains, there is also a very direct and simple way of getting to the simply-evolving states, without the use of this framework. This is the slightly more straightforward coordinate Bethe ansatz. I will start with the latter.

4.1 Coordinate Bethe ansatz: "quantum hyper-tori" in the Hilbert space

We delve directly into the calculations of some energies / eigenstates. Note that

$$\vec{\sigma}_j \cdot \vec{\sigma}_{j+1} = \sigma_j^z \sigma_{j+1}^z + 2(\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+). \tag{4.5}$$

Then it is clear that the the vector $|\Omega\rangle$ where all spins are in the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of H:

$$H|\Omega\rangle = 0. \tag{4.6}$$

However this cannot be the ground state, as the state with all spin up must have the highest energy (the chain is anti-ferromagnetic). In order to find other states: consider a state $|v_j\rangle$ where there is only one spin down, at position j. Since spin is preserved, there will be eigenstates of the form

$$|\psi_1\rangle = \sum_j C_j |v_j\rangle. \tag{4.7}$$

Applying the Hamiltonian on it in order to have

$$H|\psi_1\rangle = E|\psi_1\rangle \tag{4.8}$$

we find

$$H|\psi_1\rangle = \sum_j C_j \left(\frac{1}{2}|v_{j+1}\rangle + \frac{1}{2}|v_{j-1}\rangle - |v_j\rangle \right) \tag{4.9}$$

Note that the parenthesis in (4.5) generates a motion of the spin down: one term deletes a spin down at j and creates one at $j+1$, and the other does the opposite. This is what is seen in (4.9). Physically, we may then expect that the solution will be wave-like: the spin down is like a particle, and the energy eigenstate is one where the particle is at many positions with oscillatory exponential factors.

From (4.9), shifting the summation index, we must have

$$C_{j-1} + C_{j+1} - 2C_j = 2EC_j \tag{4.10}$$

assuming periodicity, $C_{N+1} = C_1$. This can be solved in a standard fashion: $C_j = x^j$, and we have

$$1 + x^2 - 2(1+E)x = 0. \tag{4.11}$$

But the condition of periodicity means that $x^N = 1$, whence x must be a root of unity. Writing $x = e^{ip}$ for $p \in \{0, 2\pi/N, 4\pi/N, \ldots, 2\pi(1-1/N)\}$, we find

$$E = E(p) = \cos(p) - 1. \tag{4.12}$$

Hence for every p there is an energy $E(p)$. The number p is like a momentum, and the corresponding $E(p)$ is the dispersion relation for a single particle propagating with momentum p.

Of course, we could try to do something similar for the two-particle solution, using vectors $v_{j,k}$ with two spins down, at positions j and k. However, the equations become more complicated, and ever more so as the number of spins down increase. The Bethe ansatz is an ansatz, based on the one-particle (one-spin-down) solution above, for what the multi-particle states look like. In a sense, the fact that the Hamiltonian eigenvectors take the very specific form of the Bethe ansatz is the quantum-chain equivalent of the fact that the trajectories of classical dynamical integrable systems lie on hyper-tori; the Bethe ansatz states are the simply-evolving degrees of freedom (like the angle variables).

For the two-particle states, the Bethe ansatz says that eigenvectors have the form

$$|\psi_2\rangle = \sum_{j \neq k} e^{ip_1 j + p_2 k} S_{\mathrm{sign}(j-k)} |v_{j,k}\rangle. \tag{4.13}$$

Of course, there is some redundancy as $|v_{j,k}\rangle = |v_{k,j}\rangle$, but this is a convenient notation.

Remark 4.2 *Physically, this can be understood by saying that the particles at positions j and k, with momenta p_1 and p_2 respectively, are just free particles like in the one-particle solution above, except that they scatter nontrivially when they cross each other, with a scattering matrix described by $S_\pm$. This is very much like the soliton solutions of the sine-Gordon model, which evolve like free particles except for simple time-shifts after encountering each other.*

We may omit the requirement $j \neq k$ in (4.13) with the definition that $\text{sign}(0) = 0$ and that $S_0 = 0$. Acting with the Hamiltonian on (4.13) gives

$$\begin{aligned} H|\psi_2\rangle &= \sum_{j,k} e^{ip_1 j + p_2 k} S_{\text{sign}(j-k)} \Big[\\ &\frac{1}{2}\Big(\bar{\delta}_{j-1,k}|v_{j-1,k}\rangle + \bar{\delta}_{j+1,k}|v_{j+1,k}\rangle + \bar{\delta}_{j,k-1}|v_{j,k-1}\rangle + \bar{\delta}_{j,k+1}|v_{j,k+1}\rangle\Big) \\ &-(\bar{\delta}_{j,k+1} + \bar{\delta}_{j+1,k})|v_{j,k}\rangle\Big] \end{aligned} \tag{4.14}$$

where $\bar{\delta}_{j,k} = 1 - \delta_{j,k}$ (where $\delta_{j,k}$ is the *periodic* Kroneker delta-function). Note that $S_{\text{sign}(j+1-k)}\bar{\delta}_{j,k} = S_{\text{sign}(j-k)}\bar{\delta}_{j+1,k}$, etc. as long as "periodicity holds" (see below). Then,

$$\begin{aligned} H|\psi_2\rangle &= \sum_{j,k} e^{ip_1 j + p_2 k} S_{\text{sign}(j-k)} \Big[\\ &\frac{1}{2}\Big((e^{ip_1} + e^{-ip_2} - 2)\bar{\delta}_{j+1,k} + (e^{-ip_1} + e^{ip_2} - 2)\bar{\delta}_{j,k+1}\Big)\Big]|v_{j,k}\rangle \end{aligned} \tag{4.15}$$

Looking at the terms with $j \neq k, k \pm 1$, we see that we must have

$$E(p_1, p_2) = E(p_1) + E(p_2). \tag{4.16}$$

Looking at the terms with $j = k \pm 1$, and putting them together thanks to $|v_{j,k}\rangle = |v_{k,j}\rangle$, we find the condition

$$e^{ip_1} S_+(e^{ip_1} + e^{-ip_2} - 2) + e^{ip_2} S_-(e^{-ip_1} + e^{ip_2} - 2)) = 2E(e^{ip_1} S_+ + e^{ip_2} S_-). \tag{4.17}$$

The solution is

$$\frac{S_+}{S_-} = \frac{2e^{ip_2} - e^{i(p_1+p_2)} - 1}{2e^{ip_1} - e^{i(p_1+p_2)} - 1} = \frac{e^{i(p_2-p_1)/2} - \cos\frac{p_1+p_2}{2}}{e^{i(p_1-p_2)/2} - \cos\frac{p_1+p_2}{2}}. \tag{4.18}$$

Clearly $|S_+/S_-| = 1$, so we can choose $S_- = 1$ and take $S_+ = S_+(p_1, p_2)$ as a pure phase $S_+ = e^{i\varphi}$. Note that $S_+(p_1, p_2) = S_+(p_2, p_1)^{-1}$.

Finally, in the derivation above we assumed a certain kind of periodicity. Physically, it is the fact that passing one particle all around the quantum chain does not change the wave function. That is:

$$e^{ip_1 N} = e^{i\varphi(p_2,p_1)}, \quad e^{ip_2 N} = e^{i\varphi(p_1,p_2)}. \tag{4.19}$$

Mathematically, it is the fact that in the sum (4.13), we can shift the indices without affecting the result – so, for instance, we can sum for j form 1 to N or from 2 to $N+1$ (we used this in the derivation above). The basis vectors used are by definition periodic, so the difference between these two ways of summing lies in the treatment of the site number $1 \equiv N + 1$. In the first way it gets the factor $e^{ip_1} S_-$, in the second, the factor $e^{ip_1(N+1)} S_+$. The equality of these factors is the first equation of (4.19).

The log of equations (4.19) are

$$Np_1 = \varphi(p_2, p_1) + 2\pi I_1, \quad Np_2 = \varphi(p_1, p_2) + 2\pi I_2 \tag{4.20}$$

where I_1 and I_2 are integers. These are the Bethe ansatz equations, the solutions of which give the set of momenta for which there is a two-particle eigenstate. It turns out that multiparticle states are obtained similarly, with the same two-particle S-matrix. The energy is additive:

$$E(p_1, \ldots, p_n) = \sum_{j=1}^{n} E(p_j) \tag{4.21}$$

and the Bethe ansatz equations are most easily written using the parameters λ instead of the momenta p,

$$e^{ip} = \frac{\lambda + i/2}{\lambda - i/2}. \tag{4.22}$$

They are

$$\left(\frac{\lambda_j + i/2}{\lambda_j - i/2}\right)^N = \prod_{k \neq j}^{n} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i} \tag{4.23}$$

Then one may continue in a similar fashion for multi-particle states. In order to reach the ground state, we will have to get to the lowest spin subspace. The Hamiltonian is $SU(2)$ invariant, so the energy eigenstates organize themselves into $SU(2)$ representations; for N even, these are spins $0, 1, 2, \ldots$, for N odd, these are $1/2, 3/2, 5/2, \ldots$. The lowest energy, from the physical intuition of anti-ferromagnetism, is the subspace with the lowest spin. For N even, it is unique (spin-0 space is one-dimensional), for N odd it is twice degenerate (z-component of spin being $1/2$ or $-1/2$). In order to reach it, we will need to reach a state with either z-spin 0 (N even) or z-spin $1/2$ or $-1/2$. This means wee need to put $N/2$ or $(N \pm 1)/2$ particles. Below we show a more systematic way of doing this.

Exercise 4.1 *Formulate the Bethe ansatz for three particles, and check that with the scattering matrix above this is an eigenstate of the Hamiltonian.*

4.2 Algebraic Bethe ansatz

This is based on [2, Chaps 3,4], and on parts of [4] (here mostly following the notation of [2]).

4.2.1 Local conserved quantities

We start the formulation with the quantum Lax operator. This is an operator acting on the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$ of two 2-dimensional spaces, and depending on a spectral parameter λ. It will be seen as acting on the tensor product of the space of site n and an auxiliary space denoted a. Hence we will denote the operator $L_{n,a}$. For the Heisenberg model, the operator is defined by

$$L_{n,a}(\lambda) = \lambda 1_n \otimes 1_a + \frac{i}{2} \vec{\sigma}_n \cdot \vec{\sigma}_a \tag{4.24}$$

where $\vec{\sigma}_n$ acts on site n and $\vec{\sigma}_a$ on the auxiliary space (and $\vec{\sigma}_n \cdot \vec{\sigma}_a = \sum_j \sigma_n^j \sigma_a^j$). As a 2 by 2 matrix in the auxiliary space, with elements that are operators acting on site n, this is

$$L_{n,a} = \begin{pmatrix} \lambda + \frac{i}{2}\sigma_n^z & i\sigma_n^- \\ i\sigma_n^+ & \lambda - \frac{i}{2}\sigma_n^z \end{pmatrix} \tag{4.25}$$

Using the permutation operator

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\left(1 + \sum_j \sigma^j \otimes \sigma^j\right) \tag{4.26}$$

it can be written

$$L_{n,a}(\lambda) = \left(\lambda - \frac{i}{2}\right) 1_{n,a} + iP_{n,a}. \tag{4.27}$$

It is not obvious a priori that this Lax operator has anything to do with the Heisenberg Hamiltonian. But the Hamiltonian will appear as we construct the set of local quantities in involution – hence these will be a set of local conserved quantities in involution for the Heisenberg dynamics.

The main property of the Lax operator is that forming the 4 by 4 matrix of operators where the various matrix elements in (4.25) are multiplied two by two in every combination, following a tensor product structure, and with different spectral parameters, $L_{n,a_1}(\lambda)L_{n,a_2}(\mu)$, and forming something similar the 4 by 4 matrix where something similar is done with exchanged spectral parameter and exchanged tensor product structure, $L_{n,a_2}(\mu)L_{n,a_a}(\lambda)$, we see that we can relate both 4 by 4 matrices by a similarity transformation that depends on $\lambda - \mu$:

$$R_{a_1,a_2}(\lambda - \mu)L_{n,a_1}(\lambda)L_{n,a_2}(\mu) = L_{n,a_2}(\mu)L_{n,a_1}(\lambda)R_{a_1,a_2}(\lambda - \mu). \tag{4.28}$$

These are the *fundamental commutation relations*. The explicit form of the similarity transformation, the R-matrix $R_{a_1,a_2}(\lambda)$ acting on a tensor product of two auxiliary spaces a_1 and a_2, is

$$R_{a_1,a_2}(\lambda) = \lambda 1_{a_1,a_2} + iP_{a_1,a_2}. \tag{4.29}$$

This is very similar to the L operator.

Now we construct the monodromy matrix:

$$T_a = L_{N,a}(\lambda) \cdots L_{1,a}(\lambda). \tag{4.30}$$

Comparing with the construction in the integrable PDE case, the L matrix relates to the transport matrix for transport along one site, from which we form the monodromy matrix for transport along the whole chain (here, the meaning of monodromy is clearer: we are transporting along the a periodic path, the chain). Thanks to (4.28), it is a simple matter to check that

$$R_{a_1,a_2}(\lambda - \mu)T_{a_1}(\lambda)T_{a_2}(\mu) = T_{a_2}(\mu)T_{a_1}(\lambda)R_{a_1,a_2}(\lambda - \mu). \tag{4.31}$$

Exercise 4.2 *Derive (4.31) from (4.28).*

The operator T_a is a 2 by 2 matrix in the auxiliary space, with elements that act on the whole chain:

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{4.32}$$

One can check that $T_a(\lambda)$ is a polynomial in λ starting with λ^N (coefficient 1). Now, from (4.31), one obtains

$$R_{a_1,a_2}(\lambda - \mu)T_{a_1}(\lambda)T_{a_2}(\mu)R_{a_1,a_2}(\lambda - \mu)^{-1} = T_{a_2}(\mu)T_{a_1}(\lambda) \tag{4.33}$$

and tracing over the two auxiliary space a_1 and a_2 and using cylicity of the trace, one finds that

$$F(\lambda) := \mathrm{Tr}_a(T_a(\lambda)) = A(\lambda) + D(\lambda) \tag{4.34}$$

generates quantities in involution:

$$[F(\lambda), F(\mu)] = 0. \tag{4.35}$$

It turns that $F(\lambda)$ has the form

$$F(\lambda) = 2\lambda^N + \sum_{l=0}^{N-2} \tilde{Q}_l \lambda^l. \tag{4.36}$$

We can look at another type of expansion than that above. Instead of expanding $F(\lambda)$ about 0, we can expand $\log F(\lambda)$ about $i/2$ (why not...), and consider

$$Q_1 = \frac{dF(\lambda)}{d\lambda} F(\lambda)^{-1}\Big|_{\lambda=i/2}, \quad Q_2 = \frac{d^2F(\lambda)}{d\lambda} F(\lambda)^{-1}\Big|_{\lambda=i/2} - Q_1^2, \quad \dots \tag{4.37}$$

Two facts:

1. the Q_l are *local quantities*;
2. $Q_1 = \frac{1}{i}(2H + N)$ where H is the Heisenberg Hamiltonian.

These two facts show that the Heisenberg model is integrable (according to our definition). Showing these two facts is somewhat involved (but possible!), and is based on the important remark that

$$L_{n,a}(i/2) = iP_{n,a}, \quad \frac{dL_{n,a}(\lambda)}{d\lambda} = 1_{n,a}. \tag{4.38}$$

Basically, the kth derivative w.r.t. λ of $T_a(\lambda)$ at $\lambda = i/2$ produces a string of permutation operators except for k identity matrices at selected sites (we sum over all possible selection of k sites); tracing over the auxiliary space makes the string of permutation operators into a cyclic permutation running over all sites but skipping the selected ones; and multiplying by the inverse full cyclic permutation of the chain gives contributions around the selected sites. Taking the log guarantees that we take only the "connected part" of this, where only the terms where selected sites are next to each other survive. Since we sum over all selected sites, we see that we get local conserved quantities (summations over all sites of local densities).

Exercise 4.3 *Evaluate Q_1 and show Fact 2 above.*

4.2.2 Eigenvalues and eigenvectors

The set-up above, from which the conserved quantities were find, can also be used to find the Bethe ansatz eigenstates (those we found using the coordinate Bethe ansatz – but here we will work it out for any number of particles). This is in the same spirit as what happened in the case of the sine-Gordon integrable PDE, where we used the zero-curvature formulation for both purposes.

The idea here is to follow more or less the construction of eigenstates of the harmonic oscillator, or of the $SU(2)$ eigenstates. We start with a "highest-weight" vector, and from it go down by applying lowering operators. In the case of the harmonic oscillator the highest-weight vector is that of the canonical algebra; for the $SU(2)$, it is an $SU(2)$-highest weight. There are similar concepts of highest weights for any semisimple Lie algebra, as well as for more general infinite-dimensional Lie algebra (Kac-Moody, Virasoro). Here, we use this concept for yet another algebra, not a Lie algebra however, rather a set of "exchange relations". It is the algebra generated by the operators $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ via their exchange relation (4.31).

The explicit form of the R matrix (where the internal matrices act on the a_1 space, and the external acts on the a_2 space) is

$$R_{a_1,a_2}(\lambda) = \begin{pmatrix} \begin{pmatrix} a(\lambda) & 0 \\ 0 & b(\lambda) \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ c(\lambda) & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & c(\lambda) \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} b(\lambda) & 0 \\ 0 & a(\lambda) \end{pmatrix} \end{pmatrix} \tag{4.39}$$

where

$$a(\lambda) = \lambda + i, \quad b(\lambda) = \lambda, \quad c(\lambda) = i. \tag{4.40}$$

The explicit forms of the matrices T_{a_1} and T_{a_2} are, on the other hand,

$$T_{a_1}(\lambda) = \begin{pmatrix} \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \end{pmatrix} \tag{4.41}$$

and

$$T_{a_2}(\mu) = \begin{pmatrix} \begin{pmatrix} A(\lambda) & 0 \\ 0 & A(\lambda) \end{pmatrix} & \begin{pmatrix} B(\lambda) & 0 \\ 0 & B(\lambda) \end{pmatrix} \\ \begin{pmatrix} C(\lambda) & 0 \\ 0 & C(\lambda) \end{pmatrix} & \begin{pmatrix} D(\lambda) & 0 \\ 0 & D(\lambda) \end{pmatrix} \end{pmatrix}. \tag{4.42}$$

Using these we find that (4.31) implies

$$\begin{aligned} B(\lambda)B(\mu) &= B(\mu)B(\lambda) \\ A(\lambda)B(\mu) &= f(\lambda-\mu)B(\mu)A(\lambda) + g(\lambda-\mu)B(\lambda)A(\mu) \\ D(\lambda)B(\mu) &= h(\lambda-\mu)B(\mu)D(\lambda) + k(\lambda-\mu)B(\lambda)(D\mu) \end{aligned} \tag{4.43}$$

where

$$f(\lambda) = \frac{\lambda - i}{\lambda}, \quad g(\lambda) = \frac{i}{\lambda}, \quad h(\lambda) = \frac{\lambda + i}{\lambda}, \quad k(\lambda) = -\frac{i}{\lambda}. \tag{4.44}$$

These are the only exchange relations that we will use.

The highest-weight vector that we take is the vector $|\Omega\rangle$ defined above (4.6). We note that if two auxiliary-space matrices $L_a = \begin{pmatrix} u & v \\ w & z \end{pmatrix}$ and $L'_a = \begin{pmatrix} u' & v' \\ w' & z' \end{pmatrix}$, with entries being operators on the spin chain, are such that $w|\Omega\rangle = w'|\Omega\rangle = 0$ and $[w, u'] = 0$, then $L_a L'_a = \begin{pmatrix} u'' & v'' \\ w'' & z'' \end{pmatrix}$ is such that $w''|\Omega\rangle = 0$. This is exactly what happens in constructing T via (4.30). Hence, we deduce that

$$C(\lambda)|\Omega\rangle = 0. \tag{4.45}$$

So, $|\Omega\rangle$ can be taken as a highest-weight vector, and $C(\lambda)$ as a raising operator. Further, by similar arguments we can evaluate what's on the diagonal of the T operator:

$$T_a(\lambda)|\Omega\rangle = \begin{pmatrix} \alpha(\lambda)^N & * \\ 0 & \delta(\lambda)^N \end{pmatrix} |\Omega\rangle \tag{4.46}$$

where

$$\alpha(\lambda) = \lambda + \frac{i}{2}, \quad \delta(\lambda) = \lambda - \frac{i}{2}. \tag{4.47}$$

Hence, the vector $|\Omega\rangle$ is an eigenvector of $A(\lambda)+D(\lambda)$ (the trace of $T_a(\lambda)$); we already knew that it was an eigenvector of the Hamiltonian, now we find it is an eigenvector simultaneously of all local conserved quantities. It turns out that $B(\lambda)$ can be seen as a lowering operator, so that we may look for Hamiltonian eigenstates of the form

$$|\psi\rangle = B(\lambda_1)\cdots B(\lambda_l)|\Omega\rangle. \tag{4.48}$$

The technique is to write down the the eigenvalue equation for $A(\lambda)+D(\lambda)$ (that is, diagonalizing all local conserved quantities simultaneously), use the exchange relations to bring $A(\lambda)+D(\lambda)$ to the right, and find the values of λ_i that solve the resulting equation.

We start by looking at the action of $A(\mu)$. In exchanging A with B, we can use either of the two terms on the r.h.s. of (4.43). Let us write explicitly the result of using the first term always, and of using the second term on $B(\lambda_1)$ and then the first term always:

$$\begin{aligned} A(\mu)B(\lambda_1)\cdots B(\lambda_l)|\Omega\rangle &= \prod_{i=1}^{l} f(\mu-\lambda_i)\,\alpha(\lambda)^N B(\lambda_1)\cdots B(\lambda_l)|\Omega\rangle + \\ &\quad + g(\mu-\lambda_1)\prod_{i=2}^{l} f(\lambda_1-\lambda_i)\,\alpha(\lambda_1)^N B(\mu)B(\lambda_2)\cdots B(\lambda_l)|\Omega\rangle + \\ &\quad + \dots \end{aligned} \tag{4.49}$$

Other terms in this expression have a similar form to that of the second term written there, but with $B(\lambda_j)$ replaced by $B(\mu)$ for $j=2,\dots,l$. Note that the order of the B's doesn't matter thanks to the first equation in (4.43). This means that the l.h.s. is symmetric under exchange of λ_js, hence we know exactly what the other terms look like:

$$g(\mu-\lambda_j)\prod_{\substack{i=1\\ i\neq j}}^{l} f(\lambda_j-\lambda_i)\,\alpha(\lambda_j)^N B(\mu)B(\lambda_1)\cdots B(\lambda_{j-1})B(\lambda_{j+1})\cdots B(\lambda_l)|\Omega\rangle \tag{4.50}$$

The first term in (4.49) is what we want for an eigenvalue equation. We can do a similar calculation for the action of $D(\mu)$ and we find something of exactly the same form, with the replacements $g\mapsto k$, $f\mapsto h$ and $\alpha\mapsto\delta$. Adding both contributions, we only need to cancel all terms except those that look like the eigenvalue equation, so that we get

$$(A(\mu)+D(\mu))B(\lambda_1)\cdots B(\lambda_l)|\Omega\rangle = \alpha(\lambda)^N\prod_{i=1}^{l} f(\mu-\lambda_i) + \delta(\lambda)^N\prod_{i=1}^{l} h(\mu-\lambda_i) \tag{4.51}$$

as long as the following set of equations hold (here we uses the fact that $g(\lambda)=-k(\lambda)$):

$$\prod_{\substack{i=1\\ i\neq j}}^{l} f(\lambda_j-\lambda_i)\,\alpha(\lambda_j)^N = \prod_{\substack{i=1\\ i\neq j}}^{l} h(\lambda_j-\lambda_i)\,\delta(\lambda_j)^N. \tag{4.52}$$

This is exactly the equations (4.23), and one can check that we do recover exactly the correct eigenvalues of the Hamiltonian as predicted with the coordinate Bethe ansatz.

4.2.3 Solution to the inverse problem

Again much like in the case of the integrable PDEs where the inverse scattering method could be used in conjunction with the zero-curvature formulation, our algebraic Bethe ansatz not only allows us to find the

conserved charges, and the Hamiltonian eigenstates (solution of the "direct problem"), but also allows us to solve the "inverse problem" of finding the physical observables in terms of the "simply-evolving" (or action-angle-like) observables.

We have seen that eigenstates of $F(\lambda) = \mathrm{Tr}T(\lambda)$ (which generates all conserved quantities including the Hamiltonian) can be constructed by using the algebra generated by $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$. Hence, a solution to the inverse problem here will be understood as an inversion of the expression of these operators in terms of local spin operators: a writing of $\sigma_n^{\pm}$ and σ_n^z in terms of $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$.

The result and the proof are in fact rather simple. We have:

$$\begin{aligned} \sigma_n^- &= (A+D)^{n-1}\, B\, (A+D)^{-n} \\ \sigma_n^+ &= (A+D)^{n-1}\, C\, (A+D)^{-n} \\ \sigma_n^z &= (A+D)^{n-1}\, (A-D)\, (A+D)^{-n} \end{aligned} \tag{4.53}$$

where A, B, C and D are all evaluated at $\lambda = i/2$.

The proof goes as follows. Recall the first equation of (4.38). This means that

$$A + D = \mathrm{Tr}_a T_a = i^N \mathrm{Tr}_a(P_{N,a} \cdots P_{1,a}) = i^N \mathrm{Tr}_a(P_{1,a} P_{N,1} \cdots P_{2,1}) = i^N P_{N,1} \cdots P_{2,1} \tag{4.54}$$

using $\mathrm{Tr}_a(P_{j,a}) = 1$. Note that $P_{N,1} \cdots P_{2,1} = U$ is just the cyclic permutation operator. Then we find

$$\begin{aligned} B &= \mathrm{Tr}\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right] \\ &= \mathrm{Tr}_a(\sigma_a^- T_a) \\ &= i^N \mathrm{Tr}_a(\sigma_a^- P_{N,a} \cdots P_{1,a}) \\ &= i^N \mathrm{Tr}_a(P_{1,a} \sigma_1^- P_{N,1} \cdots P_{2,1}) \\ &= i^N \sigma_1^- P_{N,1} \cdots P_{2,1} \\ &= \sigma_1^- \, (A+D) \end{aligned} \tag{4.55}$$

so that $\sigma_1^- = B(A+D)^{-1}$. In order to get to other sites, we just have to conjugate with the cyclic permutation operator $U = i^{-N}(A+D)$ (this is a translation operator on the chain!):

$$\sigma_n^- = (A+D)^{n-1} B (A+D)^{-n}. \tag{4.56}$$

Similarly for other operators.

Writing local sigma matrices in terms of A, B, C and D is very useful, as one can then use the exchange relations (4.43) and get, in combination with an analysis of the Bethe ansatz equations, useful and quite powerful integral equations for correlation functions.

4.2.4 Yang-Baxter equations

As is clear form the derivation above, the crucial relation is (4.28) (or equivalently (4.31). Such an equation can only hold if the R matrix satisfies certain consistency conditions. These are the Yang-Baxter equations. They are obtained as follows (here we keep implicit the spectral parameters, as they

are associated with the space index):

$$\begin{aligned} T_{a_1}T_{a_2}T_{a_3} &= \mathrm{Ad}(R_{a_2,a_1})T_{a_2}T_{a_1}T_{a_3} \\ &= \mathrm{Ad}(R_{a_2,a_1}R_{a_3,a_1})T_{a_2}T_{a_3}T_{a_1} \\ &= \mathrm{Ad}(R_{a_2,a_1}R_{a_3,a_1}R_{a_3,a_2})T_{a_3}T_{a_2}T_{a_1} \end{aligned}$$

and

$$\begin{aligned} T_{a_1}T_{a_2}T_{a_3} &= \mathrm{Ad}(R_{a_3,a_2})T_{a_1}T_{a_3}T_{a_2} \\ &= \mathrm{Ad}(R_{a_3,a_2}R_{a_3,a_1})T_{a_3}T_{a_1}T_{a_2} \\ &= \mathrm{Ad}(R_{a_3,a_2}R_{a_3,a_1}R_{a_2,a_1})T_{a_3}T_{a_2}T_{a_1} \end{aligned}$$

where we use the group Adjoint action for conciseness,

$$\mathrm{Ad}(R)Q = RQR^{-1} \tag{4.57}$$

(and note that $R_{a_1,a_2} \equiv R_{a_1,a_2}(\lambda_1 - \lambda_2)$, etc.). Consistency is guaranteed if

$$R_{a_2,a_1}R_{a_3,a_1}R_{a_3,a_2} = R_{a_3,a_2}R_{a_3,a_1}R_{a_2,a_1} \tag{4.58}$$

These are the *Yang-Baxter equations*. Solutions to the Yang-Baxter equations correspond to integrable models, for which the methods above often can be applied.

5 Quantum field theory: factorized scattering theory

Now we combine both difficulties coming from the infinite number of degrees of freedom and quantum mechanics: we consider QFT. Relativistic QFT can be understood in various fashions.

Note: This section is mostly taken from my lecture notes for a set of lectures on integrable QFT. There is much more material in this section than what is taught in the present 2+1-hour lecture. However, most subjects should be covered in class, if not in as many details as there are here. The basic reference on factorized scattering theory is [6].

5.1 Scaling limit: local fields description

The first is as a quantization of a field equation, like the sine-Gordon equation. This however causes problems: from standard QFT material we know that renormalization is necessary. The underlying cause is that in quantum mechanics, thanks to Feynman's description we know that the possible trajectories of the particles are very "erratic" (a bit like Brownian motions). But when each particle is a part of a continuous field (like the sine-Gordon field), the erratic trajectories mean that parts of the field at nearby point x and $x + \Delta x$ will have a good probability of being far apart: there is no reason for the resulting field to be differentiable, hence the energy (which involves the derivative of the field) will be infinite. This infinity is at the basis of the need for renormalization: one needs to first consider the field to be divided into separate parts, with a nonzero separation "distance", then one quantizes in the usual way, and then to take the limit where this distance goes to zero and at the same time the parameters (for instance the "elastic constant" telling us how much energy there is when nearby points are far apart)

of the Hamiltonian are readjusted in order to keep the ground state and low-lying excited state energies finite.

The renormalization process can be put in a more solid and much deeper framework through the concept of scaling limit. Note that the classical equation has to be discretized then quantized (this is the "regularization" process) before renormalization. But renormalization can be applied to a wealth of quantum models with infinitely-many degrees of freedom. The leading idea behind it is that of quantum critical point. Let us give an example with a model similar to that studied in the previous section.

Consider the so-called "XXZ" Heisenberg spin chain: the Hamiltonian is a slight generalization of the Heisenberg spin chain, still integrable,

$$H = J \sum_j \left(\sigma_j^{\mathrm{x}} \sigma_{j+1}^{\mathrm{x}} + \sigma_j^{\mathrm{y}} \sigma_{j+1}^{\mathrm{y}} + \Delta \sigma_j^{\mathrm{z}} \sigma_{j+1}^{\mathrm{z}} \right)$$

(again, here we take $J > 0$). The parameter Δ is an anisotropy: the z direction of the magnetic interaction is different.

Many of the quantities of physical interest are obtained from *correlation functions*, averages of products of observables of local degrees of freedom at different sites of the chain. For $\Delta > 1$, complicated calculations using integrability for instance show that correlation functions for two separated sites decrease exponentially at large separations, for instance:

$$\langle \sigma_j^{\mathrm{y}} \sigma_k^{\mathrm{y}} \rangle_{XXZ} \propto e^{-\frac{|j-k|}{\xi}} \quad \text{as} \quad |j-k| \to \infty$$

for some *correlation length* $\xi = \xi(\Delta)$. This correlation length is a function of the parameter Δ. Hence if we look at very large distances, we see no quantum correlations at all; this seems natural, since the interaction is just between nearest neighbors.

But the exact expression of $\xi(\Delta)$ can be obtained, and shows that the correlation length tends towards infinity as $\Delta \to 1^+$. Since correlation functions summed over all sites give rise to *response functions*, this will imply singularities in the magnetic susceptibility, for instance, so that this is a second order (quantum) phase transition. Exactly at $\Delta = 1$, we have

$$\langle \sigma_j^{\mathrm{y}} \sigma_k^{\mathrm{y}} \rangle_{XXX} \propto |j-k|^{-2d} \quad \text{as} \quad |j-k| \to \infty$$

where d is come positive number (known exactly). The point $\Delta = 1$ is a quantum critical point of the family of XXZ models.

The scaling limit is obtained by looking at very large distances $|j-k|$ while at the same time taking the limit $\Delta \to 1^+$. In general, take any correlation function of local operators $\mathcal{O}_{j_1}^{(1)}, \mathcal{O}_{j_2}^{(2)}$ at positions $j_1, j_2, \ldots$. Then take the limit where all positions j_1, $j_2, \ldots$ are scaled up while at the same time the correlation length is sent to infinite (i.e. $\Delta \to 1^+$), with constant ratios j_1/ξ, $j_2/\xi, \ldots$. This limit as such is zero; but now take the same limit, where the correlation function is multiplied by $\xi^{d_1} \xi^{d_2} \cdots$ for appropriate constants $d_1, d_2, \ldots$ that depend on which operators are at positions $j_1, j_2, \ldots$ respectively[3]. Then the result of this limit is nonzero. This is the continuum or scaling limit, and the result of the limit is described by QFT. The result is largely independent of the details of the underlying quantum chain: different quantum chains will give rise to the same limits of correlation functions. This is universality of

[3]In fact, more generally, one has to choose families of local operators parametrized by ξ, corresponding to general renormalization; here we are choosing the families $(\xi^{2d_n} \mathcal{O}_{j_n}^{(n)} : \xi > 0)$, corresponding to multiplicative renormalization.

the scaling limit: models fall into universality classes. QFT is a theory describing the universality classes. In equation, for instance for the 2-point function

$$\lim_{\Delta\to 1^+} \xi^{2d}\langle\sigma^{\rm y}_{[mx\xi]}\sigma^{\rm y}_{[mx'\xi]}\rangle_{XXZ} = \langle{\rm vac}|\mathcal{O}(x)\mathcal{O}(x')|{\rm vac}\rangle$$

where the limit is taken keeping the real parameters mx and mx' fixed (and $[mx\xi]$ mean that we must take the integer part of $mx\xi$, just so that we sit on a site of the chain). On the right-hand side, we have a correlation function in a QFT with a mass m, and the operator $\mathcal{O}(x)$ is now an appropriate local field (see below) "representing" $\sigma^{\rm y}$ in the QFT. The unique (positive) number d making the limit finite is called the dimension of the field $\mathcal{O}$. The QFT correlation function is a "scaling function". In this case, the QFT is integrable: it is the quantum sine-Gordon model (at a very special point, equivalent (up to a free boson) to the $SU(2)$-Thirring model).

This local-field point of view leads to the following: a QFT model will be completely specified (hence all physical information can be extracted, at least in principle) once we have

- **Hilbert space:** a Hilbert space $\mathcal{H}$.
- **Hamiltonian:** an operator $\hat{H}$ acting on $\mathcal{H}$, called the Hamiltonian, which is diagonalizable and whose eigenvalues are bounded from below; the vector associated with the lowest eigenvalue is the vacuum, $|{\rm vac}\rangle \in \mathcal{H}$, and the lowest eigenvalue is set to 0.
- **Relativistic invariance:** operators $\hat{P}$ (the momentum operator) and $\hat{B}$ (the boost operator) which satisfy
$$[\hat{H},\hat{P}] = 0\ ,\quad [\hat{B},\hat{P}] = i\hat{H}\ ,\quad [\hat{B},\hat{H}] = i\hat{P} \tag{5.1}$$
as well as
$$\hat{H}|{\rm vac}\rangle = \hat{P}|{\rm vac}\rangle = \hat{B}|{\rm vac}\rangle = 0. \tag{5.2}$$
- **Locality:** local fields $\hat{h}(x)$ and $\hat{p}(x)$ such that
$$\hat{H} = \int dx\,\hat{h}(x)\ ,\quad \hat{P} = \int dx\,\hat{p}(x)$$

Above we say that an x-dependent operator $\mathcal{O}(x)$ is a local field if $[\mathcal{O}(x), h(x')] = 0$ for $x \neq x'$ and if additionally

$$[\hat{P},\mathcal{O}(x)] = i\frac{d}{dx}\mathcal{O}(x). \tag{5.3}$$

5.2 Relativistic particles: asymptotic state description

The second way of understanding QFT is as what it was invented for: a theory for relativistic particles in interaction. In this way, we describe the Hilbert space by giving the type of particles we can have: the set of masses $\{m_a : a \in I\}$ where I is the set of particle types, and their associated quantum numbers $\{q_a : a \in I\}$, like their $U(1)$ charges, etc. For instance, if we have a positron and an electron, then $I = \{p, e\}$ and both particles have the same mass, $m_e = m_p$, while different $U(1)$ charges, $q_e = -1$, $q_p = 1$. Physically, what QFT is describing here are all possible scattering experiments. So, we have a basis of states which correspond to particles coming from far in the past (the *in* states), and a basis which correspond to particles going far in the future (the *out* states). These are bases of *asymptotic particles*.

Since all particles are supposed to scatter around a finite region of space, in order to specify a vector in the *in* basis, say, we only need to provide the particle types involved and the corresponding set of momenta; we know that the one with highest positive momentum comes from the far past and far left, while the one with the highest negative momentum comes from the far past and far right; similarly for the *out* basis. The two bases are describing the same Hilbert space, and are written as follows:

$$|\theta_1,\dots,\theta_n\rangle^{in,out}_{a_1,\dots,a_n} : \begin{array}{ll} \theta_1 > \cdots > \theta_n \in \mathbb{R} & (in) \\ \theta_1 < \cdots < \theta_n \in \mathbb{R} & (out) \end{array}, \ a_1,\dots,a_n \in I. \tag{5.4}$$

We denote $|\text{vac}\rangle$ the state without any particle, common to both bases. Here we use the rapidity θ, which parametrizes the on-shell energy and momentum of the particles found in the far part of far future:

$$E_{\theta,a} = m_a \cosh\theta, \quad p_{\theta,a} = m_a \sinh\theta. \tag{5.5}$$

We have also chosen, for convenience, an ordering of the rapidity that matches our physical intuition about where the particles are in the far past or far future.

This particle description can be connected to the quantum field idea: we can construct asymptotic states using local fields. This is often done via the so-called LSZ formula (see any QFT textbook). The intuition is that asymptotic states should be understood as particular, complicated linear combinations of field-configuration states, which are such that, evolved back towards the far past or forward towards the far future, they look, in a sense, like Gaussian envelops of free waves. We may describe asymptotic states using creation / destruction operators (this is a reformulation of the LSZ formula) $A^\dagger(\theta)^{(in/out)}$, $A(\theta)^{(in/out)}$ defined by

$$A_a(\theta)^{(in,out)} = i \lim_{L\to\infty} \lim_{t\to\mp\infty} \int dx \, (f_\theta(x,t)\partial_t \Psi_a(x,t) - \partial_t f_\theta(x,t)\Psi_a(x,t)) \tag{5.6}$$

with

$$f_\theta(x,t) = \exp\left[im\cosh(\theta)\, t - im\sinh(\theta)\, x - \frac{(x-\coth(\theta)t)^2}{L^2} \right], \tag{5.7}$$

and where Ψ_a is (here bosonic with spin 0 for simplicity) a fundamental local field associated to the particle of type a being created / destroyed. Asymptotic *in* states, for instance, are

$$A^\dagger_{a_1}(\theta_1)^{(in)} \cdots A^\dagger_{a_n}(\theta_n)^{(in)} |\text{vac}\rangle. \tag{5.8}$$

Finally, we introduce the notion of *impact parameters*, as this will be useful below. The impact parameters are essentially the absolute positions of the particles with respect to each other as they come to time 0 and all scatter with each other. Of course, at time 0 there is no well-defined notion of particle, because the fields become very complicated, but we can define impact parameters as the absolute positions associated to straight trajectories of imaginary, noninteracting particles that would follow the physical particles from the far past say. This is most easily implement into the creation / destruction operators: we modify the wave-packet factors into

$$f_\theta(x,t) = \exp\left[im\cosh(\theta)\, t - im\sinh(\theta)\, x - \frac{(x-x_0-\coth(\theta)t)^2}{L^2} \right], \tag{5.9}$$

where x_0 is the impact parameter associated to the particle created.

The nontrivial physical information from this viewpoint is the *scattering matrix*, the overlap between the *in* states and the *out* states:

$$\prescript{(in)}{a_1,\dots,a_n}{\langle}\theta_1,\dots,\theta_n|\theta'_1,\dots,\theta'_n\rangle^{out}_{a'_1,\dots,a'_n} \tag{5.10}$$

This scattering viewpoints leads to the following: a QFT model will be completely specified (all physical information can be extracted) once we have

- **Hilbert space:** a Hilbert space $\mathcal{H}$, described as above.
- **Hamiltonian:** an operator $\hat{H}$ acting on $\mathcal{H}$, called the Hamiltonian, whose eigenvalues on the vectors in the bases above is just the sum of the energies $\sum_j m_{a_j} \cosh\theta_j$.
- **Relativistic invariance:** operators $\hat{P}$ (the momentum operator) and $\hat{B}$ (the boost operator) which satisfy
$$[\hat{H}, \hat{P}] = 0 \ , \quad [\hat{B}, \hat{P}] = i\hat{H} \ , \quad [\hat{B}, \hat{H}] = i\hat{P} \tag{5.11}$$
as well as
$$\hat{H}|\text{vac}\rangle = \hat{P}|\text{vac}\rangle = \hat{B}|\text{vac}\rangle = 0. \tag{5.12}$$
- **Scattering matrix:** the overlaps (5.10) between *in* and *out* states.

Hence, quite interestingly, we are saying that the knowledge of the scattering matrix is equivalent to the knowledge of the local energy and momentum densities $h(x)$ and $p(x)$. In the direction from local fields to scattering states this is implemented by the usual LSZ formula, but in the opposite direction it is less obvious. In integrable models, the opposite direction is the inverse problem, and is very natural.

5.3 Integrable QFT, and the free boson example

The notion of integrability in QFT follows directly from that of integrability both in quantum spin chains and in classical field theory.

Definition 5.1 *A model of QFT is integrable if there exists enough, which will be infinitely many, conserved quantities $Q_j = \int dx\, q_j(x)$ in involution, with densities $q_j(x)$ that are local (Q_j are sometimes called "local conserved charges").*

Making connection with the previous concepts:

- The *direct* problem will be to provide a the full scattering-state description of the model, like in the case of spin chains where we described the Bethe ansatz eigenstates. All we need to do is to provide the particle types and the scattering matrix. We will see how integrability has strong consequences on the type of scattering matrices we can have. This direct problem is like the construction of the action-angle variables, and the simplification of the scattering matrix that ensues from the presence of the conserved quantities is the QFT equivalent of the Bethe ansatz form of the eigenstates in quantum chains, or the fibration of the classical phase space into tori.
- The *inverse* problem will be to provide matrix elements, in the scattering-state basis, for local operators. This is like the re-construction of the local spin operators in terms of $A(\lambda), B(\lambda), C(\lambda), D(\lambda)$ operators in the quantum inverse scattering method, or like the GLM equation in order to reconstruct the field in terms of the scattering data in the classical inverse scattering method. This inverse problem is generally more difficult, and is performed via Karowski's and Smirnov's system

of equations for form factors (analytic continuation in rapidity space of matrix elements). This inverse problem gives a strong relation between the two formulations of QFT (local fields and scattering states): the form factor equations contain, as an input data, the scattering matrix, and the set of solutions to the form factor equations is assumed to be equivalent to the set of local fields of the QFT, including in particular the energy and momentum densities (energy-momentum tensor). Hence in integrable QFT, the inverse problem has the deep meaning of completing the equivalence between the local-field and scattering-state descriptions of QFT.

5.3.1 Local conserved charges: case of the free boson

An integrable model has an infinity of local conserved charges. It is very instructive to see what they look like in a simple example. The most simple example possible of conserved charges in an integrable model are those of the free boson. The Hamiltonian of the free boson is

$$H = \int dx \, \frac{1}{2} \left[\pi^2(x) + \left(\frac{\partial}{\partial x} \phi(x) \right)^2 + m^2 \phi^2(x) \right]$$

with the two local fields $\phi(x)$ and $\pi(x)$ satisfying

$$[\phi(x), \pi(x')] = i\delta(x - x') \ .$$

The momentum operator is simply expressed as

$$P = - \int dx \, \pi(x) \frac{\partial}{\partial x} \phi(x) \ .$$

Now, there are many very simple conserved charges in involution that can be constructed:

$$P_{(n)} = (-1)^{\frac{n+1}{2}} \int dx \, \pi(x) \left(\frac{\partial}{\partial x} \right)^n \phi(x)$$

for any n positive and odd, which have action

$$[P_{(n)}, \phi(x)] = \left(i \frac{\partial}{\partial x} \right)^n \phi(x) \ , \quad [P_{(n)}, \pi(x)] = \left(i \frac{\partial}{\partial x} \right)^n \pi(x) \ .$$

A simple way to check that these are indeed conserved charges in involution is as follows. First, we can check explicitly, from the equations of motion for the fields, that

$$[P_{(n)}, [H, \phi(x)]] = [H, [P_{(n)}, \phi(x)]] \ , \quad [P_{(n)}, [H, \pi(x)]] = [H, [P_{(n)}, \pi(x)]] \ .$$

This means, using Jacobi idendities, that

$$[[H, P_{(n)}], \phi(x)] = [[H, P_{(n)}], \pi(x)] = 0 \ .$$

Hence, in general

$$[[H, P_{(n)}], \mathcal{O}] = 0$$

for any $\mathcal{O}$ formed out of linear combinations of products of fields ϕ and π. We can in particular form products of creation and annihilation operators this way. If additionnaly $[H, P_{(n)}]|\text{vac}\rangle = 0$, for instance it is sufficient that $P_{(n)}|\text{vac}\rangle = 0$, then

$$[H, P_{(n)}] = 0$$

on the Hilbert space. The same arguments can be used for proving involution.

Other charges are

$$H_{(n)} = (-1)^{\frac{n-1}{2}} \int dx \, \frac{1}{2} \left[\pi(x) \left(\frac{\partial}{\partial t} \right)^n \phi(x) - \phi(x) \left(\frac{\partial}{\partial t} \right)^n \pi(x) \right]$$

for odd positive n, where the time derivatives $\partial/\partial t$ should really be replaced by what is obtained from the equation of motion: $\partial/\partial t \cdot = i[H, \cdot]$. These charges have actions

$$[H_{(n)}, \phi(x)] = \left(-i \frac{\partial}{\partial t} \right)^n \phi(x) \ , \quad [H_{(n)}, \pi(x)] = \left(-i \frac{\partial}{\partial t} \right)^n \pi(x)$$

with the same meaning for the time derivatives. This can be obtained by using explicitlt the equations of motion $(\partial/\partial x)^2 - (\partial/\partial t)^2 = m^2$ on ϕ and π, and doing integration by part. In similar ways, one can imagine how to construct local conserved charges whose action on the fields $\phi(x)$ and $\pi(x)$ are of the type

$$[Q_{(n,k)}, \phi(x)] = \left(i \frac{\partial}{\partial x} \right)^n \left(-i \frac{\partial}{\partial t} \right)^k \phi(x) \ , \quad [Q_{(n,k)}, \pi(x)] = \left(i \frac{\partial}{\partial x} \right)^n \left(-i \frac{\partial}{\partial t} \right)^k \pi(x)$$

with $n + k$ positive and odd. They are not all independent, because of the equations of motion. Indeed, we always have, for instance, $(-\partial^2/\partial t^2 + \partial^2/\partial x^2)\phi(x) = m^2 \phi(x)$. In order to get the set of independent charges, it is convenient to use the coordinates $z = x - t$ and $\bar{z} = x + t$. Then, one can combine the charges $Q_{(n,k)}$ to form charges that act through

$$\left(\frac{\partial}{\partial z} \right)^{n'} \left(\frac{\partial}{\partial \bar{z}} \right)^{k'} \ .$$

But clearly, $\partial/\partial z \, \partial/\partial \bar{z} = m^2/4$ as a consequence of the equations of motion, so that we are left with only $\partial/\partial z$ derivatives or $\partial/\partial \bar{z}$ derivatives. Hence, the independent conserved charges are Q_s for odd (positive or negative) integer s whose action on the fields $\phi(x)$ and $\pi(x)$ are examplified by

$$[Q_s, \phi(x)] = \left(\frac{\partial}{\partial z} \right)^s \phi(x) \ (s > 0) \ , \quad [Q_s, \phi(x)] = \left(\frac{\partial}{\partial \bar{z}} \right)^s \phi(x) \ (s < 0) \ .$$

Under the action of the boost operator B, these charges have spin s, that is, $[B, Q_s] = sQ_s$. This is most intuitively seen using mode operators that we saw earlier in the most general constext, with commutation relations $[A(\theta), A^\dagger(\theta')] = 4\pi\delta(\theta - \theta')$. Here, the Hamiltonian and momentum operators are

$$H = \int \frac{d\theta}{4\pi} \, m \cosh(\theta) A^\dagger(\theta) A(\theta) \ , \quad P = \int \frac{d\theta}{4\pi} \, m \sinh(\theta) A^\dagger(\theta) A(\theta) \ ,$$

and the conserved charges of well-defined spins that we alluded to are simply written, in terms of modes, as

$$Q_s = \int \frac{d\theta}{4\pi} \, e^{s\theta} A^\dagger(\theta) A(\theta) \ .$$

In particular, $Q_{\pm 1} = m^{-1}(H \pm P)$.

Here it seems not to be possible to construct local conserved charges of even spins. For free fermions, similar arguments can be made, but in this case we will have conserved charges of all integer spins. That is, in the free theories, we have infinitely many independent local Hermitian conserved charges indexed by integer spins (we just constructed even ones for real bosons). The same type of charge occurs in any free theory, with or without internal degrees of freedom.

5.4 Elastic scattering

Let us now derive one consequence of the presence of an infinity of local conserved charges in involution in interacting models. Recall that those are local charges that commute with the Hamiltonian and with each other. Besides the Hamiltonian, there is one that is trivial: the momentum. From these two charges, it is convenient, as above, to build two different, with better transformation properties under boost, and without scale:

$$Q_1 = m^{-1}(H+P) \ , \quad Q_{-1} = m^{-1}(H-P)$$

where m is the mass of one of the particles of the theory (arbitrarily chosen, this is just a normalisation of the operator; for instance, take the lowest mass). These charges have eigenvalues

$$Q_{\pm 1}|\theta_1,\theta_2,\ldots\rangle^{in,out}_{a_1,a_2,\ldots} = \sum_k q_{a_k} e^{\pm\theta_k}|\theta_1,\theta_2,\ldots\rangle^{in,out}_{a_1,a_2,\ldots}$$

for $q_{a_k} = m_{a_k}/m$ where m_{a_k} is the mass of the particle of type a_k, and transform with spin ± 1 under boost.

Remember what locality of a charge essentially means: the charge density $q(x)$ commutes with the Hamiltonian density $h(x')$ at different points $x \neq x'$. In order to derive consequences of conserved charges on the scattering matrix, we need something just slightly stronger: the charge density $q(x)$ must commute with a "fundamental" fields creating particles $\Psi(x')$ at different points:

$$Q = \int dx\, q(x) \ , \quad [q(x),\Psi(x')] = 0 \quad (x \neq x') \ .$$

We say that not only the charge Q is local (that is, its density $q(x)$ is a local field), but also that it is local *with respect to a fundamental fields* (that is, its density is a field local with respect to the fundamental fields). I will come back to the various extension of the concept of locality a bit later in the course. What we say here, in other words, is that the densities of the conserved charges must be in the consistent set of local observables that include fields that create the particles.

Then, a conserved charge applied on an asymptotic state with many particles will act independently on each of them. In fact, the conserved charges that we are interested in are Hermitian local conserved charges that act on asymptotic states (essentially very separated particles propagating freely) in the same way as conserved charges of free models (see previous sub-section) act on free particles. That is, they are conserved charges corresponding to some space-time symmetries, and can always be made to commute with possible internal symmetry charges of the model. As we saw, we can always arrange the charges to have definite spin, so that by relativistic covariance, we have, for some infinite set of spins s,

$$Q_s|\theta_1,\theta_2,\ldots\rangle^{in,out}_{a_1,a_2,\ldots} = \sum_k q^{(s)}_{a_k} e^{s\theta_k}|\theta_1,\theta_2,\ldots\rangle^{in,out}_{a_1,a_2,\ldots} \ .$$

Note that these charges are automatically in involution.

We can now understand the physical meaning of the numbers $q^{(s)}_{a_k}$. Of course, if there is only one type of particle, we can normalise the charges to put all these numbers to 1. But if there are many types, this is not so. However, thanks to the fact that the Q_s commute with any internal symmetry, these numbers will be the same if different particle types fall into a symmetry multiplet. In general, we expect also that they will be different if not. Hence, these numbers tell us about the belonging to symmetry multiplets.

A comment about impact parameters is in order here. We are not writing them explicitly in the states above, but the action of a charge may involve a modification of the impact parameters (something which

we do not see in free theories because the states are independent of impact parameters). We will analyse this in more details below, but for now we just keep in mind that we can choose them as we wich, in such a way that the conditions we derive below are valid for scattering matrices of any impact parameters.

The values of the spins s for which there is a conserved charge and the numbers $q_{a_k}^{(s)}$ are usually good fingerprint of a given integrable model (although there isn't a one-to-one relation). Now, in the equation above, take the *in* states, and sandwich the whole equation with the conjugate *out* state

$$ {}^{out}_{a'_1,a'_2,\dots}\langle\theta'_1,\theta'_2,\dots| \ .$$

We see that the overlap between an *in* state and an *out* state is non-zero only if the following equation is satisfied:

$$\sum_k q_{a_k}^{(s)} e^{s\theta_k} = \sum_k q_{a'_k}^{(s)} e^{s\theta'_k} \ .$$

This is an infinite set of equations (for all spins s occuring in the model) for an indefinite number of the rapidities θ'_k and particle types a'_k, being given the rapidities θ_k and the particle types a_k. One solution is obviously

$$\{\theta'_k\} = \{\theta_k\} \ , \quad q_{a'_k}^{(s)} = q_{a_k}^{(s)} \text{ for } \theta'_k = \theta_k \ .$$

In fact, if we require the solution for the out-going rapidities to be $\{\theta'_k\} = \{\theta_k\}$ for all sets of in-going rapidities $\{\theta_k\}$, then the only possibility for the particle types is $q_{a'_k}^{(s)} = q_{a_k}^{(s)}$ for $\theta'_k = \theta_k$. The meaning of these equalities is that particle types must be the same in the *in* and *out* states, and particles can exchange their rapidities only if they are part of a symmetry multiplets. I am talking of exchanging rapidities, because if particles are part of a symmetry multiplets, conservation of this symmetry implies that particle types *in that symmetry multiplet* must also be the same in the *in* and *out* states. This is elastic scattering: the number of particles going out is the same as that coming in, and they have the same rapidities, with possible exchanges and possible modifications of the particle types.

We would indeed like to conclude elastic scattering, but for this we need to do a little bit more analysis from this argument. In principle there may be other solutions than $\{\theta'_k\} = \{\theta_k\}$, depending on how the numbers $q_{a_k}^{(s)}$ behave as function of s. The simplest way to see this is to construct the following analytic function of α, defined by its expansion at large α:

$$f_{a_k}(\alpha) = \sum_{s>0} e^{-s\alpha} q_{a_k}^{(s)}$$

where the sum is over all spins that occur in the model; we assume this series to be convergent for α large enough. Our condition, for positive spins, then becomes

$$\sum_k f_{a_k}(\alpha - \theta_k) = \sum_k f_{a'_k}(\alpha - \theta'_k) \ \forall\, \alpha$$

(we can do something similar for negative spins; if we have parity symmetry, it is sufficient to look at positive spins). If the function $f_{a_k}(\alpha)$ has only one singular point on the real line at the same position for all particle types a_k, then the only possibility, for any real $\{\theta_k\}$, is indeed $\{\theta'_k\} = \{\theta_k\}$.

Take, for instance, the situation where there is only one particle type, or when there are many but thay all fall into one multiplet of some internal symmetry. Then, we must have $q_{a_k}^{(s)} = q_{a'_k}^{(s)}$ so that they can all be set to 1. If, for instance, only odd spins are involved (this is the situation in the $SU(2)$-Thirring model, for instance), the analytical function $f(\alpha)$ is

$$f(\alpha) = \frac{1}{2\sinh(\alpha)} \ .$$

This indeed only has one pole on the real line, and the conclusion follows.

But take, for instance, a fictitious case where we only have odd spins, where for a particle type 1, we have $q_1^{(s)} = 1$, and where for a particle type 2, we have $q_2^{(s)} = 1 + e^{-s\beta}$ for $s > 0$. Then clearly, if there are just two *in* rapidities and if we have with $\theta_1 = \theta_2 + \beta$ and $a_1 = a_2 = 2$, then we can choose to have only one *out* rapidity with $\theta_1' = \theta_1$ and $a_1' = 2$, and the equation is satisfied for all α.

This fictitious situation has never been seen in any model, to my knowledge. In order to establish elastic scattering from this argument, we must check in explicit models what the $q_a^{(s)}$ look like. Admittedly, then, this argument is not as strong as we would like it to be, but generically, we see a singularity only at $\alpha = 0$ in $f(\alpha)$ for $\alpha \in \mathbb{R}$. From this we may conclude that the obvious solution is the only one for real rapidities:

$$\{\theta_k'\} = \{\theta_k\} \ , \quad q_{a_k'}^{(s)} = q_{a_k}^{(s)} \text{ for } \theta_k' = \theta_k \ .$$

5.5 Factorized scattering: Yang-Baxter equations

We consider again the overlap

$${}^{out}_{a_1',a_2',\dots}\langle\theta_1',\theta_2',\dots;\text{imp. param.}|\theta_1,\theta_2,\dots;\text{imp. param.}\rangle^{in}_{a_1,a_2,\dots}$$

and look at the dependence on the impact parameters. It will be convenient and sufficient for our purposes to choose the "same" impact parameters for *in* and *out* particles; that is, for given impact parameters of *in* states, we choose the impact parameters of *out* particles to be those obtained by an extension of the trajectory of the *in* particles all the way to the *out* particles as if they were not interacting. The main point is to write the simple identity:

$$\begin{aligned}&{}^{out}_{a_1',a_2',\dots}\langle\theta_1',\theta_2',\dots;\text{imp. param.}|e^{i\alpha Q_s}e^{-i\alpha Q_s}|\theta_1,\theta_2,\dots;\text{imp. param.}\rangle^{in}_{a_1,a_2,\dots} = \\ &{}^{out}_{a_1',a_2',\dots}\langle\theta_1',\theta_2',\dots;\text{imp. param.}|\theta_1,\theta_2,\dots;\text{imp. param.}\rangle^{in}_{a_1,a_2,\dots} \ . \end{aligned} \tag{5.13}$$

We can exploit this formula by analysing in more details than above the action of a conserved charge on asymptotic states. More precisely, we will see that the scattering matrix is in fact independent of impact parameters, and this will lead to Yang-Baxter equation.

From the formula (5.6) for the operators destroying/creating asymptotic *in*-states, let us compute

$$[Q_s, A(\theta)^{(in)}] = \lim_{L\to\infty}\lim_{t\to-\infty}\int dx\,(f_\theta(x,t)\partial_t[Q_s,\Psi(x,t)] - \partial_t f_\theta(x,t)[Q_s,\Psi(x,t)])$$

where the function $f_\theta(x,t)$ is given in (5.7). Let us consider for now L finite but large, and take the limit on t. We can evaluate the commutators involved, because the conserved charges act like higher-spin conserved charges on free particles as $t \to -\infty$. For instance, for positive spins we have

$$-\frac{(-i)^s q^{(s)}}{m^s}\lim_{t\to-\infty}\int dx\,(f_\theta(x,t)\partial_t(\partial_x-\partial_t)^s\Psi(x,t) - \partial_t f_\theta(x,t)(\partial_x-\partial_t)^s\Psi(x,t)) \ .$$

We can move the derivatives $(\partial_x - \partial_t)$ on the fields to the same derivatives on the wave packet $f_\theta(x,t)$ with an overall sign $(-1)^s$. This is obvious for the space derivatives: we do this by integration by part. For the time derivatives, consider the following fact: if a product fg is time independent, then certainly $f\partial_t g = -(\partial_t f)g$. Now for large times, the integral we have is a free-evolving wave packet. We know from quantum mechanics that these evolve in two ways: the centre evolves following a straight trajectory, and

the wave packet spreads. The first is taken into account in $f_\theta(x,t)$, but the second is not. However, the effect of the spreasing is a term in the exponential in (5.7) of the form $(x-\coth(\theta)t)^2/L^3\partial L(t)/\partial t$. Now, from $L(t)=\sqrt{L^2+(t-t_0)^2}$ (with $L(t_0)=L$), we have $\partial L(t)/\partial t \sim 1/(mL)$ for any finite $(t-t_0)$ (that is, of the order of m^{-1}). Since $x-\coth(\theta)t$ is of the order of mL in a wave packet, this correction term is $O((mL)^{-2})$, and we will neglect these terms. Hence, we can move the time derivatives in the limit on t, up to these terms for large L. We must be careful to use time independednce on the difference of the two terms in the definition asymptotic state operators. This allows us to move time derivatives except for terms with only one time derivative, but these don't need to be moved. We then get

$$-\frac{i^s q^{(s)}}{m^s}\lim_{t\to-\infty}\int dx\,((\partial_x-\partial_t)^s f_\theta(x,t)\partial_t\Psi(x,t)-\partial_t(\partial_x-\partial_t)^s f_\theta(x,t)\Psi(x,t))\ .$$

Now consider the large L asymptotic: we will keep terms up to in $1/(mL)$, while always considering $x-\coth(\theta)t$ of the order of L. We have

$$-\frac{i^s q^{(s)}}{m^s}(\partial_x-\partial_t)^s f_\theta(x,t)=-q^{(s)}e^{s\theta}\left[1+\frac{2s}{imL\sinh(\theta)}\frac{x-\coth(\theta)t}{L}+O((mL)^{-2})\right]f_\theta(x,t)\ .$$

We recognize the leading term as giving the correct eigenvalue of Q_s on asymptotic states, but we want to analyse the effect of the additional term. Let us write it as

$$-\frac{i^s q^{(s)}}{m^s}(\partial_x-\partial_t)^s f_\theta(x,t)=-q^{(s)}\exp\left[s\theta+\frac{2s}{imL\sinh(\theta)}\frac{x-\coth(\theta)t}{L}+O((mL)^{-2})\right]f_\theta(x,t)\ .$$

In order to calculate

$$e^{i\alpha Q_s}A(\theta)^{(in)}e^{-i\alpha Q_s}\ ,$$

we compute

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{1}{n!}\left(-i\alpha\frac{i^s q^{(s)}}{m^s}\right)^n(\partial_x-\partial_t)^{ns}f_\theta(x,t)\\&=\exp\left(-i\alpha q^{(s)}\exp\left[s\theta+\frac{2s}{imL\sinh(\theta)}\frac{x-\coth(\theta)t}{L}+O((mL)^{-2})\right]\right)f_\theta(x,t)\end{aligned}$$

Expanding at large L, the right-hand side can be written

$$\exp\left(-i\alpha q^{(s)}e^{s\theta}\right)\exp\left[iM\cosh(\theta)\,t-iM\sinh(\theta)\,x-\frac{(x-x_0-\coth(\theta)(t-t_0))^2}{L^2}+O((mL)^{-2})\right]$$

with

$$m\cosh(\theta)t_0-m\sinh(\theta)x_0=\alpha q^{(s)}se^{s\theta}\ .$$

It is convenient to choose have $x_0=-t_0\equiv l$.

We recognize this as, up to the factor $\exp\left(-i\alpha q^{(s)}e^{s\theta}\right)$ containing the eigenvalue of Q_s, simply the wavepacket (5.9) with non-zero impact parameters. The impact parameter is given by

$$ml=-\alpha q^{(s)}se^{(s-1)\theta}\quad(s>0)\ .$$

If $s=1$, it is clearly independent of θ, but otherwise, the impact parameters depend on the rapidity θ.

A similar derivation holds for negative spins. The action of Q_s on a field $\Psi(x,t)$ in this case is

$$[Q_s,\Psi(x,t)]=-\frac{(-i)^s q^{(s)}}{m^{|s|}}(\partial_x+\partial_t)^{|s|}\Psi)(x,t)\quad(s<0)$$

and we will have $x_0 = t_0 \equiv l$. The impact parameter then takes the value

$$ml = -\alpha q^{(s)} s e^{(s+1)\theta} \quad (s < 0) \ .$$

Hence, we found that

$$e^{i\alpha Q_s} A(\theta)^{(in)} e^{-i\alpha Q_s} = e^{-i\alpha q^{(s)} e^{s\theta}} \ A(\theta)^{(in)}\Big|_{\text{impact parameter } l}$$

and the same thing holds, for conjugate eigenvalue, for the operator $A^\dagger(\theta)^{(in)}$. Looking back at (5.13), we realize that the scattering matrix is invariant under such changes of impact parameters.

It is not *a priori* clear that using enough conserved charges, we can bring the impact parameters to any value we want, hence showing that the scattering matrix is independent of impact parameters. However, there is a simple and strong argument that can do this. First, we must realise that the 2-particle S-matrix does not depend on the impact parameters. Indeed, by convention, we chose the *out* impact parameters to agree with the *in* impact parameters (which can always be done in elastic scattering). Then, space and time translation invariance means that the 2-particle S matrix is independent of impact parameters. Consider then the 3-particle to 3-particle scattering, with scattering matrix

$$S^{a_1',a_2',a_3'}_{a_1,a_2,a_3}(\theta_1,\theta_2,\theta_3;l_1,l_2,l_3) \ .$$

Here we use elastic scattering and write only the three rapidities involved. The particle types a_1, a_2, a_3 are for the *in*-particles of rapidity $\theta_1, \theta_2, \theta_3$ respectively, and similarly for a_1', a_2', a_3' for the *out*-particles. The parameters l_1, l_2, l_3 are the associated impact parameters. Consider the set of rapidities $(\theta_1, \theta_2, \theta_3)$ for which the three impact parameters resulting from the action of Q_s,

$$-\alpha m_{a_i}^{-1} q^{(s)}_{a_i} s e^{(s-1)\theta} \ , \quad i = 1,2,3$$

are not all equal to each other. There is only possibly one set $(\theta_1, \theta_2, \theta_3)$ for which this is not so for a given spin s; for this one, we must use another conserved charge of a different spin and verify that the inequality holds. If we have parity invariance in the theory, we can use the conserved charge of spin $-s$ (we have $q_a^{(-s)} = q_a^{(s)}$ in parity invariant models), and we are guaranteed that it will work. Then, we can use invariance under shift of impact parameters for each particle, and with α large enough, the impact parameters become so different that the particle will only meet pair by pair at points very distant from each other: say particles 1 and 2 first, then particle particles 1 and 3, finally particles 2 and 3. Since these meeting points are very far from each other, by locality the 3-particle S-matrix decomposes itself into products of 2-particle S-matrices:

$$S^{a_1',a_2',a_3'}_{a_1,a_2,a_3}(\theta_1,\theta_2,\theta_3;l_1,l_2,l_3) = S^{c,b}_{a_1,a_2}(\theta_1,\theta_2) S^{a_1',d}_{c,a_3}(\theta_1,\theta_3) S^{a_3',a_2'}_{b,d}(\theta_2,\theta_3)$$

(with implicit sum over repeated indices). Since this does not depend anymore on the initial impact parameters, this shows that the scattering matrix does not depend on them. Hence, we can also choose them so that we get the opposite situation: particles 2 and 3 meet first, then particles 1 and 3, finally particles 1 and 2 (equivalently, we could have taken α large with opposite sign). The two ways of decomposing it give the same value:

$$S^{c,b}_{a_1,a_2}(\theta_1,\theta_2) S^{a_1',d}_{c,a_3}(\theta_1,\theta_3) S^{a_2',a_3'}_{b,d}(\theta_2,\theta_3) = S^{b,c}_{a_2,a_3}(\theta_2,\theta_3) S^{d,a_3'}_{a_1,c}(\theta_1,\theta_3) S^{a_1',a_2'}_{d,b}(\theta_1,\theta_2) \tag{5.14}$$

This equation (or set of equations) is called *Yang-Baxter equation*.

Now consider the multi-particle scattering. We can apply a similar argument and write it as a product of a $n-1$-particle S-matrix times a 2-particle S-matrix. Recursively, we then get a product of 2-particle S-matrices. Again this is true for any initial impact parameters, so that it does not depend on them. Repeated use of Yang-Baxter equation insures that we can write it as any decomposition into 2-particle S-matrices.

Note that we only needed *two* conserved charges of spin higher than 1 to do these manipulations! Using similar arguments, Parke (1980) was able to actually prove *elastic scattering* as well as factorizability, using two local conserved charges of higher spin.

It is worth noting that similar arguments could be attempted in higher dimension. The presence of higher-spin conserved charges indeed give there the independence form impact parameters, and in higher than on space dimension, this means that we can choose the trajectory to all avoid each other. Hence the theory has to be trivial: free fermions or free bosons. This is the essence of Coleman-Mandula theorm (1967).

5.6 Analytic and other properties of the two-particle S-matrix

Besides satisfying the Yang-Baxter equation (5.14), the two-particle S-matrix must satisfy other, simpler equations that are consequences of general properties of QFT, as well as "bootstrap" equations, which are like Yang-Baxter equations but which concern bound states.

The first equation comes from Lorentz invariance, and says that only the difference of rapidities matters:

$$S^{b_1,b_2}_{a_1,a_2}(\theta_1,\theta_2) = S^{b_1,b_2}_{a_1,a_2}(\theta_1-\theta_2) \ .$$

The second is about the fact that the transformation from *in*-states to *out*-states is unitary. From $|\theta_1,\theta_2\rangle^{(in)}_{a_1,a_2} = \sum_{b_1,b_2} S^{b_1,b_2}_{a_1,a_2}(\theta_1-\theta_2)|\theta_1,\theta_2\rangle^{(out)}_{b_1,b_2}$ and orthonormality of both *in*-states and *out*-states, we have

$$\sum_{b_1,b_2} S^{b_1,b_2}_{a_1,a_2}(\theta)\left(S^{b_1,b_2}_{c_1,c_2}(\theta)\right)^* = \delta^{c_1}_{a_1}\delta^{c_2}_{a_2} \tag{5.15}$$

(for real θ). Note that the unitarity relation for non-integrable models is not this, because there is in general more intermediate states, with more than 2 particles.

With time-independent Hermitian Hamiltonian, there is time-reversal symmetry. This says that if we reverse the direction of time and if all particles are replaced by their corresponding anti-particle, then the scattering amplitudes are the same. The consequence for the two-particle scattering matrix is

$$\text{(time-reversal symmetry)} \qquad S^{b_1,b_2}_{a_1,a_2}(\theta) = S^{\bar{a}_2,\bar{a}_1}_{\bar{b}_2,\bar{b}_1}(\theta)$$

(for real rapidities). The exchange of horizontal position of the indices occurs because if a particle is travelling to the right, then under time reversal, its anti-particle travels to the left. The notation $\bar{a}$ means that we must take the particle index corresponding to the anti-particle of that of particle type a. Generically, there will be a conjugation matrix $C_{a,b}$ such that

$$A^{\bar{a}} = C_{a,b}A^b \ , \quad A_{\bar{a}} = C^{a,b}A_b \ ,$$

with property

$$C_{a,b}C^{b,c} = \delta^c_a \ .$$

The most important properties of the scattering matrix is its analytical structure. General principles of QFT (or of the theory of analytic S-matrices) say the following:

In a general model of (1+1-dimensional) QFT, as an analytic function of the Mandelstam variable

$$s = m_1^2 + m_2^2 + 2m_1 m_2 \cosh(\theta_1 - \theta_2) \ ,$$

the two-particle to two-particle scattering matrix (preserving the masses – such a scattering is always elastic from the kinematics) is a multi-valued function, and possesses a Riemann sheet with only three branch points where it is otherwise meromorphic. On this sheet, called the physical sheet, the branch points are at $(m_1 + m_2)^2$, $(m_1 - m_2)^2)$ and ∞, the cuts are on the real line avoiding the interval $[(m_1 - m_2)^2, (m_1 + m_2)^2]$, and the only possible poles are on this interval. This sheet is characterised by the fact that the physical values of the scattering matrix are just above the cut on the interval $[(m_1 + m_2)^2, \infty]$ of the real line.

The prescription "just above the cut" for the physical values of the scattering can be understood as coming from Feynmann's prescription for the propagator. The poles on the real line, between the cuts, correspond to possible bound states between the two particles involved in the scattering. The restriction to 1+1-dimensional QFT implies that the 2-particle scattering matrix only depends on the Mandelstam variable s.

Back in the $\theta = \theta_1 - \theta_2$ plane, the physical sheet corresponds to the strip $\mathrm{Im}(\theta) \in [0, \pi]$, the branch points are $\theta = 0, i\pi, \infty$, and the cuts run along the lines $\mathrm{Im}(\theta) = 0, \pi$ connecting to infinity. They may be chosen to run towards the left or towards the right. The poles corresponding to bound states lie on the line $\mathrm{Re}(\theta) = 0$ in the physical strip.

An important relation valid on the physical sheet is called "Hermitian analyticity", or when there is parity invariance, "real analyticity". It simply says that the complex conjugate of the scattering matrix on the physical sheet is the scattering matrix on the complex conjugate argument still on the physical sheet:

$$\big(S_{a_1,a_2}^{b_1,b_2}(\theta)\big)^* = S_{b_2,b_1}^{a_2,a_1}(-\theta^*) \ . \tag{5.16}$$

Notice how the real part of the rapidity changes its sign. This is because $s \mapsto s^*$ on the physical sheet corresponds to $\theta \mapsto -\theta^*$.

This relation can be understood as follows. Consider the path-integral formulation of the scattering matrix, with real rapidities and $\theta_1 > \theta_2$:

$$S_{a_1,a_2}^{b_1,b_2}(\theta) = \int_{\substack{\Psi_{a_k} \propto A(x-c_k)e^{iE_k t - ip_k x} \ (c_1 < c_2, \ t \to -\infty) \\ \Psi_{b_k} \propto A(x-c'_k)e^{iE_k t - ip_k x} \ (c'_1 > c'_2, \ t \to +\infty)}} [d\Psi] e^{iS[\Psi]} \ .$$

Here, the Ψ_k are the well-separated wave packets, and A is a Gaussian envelope, as before. The numbers c_k, c'_k are the centers of the envelopes. Taking the complex conjugate of this gives

$$\big(S_{a_1,a_2}^{b_1,b_2}(\theta)\big)^* = \int_{\substack{\Psi_{a_k} \propto A(x-c_k)e^{-iE_k t + ip_k x} \ (c_1 < c_2, \ t \to -\infty) \\ \Psi_{b_k} \propto A(x-c'_k)e^{-iE_k t + ip_k x} \ (c'_1 > c'_2, \ t \to +\infty)}} [d\Psi] e^{-iS[\Psi]} \ .$$

On the other hand, time-reversal invariance can be written

$$S_{a_1,a_2}^{b_1,b_2}(\theta) = \int_{\substack{\Psi_{a_k} \propto A(x-c_k)e^{-iE_k t - ip_k x} \ (c_1 < c_2, \ t \to +\infty) \\ \Psi_{b_k} \propto A(x-c'_k)e^{-iE_k t - ip_k x} \ (c'_1 > c'_2, \ t \to -\infty)}} [d\Psi] e^{-iS[\Psi]} \quad (\theta_1 > \theta_2) \ .$$

Comparing gives (5.17) for θ real and positive, if we consider that the region $\theta_1 < \theta_2$ of the latter expression is obtained by analytic continuation in θ. Analytic continuation gives it for all θ on the physical sheet.

If there is parity invariance:

$$\text{(parity invariance)} \qquad S_{a_1,a_2}^{b_1,b_2}(\theta) = S_{a_2,a_1}^{b_2,b_1}(\theta)$$

then Hermitian analyticity implies real analyticity,

$$\left(S_{a_1,a_2}^{b_1,b_2}(\theta)\right)^* = S_{a_1,a_2}^{b_1,b_2}(-\theta^*) \ , \tag{5.17}$$

because we always have CPT invariance (charge-parity-time-reversal invariance) from general principles of QFT:

$$\text{(CPT invariance)} \qquad S_{a_1,a_2}^{b_1,b_2}(\theta) = S_{b_1,b_2}^{a_1,a_2}(\theta) \ .$$

Now we may combine Hermitian analyticity with the unitarity relation of integrable models derived above, in order to obtain

$$S_{a_1,a_2}^{b_1,b_2}(\theta) S_{b_2,b_1}^{c_2,c_1}(-\theta^*) = \delta_{a_1}^{c_1}\delta_{a_2}^{c_2} \ . \tag{5.18}$$

This relation is usually called "unitarity" for the 2-particle S-matrix in integrable models. It is important to recall that $-\theta^*$ means the analytic continuation from θ two $-\theta^*$ counterclockwise around the point 0 (that is, for physical initial $\theta > 0$, we always stay on the physical strip to reach $-\theta$).

Another relation, again consequence of general principles of QFT, is crossing symmmetry. It essentially says that quantizing the theory in a scheme where the time and space arrows are rotated by $\pi/2$ gives the same scattering amplitudes. For the 2-particle scattering $S_{a_1,a_2}^{b_1,b_2}(\theta_1 - \theta_2)$, the slope of the world line of particle k is $\coth(\theta_k)$. Making a $\pi/2$ counter-clockwise rotation amounts to $\theta_k \mapsto i\pi/2 - \theta_k$, to be understood as analytic continuation on the physical strip, and after this rotation particle 2 seems like propagating in reverse time. We can then use time-reversal symmetry in order to transform it to an anti-particle propagating correctly, and we must remember that when complex values of rapidities are involved, time-reversal symmetry involves complex conjugation of the rapidities (understood, again, on the physical strip of $\theta_1 - \theta_2$). Hence, we find

$$S_{a_1,a_2}^{b_1,b_2}(i\pi - \theta) = S_{\bar{b}_2,a_1}^{\bar{a}_2,b_1}(\theta) \tag{5.19}$$

Now it is possible to understand the nature of the branch points at $\theta = 0, i\pi$ on the physical strip. Start at $\theta > 0$ real, and use the unitarity relation (5.18) in order to travel to $\theta < 0$ real, always staying on the physical strip. Using again (5.18) with $\theta < 0$ real, we get the analytic continuation all the way around the point 0. But this is equal to the initial value:

$$S_{a_1,a_2}^{b_1,b_2}(\theta) S_{b_2,b_1}^{c_2,c_1}(-\theta^*) [S_{c_1,c_2}^{d_1,d_2}(\theta)]_{\text{anal. cont. around 0}} = S_{a_1,a_2}^{d_1,d_2}(\theta) = [S_{a_1,a_2}^{d_1,d_2}(\theta)]_{\text{anal. cont. around 0}}$$

(we used unitarity in two different ways for contracting the two different pairs of S-matrices). Hence, the point $\theta = 0$ is in fact not a branch point, it is a regular point without any singularity; this means that the corresponding branch point in the s-plane is a square-root branch point. By crossing symmetry, the same is true for the point $\theta = i\pi$.

This means that the S-matrix is a meromorphic function of θ, its poles on the physical strip can only be on the imaginary axis, and once it is determined on the physical sheet, it is determined everywhere

on the θ-plane. Also, if the poles and their residues are known on the physical strip and on the strip $\mathrm{Im}(\theta) \in [-\pi, 0]$, then it is determined everywhere.

Finally, there are additional conditions coming from possible bound states. A bound state manifests itself by a virtual particle being created in a two-particle scattering process, and gives rise to a pole in the S-matrix at the corresponding imaginary value of the rapidity. This value is sole consequence of the kinematics. For two particles of masses m_1 and m_2 forming a particle of mass m, it is

$$\cosh(\theta) = \frac{m^2 - m_1^2 - m_2^2}{2m_1 m_2} \ . \tag{5.20}$$

The residue at the pole is purely imaginary, iR. If $R > 0$, this is a bound state in the "direct" channel, if $R < 0$, it is in the "crossed" channel (as if the virtual particle were travelling faster than the speed of light!). This would not be of much use if it were not for the resulting "bound-state Yang-Baxter" relations. They are not as easy to derive from local conserved charges, and I will not go into any detail of there derivation; but they are easy to state. They can just be seen as consistency in the different ways of decomposing a 3-particle scattering, where two of the particles (say particle 1 and 2) form a virtual bound-state particle, into products of 2-particle scattering. The assumption is that any bound state is a particle that is part of the asymptotic state spectrum (this assumption is called "nuclear democracy"), hence particle 3 can scatter with the bound-state of particles 1 and 2 as if it were an asymptotic particle, and this must give the same result as when particle 3 scatters with particles 1 then 2 separately.

5.7 Zamolodchikov-Faddeev algebra

A convenient way of representing the properties of factorized scattering is using the Zamolodchikov algebra. It is the exchange algebra generated by elements $Z_a(\theta)$, which are parametrised by particle type a and rapidity θ. This algebra is used to represent states in integrable QFT by associating to a state an element of the enveloping algebra of Zamolodchikov's algebra. The horizontal position of each factor corresponds to the position of the particles in the scattering process at a given time. For instance, the *in*-state, where wave packets are ordered from the most positive rapidity at the left to the most negative one at the right, is represented by the product (an element of the enveloping algebra)

$$Z_{a_1}(\theta_1) \cdots Z_{a_n}(\theta_n) \ , \quad \theta_1 > \cdots > \theta_n \ .$$

Since, as we saw, we can choose the impact parameters so that scattering occurs 2 particles by 2 particles at points very separated from each other, we can define "intermediate states", which are neither *in* nor *out*, where wave packets travel freely and are ordered in such a way that some may never meet in the future (going in opposite directions after having scattered already). Such states would not make sense in ordinary quantum field theory, or at least would be of no interest, because in this case such a situation never happens by construction (the asymptotic states are initially defined such that particles meet in a finite region of space-time); it only happens, or is useful, thanks to independence from impact parameters. Then, in general, such intermediate states will be represented by

$$Z_{a_1}(\theta_1) \cdots Z_{a_n}(\theta_n) \ , \quad \text{any ordering of } \theta_1, \ldots, \theta_n \ .$$

Physical scattering tells us that the Zamolodchikov algebra elements must satisfy the exchange relation

$$Z_{a_1}(\theta_1) Z_{a_2}(\theta_2) = S_{a_1,a_2}^{b_1,b_2}(\theta_1 - \theta_2) Z_{b_2}(\theta_2) Z_{b_1}(\theta_1) \ , \quad \theta_1 > \theta_2 \ . \tag{5.21}$$

In order to verify associativity of this algebra, it is sufficient to check that the two ways of obtaining $Z_{a_3}(\theta_3)Z_{a_2}(\theta_2)Z_{a_1}(\theta_1)$ from $Z_{a_1}(\theta_1)Z_{a_2}(\theta_2)Z_{a_3}(\theta_3)$ are consistent. They are indeed consistent thanks to the Yang-Baxter equation (5.14).

The relation (5.21) was written for $\theta_1 > \theta_2$. The exchange relation for $\theta_1 < \theta_2$ follows from it by exchanging θ_1 and θ_2, but if we want to write it in the same form as (5.21), we need to define $S_{a_1,a_2}^{b_1,b_2}(\theta_1-\theta_2)$ for $\theta_1 < \theta_2$. The algebra tells us that it should be defined such that the following equation is sastisfied:

$$S_{a_1,a_2}^{b_1,b_2}(\theta_1-\theta_2)S_{b_2,b_1}^{c_2,c_1}(\theta_2-\theta_1) = \delta_{a_1}^{c_1}\delta_{a_2}^{c_2} \ .$$

This is just a consequence of the algebra, but we may wonder if the analytic continuation of the physical 2-particle scattering matrix $S_{a_1,a_2}^{b_1,b_2}(\theta_1-\theta_2)$ from the region $\theta_1 > \theta_2$ to the region $\theta_1 < \theta_2$ would give a function that satisfies this relation. It is not *a priori* clear that this must be so (and certainly it is not true of a 2-particle scattering matrix in non-integrable models), but since the states we constructed are actual asymptotic-like state of integrable models, we could expect that the relation above is true for the 2-particle scattering matrix as an analytical function of the rapidities. Comparison with the previous sub-section shows that it is so.

It is just a small step from representing states using the enveloping algebra of Zamolodchikov's algebra to constructing the Hilbert space as a module for a slightly extended algebra, Zamolodchikov-Faddeev algebra. It is defined by the relations

$$\begin{aligned} Z_{a_1}(\theta_1)Z_{a_2}(\theta_2) - S_{a_1,a_2}^{b_1,b_2}(\theta_1-\theta_2)Z_{b_2}(\theta_2)Z_{b_1}(\theta_1) &= 0 \\ \bar{Z}^{a_1}(\theta_1)\bar{Z}^{a_2}(\theta_2) - S_{b_1,b_2}^{a_1,a_2}(\theta_1-\theta_2)\bar{Z}^{b_2}(\theta_2)\bar{Z}^{b_1}(\theta_1) &= 0 \\ Z_{a_1}(\theta_1)\bar{Z}^{a_2}(\theta_2) - S_{b_2,a_1}^{a_2,b_1}(\theta_2-\theta_1)\bar{Z}^{b_2}(\theta_2)Z_{b_1}(\theta_1) &= 2\pi\delta_{a_1}^{a_2}\delta(\theta_1-\theta_2) \ . \end{aligned} \tag{5.22}$$

The Hilbert space can then be constructed as a Fock space over this algebra, with vacuum defined by $\bar{Z}^a(\theta)|\text{vac}\rangle = 0$, and with conjugation defined by $Z_a(\theta)^\dagger = \bar{Z}^a(\theta)$. Consistency of these relations is a consequence of Yang-Baxter relation (5.14) and unitarity (5.18), and consistency with the Hermitian structure is consequence of Hermitian analyticity (5.17).

5.8 Simple examples, CDD ambiguity

5.8.1 Recapitulation of the requirements

The problem of finding the scattering matrix is now reduced to the problem of solving a set of equations along with analytical conditions (a Riemann-Hilbert problem). We want to find a function $S_{a_1,a_2}^{b_1,b_2}(\theta)$ that is meromorphic and that satisfies:

1. Yang-Baxter equation (5.14);
2. Hermitian analyticity (5.17);
3. unitarity (5.18);
4. crossing symmetry (5.19);
5. bound state consistency relations when poles on the physical strip (5.20) are present.

The solutions to this problem can be characterized by the possible structure of solutions to the Yang-Baxter equations.

5.8.2 A diagonal example

The simplest structure is diagonal scattering: $S^{b_1,b_2}_{a_1,a_2}(\theta) = \delta^{b_1}_{a_1}\delta^{b_2}_{a_2}S_{a_1,a_2}(\theta)$ (no sum over repeated indices!). This trivially solves Yang-Baxter relations, and we are left only with the other requirements. The most general solution to requirements of points 2, 3 and 4 above is the following:

$$S_{a,b}(\theta) = \prod_{x\in X_{a,b}} \frac{\sinh\left(\frac{1}{2}(\theta + i\pi x)\right)}{\sinh\left(\frac{1}{2}(\theta - i\pi x)\right)} \tag{5.23}$$

where $X_{a,b}$ is a set of a finite number of complex numbers lying, for instance, in the strip $\mathrm{Re}(x) \in [-1,1]$. If these numbers are all in the strip $\mathrm{Re}(x) \in [-1,0]$, then there are no poles on the physical strip, and this immediately is a valid scattering matrix! An example of such a model is the so-called sinh-Gordon model. It is a model with only one particle in the spectrum of asymptotic states and, besides the mass of this particle, one additional free real parameter b. The scattering amplitude is

$$S(\theta) = \frac{\tanh\left(\frac{\theta}{2} - \frac{i\pi b^2}{2(b^2+1)}\right)}{\tanh\left(\frac{\theta}{2} + \frac{i\pi b^2}{2(b^2+1)}\right)}$$

The classical action of the sinh-Gordon model is written in terms of a single real b1osonic field ϕ:

$$\mathcal{A} = \int dx\, dt\, \left\{ \frac{1}{16\pi}\left[(\partial_t\phi)^2 - (\partial_x\phi)^2\right] - 2\mu\cosh(b\phi)\right\}\ .$$

The mass of the particle is proportional to a power of μ. The fact that the scattering amplitude above corresponds to this action is really a conjecture, but I will discuss below possible ways of going from classical action to a solution to the equations for the scattering matrix.

If poles are present on the physical strip, we must additionaly solve the consistency relations for bound states. I will not go at all into this quite extensive subject...

5.8.3 A non-diagonal example

Another class of solutions to the Yang-Baxter equations are those where two types particles (that we will characterize by two charges $+/-$), scatter into one another, with scattering matrix of the form

$$S(\theta) = \begin{array}{c} \begin{array}{cccc} ++ & +- & -+ & -- \end{array} \\ \begin{pmatrix} a(\theta) & 0 & 0 & 0 \\ 0 & b(\theta) & c(\theta) & 0 \\ 0 & c(\theta) & b(\theta) & 0 \\ 0 & 0 & 0 & a(\theta) \end{pmatrix} \end{array} \begin{array}{c} ++ \\ +- \\ -+ \\ -- \end{array}$$

The pairs of signs on the right are the lower indices, and those on the top are the upper indices of the scattering matrix $S^{b_1,b_2}_{a_1,a_2}(\theta)$. The Yang-Baxter equation then amounts to only two equations:

$$\begin{aligned} a(\theta_{12})b(\theta_{13})c(\theta_{23}) &= b(\theta_{23})c(\theta_{12})c(\theta_{13}) + a(\theta_{13})b(\theta_{12})c(\theta_{23}) \\ a(\theta_{12})a(\theta_{23})c(\theta_{13}) &= a(\theta_{13})c(\theta_{12})c(\theta_{23}) + b(\theta_{12})b(\theta_{23})c(\theta_{13}) \end{aligned}$$

where $\theta_{ij} \equiv \theta_i - \theta_j$. From this, one then needs to solve the four other requirements written above. A solution that has no pole on the physical strip is that of the so-called $SU(2)$-Thirring model. It is a model with two particles of equal mass and of opposite $SU(2)$ spin, transforming under the fundamental representation of $SU(2)$, without any other free parameter than the mass and having a scattering matrix as above with

$$a(\theta) = \frac{\Gamma\left(\frac{1}{2} - \frac{i\theta}{2\pi}\right)\Gamma\left(\frac{i\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{2} + \frac{i\theta}{2\pi}\right)\Gamma\left(-\frac{i\theta}{2\pi}\right)} \ , \quad b(\theta) = a(\theta)\frac{\theta}{i\pi - \theta} \ , \quad c(\theta) = a(\theta)\frac{i\pi}{i\pi - \theta} \ .$$

The classical action of this model is written in terms of a Dirac spinor Ψ with extra index in the fundamental representation of $SU(2)$:

$$\mathcal{A} = \int dx\, dt \left(\bar{\Psi}\gamma^\mu \partial_\mu \Psi - \frac{g}{2} \sum_m \bar{\Psi}\gamma_\mu \sigma^m \Psi \bar{\Psi}\gamma^\mu \sigma^m \Psi \right)$$

where σ^m are Pauli matrices and γ^μ are Dirac gamma-matrices. The parameter g, which is classically dimensionless, is in fact a running coupling constant under the action of the renormalisation group, and the scale associated to this running gives rise to the physical mass of the particle; this process is called "dimensional transmutation".

5.8.4 CDD ambiguity

It is important to note that in general, any solution to the 5 requirements above can always be multiplied by a factor (5.23) for $\mathrm{Re}(x) \in [-1, 0]$, giving another solution. This is called a CDD factor (from Castillejo, Dalitz and Dyson, 1956), and the resulting ambiguity for the scattering matrix is the CDD ambiguity. The sinh-Gordon example is a pure CDD factor[4]. Resolving the CDD ambiguity cannot be done solely using the techniques of factorized scattering theory; one needs to do perturbative calculations of the scattering matrix or other, non-perturbative checks.

5.9 Inverse problem: form factors

We now start with the assumption that we know the exact scattering matrix, and that it is elastic and factorizable. Of course, the scattering matrix is not directly useful for most calculations related to experimental situations. The objects that are of direct importance are the correlation functions of local fields. In particular, the two-point function,

$$\langle \mathrm{vac} | \mathcal{O}_1(x,t)\mathcal{O}_2(0,0) | \mathrm{vac} \rangle \ ,$$

is a quantity often required, as it is related to the response function of the system at one point once it is disturbed at another point. The calculation of two-point functions generated a lot of research in integrable QFT. Probably the most fruitful idea is to start from a representation of the two-point function

[4]Hence, its scattering amplitude is just a CDD factor times the scattering amplitude of a free massive Majorana fermion!

coming from inserting a complete set of energy eigenstates between the operators:

$$\begin{aligned}
&\langle \mathrm{vac}|\mathcal{O}_1(x,t)\mathcal{O}_2(0,0)|\mathrm{vac}\rangle \\
&\quad = \sum_{n=0}^{\infty}\sum_{a_1,\dots,a_n}\int \frac{d\theta_1\cdots d\theta_n}{(2\pi)^n n!}\langle \mathrm{vac}|\mathcal{O}_1(x,t)|\theta_1,\dots,\theta_n\rangle^{in}_{a_1,\dots,a_n}\ {}^{in}_{a_1,\dots,a_n}\langle\theta_1,\dots,\theta_n|\mathcal{O}_2(0,0)|\mathrm{vac}\rangle \\
&\quad = \sum_{n=0}^{\infty}\sum_{a_1,\dots,a_n}\int \frac{d\theta_1\cdots d\theta_n}{(2\pi)^n n!}e^{-iE_nt+ip_nx}\langle \mathrm{vac}|\mathcal{O}_1(0,0)|\theta_1,\dots,\theta_n\rangle^{in}_{a_1,\dots,a_n}\ {}^{in}_{a_1,\dots,a_n}\langle\theta_1,\dots,\theta_n|\mathcal{O}_2(0,0)|\mathrm{vac}\rangle
\end{aligned}$$

where

$$E_n = \sum_{k=1}^{n} m_{a_k}\cosh(\theta_k)\ , \quad p_n = \sum_{k=1}^{n} m_{a_k}\sinh(\theta_k)\ .$$

This is the basis for the usual Källén-Lehmann spectral decomposition. The *in*-states are, as usual, wave packets at minus infinite time with particles ordered from left to right by decreasing rapidity (so that above, the leftmost particle is not necessarily θ_1). Of course, we could have used as well the *out* basis.

In integrable systems, we know that states $|\theta_1,\dots,\theta_n\rangle$ can be defined for any ordering of the rapidities by asymptotic-like states that occur between two-body scattering events far apart from each other. When $\theta_1 > \cdots > \theta_n$ they agree with the *in*-state, when $\theta_1 < \cdots < \theta_n$ they agree with the *out*-states, but for other orderings, they just form a different basis for the Hilbert space. It is not hard to see, from the unitarity of the scattering matrix written as (5.15), that we can just use all these bases and symmetrise over the rapidities:

$$\begin{aligned}
&\langle \mathrm{vac}|\mathcal{O}_1(x,t)\mathcal{O}_2(0,0)|\mathrm{vac}\rangle \\
&\quad = \sum_{n=0}^{\infty}\sum_{a_1,\dots,a_n}\int \frac{d\theta_1\cdots d\theta_n}{(2\pi)^n n!}e^{-iE_nt+ip_nx}\langle \mathrm{vac}|\mathcal{O}_1(0,0)|\theta_1,\dots,\theta_n\rangle_{a_1,\dots,a_n}\ {}_{a_1,\dots,a_n}\langle\theta_1,\dots,\theta_n|\mathcal{O}_2(0,0)|\mathrm{vac}\rangle\ .
\end{aligned}$$

The matrix elements

$$F^{\mathcal{O}}_{a_1,\dots,a_n}(\theta_1,\dots,\theta_n) = \langle \mathrm{vac}|\mathcal{O}(0,0)|\theta_1,\dots,\theta_n\rangle_{a_1,\dots,a_n}$$

involved in this expression are called *form factors*. They are really a generalisation of the form factors used in usual QFT, because of the intermediate bases that we have. The main point is that these matrix elements are actually meromorphic in the rapidities. They have the following properties, which form what is called a *Riemann-Hilbert problem*:

1. Meromorphicity: as functions of the variable $\theta_i - \theta_j$, for any $i,j \in \{1,\dots,n\}$, they are analytic inside $0 < \mathrm{Im}(\theta_i - \theta_j) < 2\pi$ except for simple poles;

2. Relativistic invariance:

$$F^{\mathcal{O}}_{a_1,\dots,a_n}(\theta_1+\beta,\dots,\theta_n+\beta) = e^{s(\mathcal{O})\beta}F^{\mathcal{O}}_{a_1,\dots,a_n}(\theta_1,\dots,\theta_n)$$

   where $s(\mathcal{O})$ is the spin of $\mathcal{O}$;

3. Generalized Watson's theorem:

$$F^{\mathcal{O}}_{a_1,\dots,a_j,a_{j+1},\dots,a_n}(\theta_1,\dots,\theta_j,\theta_{j+1},\dots,\theta_n) = S^{b_j,b_{j+1}}_{a_j,a_{j+1}}(\theta_j-\theta_{j+1})F^{\mathcal{O}}_{a_1,\dots,b_{j+1},b_j,\dots,a_n}(\theta_1,\dots,\theta_{j+1},\theta_j,\dots,\theta_n)$$

4. Locality:

$$F^{\mathcal{O}}_{a_1,\dots,a_{n-1},a_n}(\theta_1,\dots,\theta_{n-1},\theta_n+2\pi i) = (-1)^{f_{\mathcal{O}}f_\Psi}e^{2\pi i\omega(\mathcal{O},\Psi)}F^{\mathcal{O}}_{a_n,a_1,\dots,a_{n-1}}(\theta_n,\theta_1,\dots,\theta_{n-1})$$

where $f_{\mathcal{O}}$ is 1 if $\mathcal{O}$ is fermionic, 0 if it is bosonic, Ψ is the fundamental field associated to the particle a_n, and $\omega(\mathcal{O},\Psi)$ is the *semi-locality index* (or mutual locality index) of $\mathcal{O}$ with respect to Ψ (to be defined below);

5. Kinematic pole: as function of the variable θ_n, there are poles at $\theta_j + i\pi$ for $j \in \{1,\dots,n-1\}$, with residue

$$iF^{\mathcal{O}}_{a_1,\dots,a_n}(\theta_1,\dots,\theta_n) \sim C_{a_n,b_j}\frac{F_{a_1,\dots,\hat{a}_j,\dots,a_{n-1}}(\theta_1,\dots,\hat{\theta}_j,\dots,\theta_{n-1})}{\theta_n-\theta_j-i\pi}\times$$
$$\Big(\delta^{b_1}_{a_1}\cdots\delta^{b_{j-1}}_{a_{j-1}}S^{b_{j+1},c_j}_{a_{j+1},a_j}(\theta_{j+1}-\theta_j)S^{b_{j+2},c_{j+1}}_{a_{j+2},c_j}(\theta_{j+2}-\theta_j)\cdots S^{b_{n-1},b_j}_{a_{n-1},c_{n-3}}(\theta_{n-1}-\theta_j)-$$
$$(-1)^{f_{\mathcal{O}}f_\Psi}e^{2\pi i\omega(\mathcal{O},\Psi)}\delta^{b_{n-1}}_{a_{n-1}}\cdots\delta^{b_{j+1}}_{a_{j+1}}S^{c_j,b_{j-1}}_{a_j,a_{j-1}}(\theta_j-\theta_{j-1})S^{c_{j-1},b_{j-2}}_{c_j,a_{j-2}}(\theta_j-\theta_{j-2})\cdots S^{b_j,b_1}_{c_3,a_1}(\theta_j-\theta_1)\Big)$$

   where a hat means omission of the argument.

6. Bound-state poles: there are additional poles in the strip $0 < \mathrm{Im}(\theta_i-\theta_j) < \pi$ if bound states are present, and these are the only poles in that strip (I will not go into any detail about this).

I will not say anything about point 6, except that something similar occurs here as for the scattering matrix when bound states are present. It is believe that the set of all solutions to points 1 to 6 form the set of all local fields of an integrable QFT. We see, then, that the scattering matrix is enough to define local fields (if we add the property of crossing symmetry, discussed below, in order to have matrix elements with excited states on both sides), and eventually to evaluate their correlation funcitons (through the form factor expansion).

Before going into an explanation of points 1 to 5, though, I have to introduce the concept of semi-locality.

5.9.1 Local fields and semi-locality

As I said above, a local field $\mathcal{O}(x)$ is by definition a field that commutes with the Hamiltonian density $h(x)$ at space-like distances:

$$[h(x),\mathcal{O}(x')] = 0 \quad (x \neq x') \ .$$

I also introduced the concept of respective locality: a field $\mathcal{O}_1(x)$ is local with respect to another field $\mathcal{O}_2(x)$ if they commute (for bosonic fields) or anti-commute (for fermionic fields) at space-like distances:

$$[\mathcal{O}_1(x),\mathcal{O}_2(x')] = 0 \quad \text{or} \quad \{\mathcal{O}_1(x),\mathcal{O}_2(x')\} = 0 \quad (x \neq x') \ .$$

In integrable QFT, a concept attached to local fields, as important as their anomalous dimension or spin, is that of semi-locality index. Two fields $\mathcal{O}_1,\mathcal{O}_2$ are *semi-local* with respect to each other with index $\omega_{1,2}$ if they satisfy

$$\mathcal{O}_1(x)\mathcal{O}_2(x') = (-1)^{f_1f_2}e^{-2\pi i\omega_{1,2}\Theta(x-x')}\mathcal{O}_2(x')\mathcal{O}_1(x) \quad (x \neq x')$$

or if they satisfy

$$\mathcal{O}_1(x)\mathcal{O}_2(x') = (-1)^{f_1f_2}e^{2\pi i\omega_{1,2}\Theta(x'-x)}\mathcal{O}_2(x')\mathcal{O}_1(x) \quad (x \neq x')$$

where $\Theta(x-x')$ is Heaviside's step function (1 for $x > x'$, zero for $x < x'$) and $(-1)^{f_1f_2}$ is (-1) if both operators are fermionic, 1 otherwise.

Recall that in Feynmann's path integral formulation of QFT, a product of operators inside a vacuum expectation value $\langle \mathrm{vac}|\mathcal{O}_1(x_1,t_1)\cdots\mathcal{O}_n(x_n,t_n)|\mathrm{vac}\rangle$ has to be time-ordered $t_1 > \cdots > t_n$ in order to be represented as the functional integral

$$\int [d\Psi] e^{iS[\Psi]} \mathcal{O}_1(x_1,t_1)\cdots\mathcal{O}_n(x_n,t_n) \ .$$

From this, it is clear that semi-locality just says that the following holds inside any functional integral with other fields at times different from 0:

$$\begin{aligned} &\lim_{\varepsilon\to 0^+} \left[\int [d\Psi] e^{iS[\Psi]} (\cdots \mathcal{O}_1(x,\varepsilon)\mathcal{O}_2(x',0)\cdots)\right] \\ &\quad = e^{\pm 2\pi i \omega_{1,2}\Theta(\pm(x'-x))} \lim_{\varepsilon\to 0^+} \left[\int [d\Psi] e^{iS[\Psi]} (\cdots \mathcal{O}_1(x,-\varepsilon)\mathcal{O}_2(x',0)\cdots)\right] \qquad (x\neq x') \end{aligned}$$

where the signs $\pm$ are synchronized (and the equation holds for one choice of sign only), this being valid for bosonic as well as fermionic fields.

Recall also that in Feynmann's path integral formalism, it is necessary to put a slight imaginary part to the time variables. This means that the equation above will be valid also when the small real parameter ε is replaced by $-i\varepsilon$ in the arguments of the fields, the correlation functions being defined by analytic continuation to imaginary time.

$$\begin{aligned} &\lim_{\varepsilon\to 0^+} \left[\int [d\Psi] e^{iS[\Psi]} (\cdots \mathcal{O}_1(x,-i\varepsilon)\mathcal{O}_2(x',0)\cdots)\right] \\ &\quad = e^{\pm 2\pi i \omega_{1,2}\Theta(\pm(x'-x))} \lim_{\varepsilon\to 0^+} \left[\int [d\Psi] e^{iS[\Psi]} (\cdots \mathcal{O}_1(x,-i\varepsilon)\mathcal{O}_2(x',0)\cdots)\right] \qquad (x\neq x') \end{aligned} \tag{5.24}$$

Then, since with an imaginary time the distance between two operators is always space-like, we can re-formulate semi-locality in a much clearer way. Consider the functional integral

$$F(x,t) = \int [d\Psi] e^{iS[\Psi]} (\cdots \mathcal{O}_1(x,t)\mathcal{O}_2(0,0)\cdots)$$

as a function of x and t, and analytically continue it in the variable t from a space-like region $|x| > |t|$ to purely imaginary values of time $t = -i\tau$, real τ. The action $S[\Psi]$ (or its analytic continuation to imaginary times) gives rise to a natural notion of local real displacement of fields in x and τ through the equations of motion. Then, semi-locality says that the function F "locally displaced" from a point (x,τ) around the point $x=0,\tau=0$ counterclockwise all the way back to x,τ gives rise to a phase:

$$F(e^{i\alpha}z, e^{-i\alpha}\bar{z})\Big|_{\alpha:0\mapsto 2\pi} = e^{2\pi i\omega_{1,2}} F(z,\bar{z})$$

where we use the variables $z = x + i\tau$, $\bar{z} = x - i\tau$. To be more precise, the function $F(z,\bar{z})$ in fact should have a cut on $z\in\mathbb{R}^+$ or on $z\in\mathbb{R}^-$ according to (5.24). But "local displacements" means that we consider its continuation through that cut obtained from the (finite) limit as we approach the cut of its normal derivative (given by the equation of motion); in other words, we consider its value on a covering of the Euclidean plane with a branch point at $(0,0)$.

How can such fields exist in a model? Not all models possess fields that exhibit the property of semi-locality. Important cases where such fields exist are when the model has a global $U(1)$ (or a subgroup thereof) symmetry. In order to understand, it is simpler to go to the Euclidean theory: consider the analytic continuation to imaginary times as described above for all fields in a correlation function. Then, correlation functions are described by the functional integral (we denote $\mathcal{O}^E(x,\tau)$ the fields in Euclidean

theory corresponding to $\mathcal{O}(x,-i\tau)$ understood by analytic continuation as explained above, and we denote by S_E the Euclidean action obtained from analytic continuation)

$$\int [d\Psi] e^{-S_E[\Psi]} \mathcal{O}_1^E(x_1,\tau_1)\cdots\mathcal{O}_n^E(x_n,\tau_n) \ .$$

Suppose the action $S_E[\Psi]$ has a $U(1)$ symmetry; take for instance Ψ to be a complex boson, the $U(1)$ symmetry acting by $\Psi \mapsto e^{i\alpha}\Psi$. Consider a field $\mathcal{O}^E$, called *twist field* (associated to the $U(1)$ symmetry); it simply inserts of a source of magnetic charge, dual to the electric charge associated to the $U(1)$ symmetry. It is defined as follows (here we must imagine that there is another magnetic charge at infinity; it doesn't matter because this other field at infinity really just affects the normalisation of correlation functions):

$$\int [d\Psi] e^{-S_E[\Psi]} \mathcal{O}^E(0,0)\mathcal{O}_1^E(x_1,\tau_1)\cdots\mathcal{O}_n^E(x_n,\tau_n) = \int_{\mathcal{C}(0,0)} [d\Psi] e^{-S_E[\Psi]} \mathcal{O}_1^E(x_1,\tau_1)\cdots\mathcal{O}_n^E(x_n,\tau_n) \quad (5.25)$$

where $\mathcal{C}(0,0)$ is a quasi-periodicity condition on the fields Ψ in the functional integral, stating that

$$\mathcal{C}(0,0) \ : \ \begin{cases} \Psi(x,0^-) = e^{2\pi i\omega}\Psi(x,0^+) \quad (x>0) \\ \Psi(x,0^-) = \Psi(x,0^+) \quad (x<0) \ . \end{cases} \quad (5.26)$$

More precisely, the functional integral is defined by taking away the ray $x>0$, evaluating the functional integral with fixed boundary conditions on each side of the cut with (5.26) satisfied, then integrating over all the configurations on each sides of the cut keeping (5.26) satisfied. It is obvious that the field $\mathcal{O}^E$ thus defined has semi-locality index ω with respect to the fundamental field Ψ, and semi-locality index 0 with respect to any field that is $U(1)$-invariant. Also, the field $\mathcal{O}^E$ is a local field since the Hamiltonian density $h(x)$ is certainly $U(1)$-invariant, hence $\mathcal{O}^E$ is local with respect to it.

A consequence of it being a local field is that the functional integral is in fact independent of the shape of the cut from 0 to ∞ on each side of which the quasi-periodicity condition is imposed, up to a $U(1)$ transformation of the fields that the cut may cross while moving. Indeed, consider cuting not only on $A: \ x>0$, but also on a path B starting and ending on the line $x>0$, isolating a region bounded by B and by the segment $C \subset A$. Fix configurations along all these cuts with the quasi-periodicity condition (5.26) across A and with a condition of continuity across B. Integrating over all configurations on these cuts certainly gives back the functional integral (5.25) (puting the cut at B with continuity condition across it and suming over the configurations on it is like puting a complete set of states, hence does not change the path integral). But, since one region is isolated, we can make a $U(1)$ transformation in that region, which preserves the action in that region and change the fixed boundary conditions in a simple way: the new conditions across the cut B are quasiperiodicity conditions similar to (5.26), and the condition across C is continuity. Integrating over all configurations on the cuts now gives a functional integral with a modified cut, $B \cup (A \setminus C)$, across which the quasi-periodicity condition holds. The only difference is in the transformation of the fields present inside the isolated region, which shows the assertion.

Further insight can be gained into such fields making a change of variable in order to make the cut disappear. Indeed, consider the path integral on the right-hand side of (5.25) and apply a $U(1)$ transformation on the fields inside a (infinite) triangular region bounded by $A: \ x>0$, by $B: \ z = e^{i\alpha}l$, $l>0$ and by infinity (the triangular region is, for instance, above the cut A). Since this is a symmetry transformation, the action is invariant under this transformation up to three changes: a contribution on the boundary B of the triangle coming from the kinetic (derivative) term in the action, the modification

of the boundary condition just above A, and the $U(1)$ transformations of fields that may be inside this triangle. We can choose the $U(1)$ transformation in such a way that the boundary condition just above A is equal to that just below it. Then, the path integral becomes a usual path integral without cut, but with insertion of the contribution of the action along B:

$$\mathcal{O}^E(0,0) = e^{2\pi i\omega \int_B dx^\mu j^\nu \epsilon_{\mu,\nu}} \tag{5.27}$$

where we denoted the contribution as an integration along B of the component of a current perpendicular to B (obviously, here we would need an appropriate regularisation of the exponential in order to define this field properly). We showed that by moving B through fields they get $U(1)$-transformed. Moving B can be done here simply by puting the exponential of the integral of the current, $e^{i\omega \int_L dx^\mu j^\nu \epsilon_{\mu,\nu}}$, along a loop L that shares a border with a segment of B and such that, on this segment, the integral goes in opposite direction as the integral in $e^{i\omega \int_B dx^\mu j^\nu \epsilon_{\mu,\nu}}$ above. Hence, we find that the insertion of

$$e^{2\pi i\omega \int_L dx^\mu j^\nu \epsilon_{\mu,\nu}}$$

has the effect of a $U(1)$ transformation of the fields surrounded by L. This means that the current j^ν is nothing else than the Noether current associated to the $U(1)$ invariance (and the derivation above may be seen as a way of defining this current). This makes it even more clear that the field is like the insertion of a magnetic charge.

Note that it is simple to extend this construction to twist fields associated to any global symmetry of the model. The form factor equations, however, have not been derived for such fields yet.

An example of such a twist field is the field in the free massive Majorana theory corresponding to the spin variable in the Ising model. The Majorana theory can be defined by its Euclidean action

$$S_E[\psi,\bar\psi] = -i\int d^2x\ \left[\psi\bar\partial\psi - \bar\psi\partial\bar\psi + 2m\bar\psi\psi\right]$$

where $\partial \equiv (1/2)(\partial/\partial x - i\partial/\partial\tau)$ and $\bar\partial \equiv (1/2)(\partial/\partial x + i\partial/\partial\tau)$. The fermion fields $\psi, \bar\psi$ are both real and are governed by the equations of motion

$$\bar\partial\psi = \frac{m}{2}\bar\psi\ , \quad \partial\bar\psi = \frac{m}{2}\psi\ .$$

This means that any correlation function $\langle \text{vac}|\mathcal{O}_1(x_1,\tau_1)\cdots\mathcal{O}_n(x_n,\tau_n)\psi(x,\tau)|\text{vac}\rangle$ will satisfy, as function of x and τ, the equation of motion $\partial_x^2 + \partial_\tau^2 - m^2 = 0$.

The action has the Z_2 symmetry $\psi \mapsto -\psi$, $\bar\psi \mapsto -\bar\psi$. We can then define a twist field σ associated to this symmetry as above. In the free-field context, the definition means that the correlation function

$$\langle \text{vac}|\sigma(0,0)\psi(x,\tau)\psi(x',\tau')|\text{vac}\rangle$$

(we put two fermion fields so that it is non-zero) is, as function of both x,τ and x',τ', a solution to the equation of motion on a double covering of the Euclidean plane with branch point at $0,0$, and with quasi-periodicity condition according to which it gains a sign when going once around $0,0$. Along with the condition that as x,τ go to $0,0$, we get the least singular behavior, the solution is unique up to a normalisation. The condition on the least singular behavior comes from the fact that many fields, beside the twist fields introduced above, have a the same twist-field effect: their descendants under the free fermion operator algebra. But they have higher scaling dimension, so that the twist field as defined above is uniquely characterised by further imposing the condition of least singular behavior on the correlation functions. The field σ corresponds to the scaling limit of the spin variable in the lattice Ising model near criticality.

5.9.2 Analyticity and Watson's theorem

Properties 2 is obvious from relativistic invariance. Property 3 is also obvious from our definition of the states $|\theta_1,\dots,\theta_n\rangle_{a_1,\dots,a_n}$ as, generically, intermediate "asymptotic-like" states. However, the real statement of Watson's theorem, which is not obvious, is a part of Property 1: the fact that the regions $\theta_j-\theta_{j+1}>0$ and $\theta_j-\theta_{j+1}<0$ are analytic continuation of one another through at least a small strip containing the real axis. In the two-particle case, this is consequence of general (assumed) analytical properties of QFT. An heuristic argument can be given on the lines of the argument given above for understanding (5.17). Consider the path-integral formulation of the following matrix element, with $\theta_1>\theta_2$:

$$ {}^{out}_{a_1,a_2}\langle\theta_1,\theta_2|\mathcal{O}|\mathrm{vac}\rangle=\int_{\Psi=\sum_k A_{a_k}e^{iE_kt-ip_kx}\ (t\to+\infty)}[d\Psi]e^{iS[\Psi]}\ .$$

Here, $\mathcal{O}$ is a local field at the point $0,0$. Again, the sum specifying the asymptotic conditions on the fundamental field has to be understood as giving the spacetime-dependent phase factors for wave packets ordered, on the space slice $t=-\infty$, from left to right with increasing index k. Taking the complex conjugate gives

$$\langle\mathrm{vac}|\mathcal{O}|\theta_1,\theta_2\rangle^{out}_{a_1,a_2}=\int_{\Psi=\sum_k A_{a_k}e^{-iE_kt+ip_kx}\ (t\to+\infty)}[d\Psi]e^{-iS[\Psi]}$$

and time-reversal invariance leads to

$$\langle\mathrm{vac}|\mathcal{O}|\theta_1,\theta_2\rangle^{out}_{a_1,a_2}=\int_{\Psi=\sum_k A_{a_k}e^{iE_kt+ip_kx}\ (t\to-\infty)}[d\Psi]e^{iS[\Psi]}$$

This is just the matrix element $\langle|\mathcal{O}|\theta_1',\theta_2'\rangle^{in}_{a_1,a_2}$ analytically continued to from $\theta_1'>\theta_2'$ to $\theta_1'=-\theta_1$, $\theta_2'=-\theta_2$ (that is, the main assumption here is that the path integral with wave-packet asymptotic prescription in the "wrong order" is just the analytic continuation of that in the "right order"). In other words, if there are no particle production (as in integrable systems, or else if the energies of the particles are small enough), then we have, for $\theta_1>\theta_2$,

$$\langle\mathrm{vac}|\mathcal{O}|\theta_1,\theta_2\rangle^{in}_{a_1,a_2}=S^{b_1,b_2}_{a_1,a_2}(\theta_1-\theta_2)\langle\mathrm{vac}|\mathcal{O}|\theta_2',\theta_1'\rangle^{in}_{a_2,a_1}\Big|_{\theta_2'\mapsto\theta_2,\theta_1'\mapsto\theta_1}$$

where the analytic continuation is now from $\theta_2'>\theta_1'$. This is Watson's theorem

$$F^{\mathcal{O}}_{a_1,a_2}(\theta_1,\theta_2)=S^{b_1,b_2}_{a_1,a_2}(\theta_1-\theta_2)F^{\mathcal{O}}_{b_2,b_1}(\theta_2,\theta_1)$$

in the 2-particle case; that is, the analytic continuation agrees with our choice of basis of asymptotic states. This was derived quite generally, but in integrable systems we have the intermediate basis, where processes occur 2-particle by 2-particle. Although the derivation above would not work as is, the important point is that if we change the rapidities in such a way that two wave packets, first in the correct order for an *in*-state, become colinear and then are in the wrong order, doing that by analytic continuation, then we get a factor of the scattering matrix associated to this 2-particle process. We can do that independently for every two-particle processes in our intermediate states of integrable systems, and this gives generalized Watson's theorem.

Another fact which is not obvious is the other part of Property 1: the fact that only simple poles may occur all the way up to the line of imaginary part 2π. This is called the "maximal analyticity assumption". That no other singularities occur all the way up to the line of imaginary part π is from standard principles of QFT, but the maximal analyticity assumption is really just an assumption, and for integrable models, the current way of thinking is to take this for granted, and to verify that the form

factors obtained under this assumption (and the other form factor properties) give rise to local fields of the theory.

5.9.3 Crossing symmetry and locality property

For explaining Points 4 and 5, it will be useful to have in mind particles that possess a $U(1)$ charge, so that we consider Ψ, the fundamental field associated to a particle, to be a complex field. We will take $\mathcal{O}$ to be the associated twist field with semi-locality index ω defined as above.

The locality property, point 4, is essentially a consequence of two applications of crossing symmetry.

Crossing symmetry, a general property of QFT, stipulates that analytically continuing the rapidity variable θ_n to $\theta_n + i\pi - i0^+$ in the matrix element

$$\langle \mathrm{vac}|\mathcal{O}|\theta_1,\dots,\theta_{n-1},\theta_n\rangle^{(in)}_{a_1,\dots,a_{n-1},a_n} \ ,$$

with initially $\theta_1 > \cdots > \theta_n$, gives the matrix element

$${}^{out}_{\bar{a}_n}\langle \theta_n - i0^+|\mathcal{O}|\theta_1,\dots,\theta_{n-1}\rangle^{(in)}_{a_1,\dots,a_{n-1}}$$

understood as analytic continuation again. The latter matrix element, without the $-i0^+$, is really a distribution for real rapidities. This distribution, in terms of θ_n, has a part supported at other rapidities $\theta_1,\dots,\theta_{n-1}$, and a part supported on the complement on $\mathbb{R}$. Essentially, the part supported on the points $\theta_1,\dots,\theta_{n-1}$ is interpreted as coming from particles going through from *in*- to *out*-state without interacting with the operator. The analytic continuation above, with the $-i0^+$, is the analytic continuation of the part supported away from these points.

Crossing symmetry can be understood by slowly bringing the wave packet associated to θ_n around the operator $\mathcal{O}$ counterclockwise from the *in*-state to the *out*-state. This motion is implemented by taking first θ_n to $-\infty$, then adding $i\pi/2$, then bringing its real part to $+\infty$, then adding again $i\pi/2$ and finally bringing its real part back to its initial value. In this final motion, we need to avoid the position of the rapidities $\theta_1,\dots,\theta_n$, by keeping a slight negative imaginary part. Straightening the path (under the assumption that there are no branch points in the rapidity plane) gives the statement above. With the operator $\mathcal{O}$ being a twist field, we need, for this to be valid, to take the cut to its left (although the shape of the cut does not affect the correlation functions, it affects some of its matrix elements on the Hilbert space), so that the wave packet does not cross it.

Moving the wave packet from an *out* position to an *in* position, a different expression of crossing symmetry allows us to start from

$${}^{out}_{\bar{a}_n}\langle \theta_1 + i0^+|\mathcal{O}|\theta_2,\dots,\theta_n\rangle^{(in)}_{a_2,\dots,a_1}$$

with $\theta_1 > \cdots > \theta_n$ and analytically continue to $\theta_1 \mapsto \theta_1 + i\pi$, giving back

$$\langle \mathrm{vac}|\mathcal{O}|\theta_1,\theta_2,\dots,\theta_{n-1},\theta_n\rangle^{(in)}_{a_1,a_2,\dots,a_n} \ .$$

In doing so, however, the wave packet will need to cross the cut of the operator $\mathcal{O}$ if it is semi-local with respect to the associated fundamental field (we assume throughout that this cut is on the left). This gives a factor with the semi-locality index. In fact, in a more precise treatment of this, the factor of

semi-locality would come from commuting mutually semi-local operators, and it is always accompanied by a factor $(-1)^{f_{\mathcal{O}} f_\Psi}$ [5].

With an appropriate analyticity assumption, included in the assumption of maximal analyticity, the two values

$$ {}^{out}_{\bar{a}_n}\langle \theta_1 \pm i0^+ | \mathcal{O} | \theta_2, \dots, \theta_n \rangle^{(in)}_{a_2,\dots,a_1} $$

are analytic continuation of each other everywhere on $\theta_1 \in [\theta_2, \theta_n]$ (no cut occurs, or at least we can correctly go around it). Then, we can apply twice crossing symmetry and we find Point 4 with the understanding that the analytic continuation of θ_n is from a value with $\theta_1 > \cdots > \theta_n$ towards $\theta_n + 2\pi i$ with $\theta_n > \theta_1 > \cdots > \theta_{n-1}$. Since in integrable models the form factors are meromorphic, this can be extended to the real part of θ_n staying the same.

5.9.4 Kinematic pole

The kinematic pole, point 5, is, in its simplest form, a consequence of general principles of QFT (although one needs the analyticity assumption of the paragraph above). Consider the two-particle case:

$$ iF^{\mathcal{O}}_{a_1,a_2}(\theta_1,\theta_2) \sim C_{a_2,a_1} \frac{1 - e^{2\pi i \omega}}{\theta_2 - \theta_1 - i\pi} $$

where we used the fact that $(-1)^{f_{\mathcal{O}} f_\Psi} = 1$ (because, even if Ψ is fermionic, in order to have a non-zero two-particle form factor, $\mathcal{O}$ must bosonic: $f_{\mathcal{O}} = 0$). In order to derive this equation, we will look at the behavior around and at $\theta_1 = \theta_2$ of the different but related matrix element

$$ g_{\bar{a}_2,a_1}(\theta_2,\theta_1) = \langle \mathrm{vac} | A_{\bar{a}_2}(\theta_2)^{(out)} \mathcal{O} A^\dagger_{a_1}(\theta_1)^{(in)} | \mathrm{vac} \rangle \ . $$

As we said above, it is generically not an analytical function, but rather a distribution. The part of the distribution supported away from $\theta_1 = \theta_2$ has an analytic continuation that may have a singularity at this point. The part supported at $\theta_1 = \theta_2$ comes from particles going through from *in*- to *out*-state without interacting with the operator, and the singularity of the part supported away comes from particles being affected only by the cut giving the semi-locality of the operator $\mathcal{O}$.

We could write this matrix element using the definition (5.6) of the operators $A(\theta)^{(in,out)}$. Instead, we will use what is sometimes called the "extrapolating fields":

$$ \begin{aligned} g_{\bar{a}_2,a_1}(\theta_2,\theta_1) \quad &= \quad \langle \mathrm{vac} | \int dx_2 \, e^{-ip_2 x_2} \left(\partial_t \tilde{\Psi}_{\bar{a}_2}(x_2,0)^{(out)} - iE_2 \tilde{\Psi}_{\bar{a}_2}(x_2,0)^{(out)} \right) \mathcal{O} \ \times \\ &\quad \times \int dx_1 \, e^{ip_1 x_1} \left(\partial_t \tilde{\Psi}^\dagger_{a_1}(x_1,0)^{(in)} + iE_1 \tilde{\Psi}^\dagger_{a_1}(x_1,0)^{(in)} \right) | \mathrm{vac} \rangle \end{aligned} \tag{5.28} $$

where, as usual, $p_1 = m \sinh(\theta_1)$, $E_1 = m \cosh(\theta_1)$, etc. The extrapolating fields are defined by

$$ \tilde{\Psi}_a(x,t)^{(in/out)} = \frac{1}{4i\pi} \int d\theta \, \left[e^{ip(\theta)x - iE(\theta)t} A_a(\theta)^{(in/out)} + e^{-ip(\theta)x + iE(\theta)t} A^\dagger_{\bar{a}}(\theta)^{(in/out)} \right] \ . $$

These fields are not generically local, and very different from the local fundamental field associated to

[5] That this factor appears in doing the latter crossing $\theta_1 + i0^+ \mapsto \theta_1 + i\pi$ or in doing the former one $\theta_n \mapsto \theta_n + i\pi - i0^+$ is really a matter of definition of the phase of the asymptotic states (which is not fixed by their normalisation).

the particle a, which would have the form, in terms of *in* operators,

$$\begin{aligned}\Psi_a(x,t) &= \int d\theta \Bigg[e^{ip(\theta)x}\Big(e^{-iE(\theta)t}A_a(\theta)^{(in)} + \\ &+ \int d\theta_1 d\theta_2 \delta(p(\theta)-p_1-p_2) e^{-i(E_1+E_2)t} u_a^{a_1,a_2}(\theta_1,\theta_2) A_{a_1}(\theta_1)^{(in)} A_{a_2}(\theta_2)^{(in)} + \\ &+ \int d\theta_1 d\theta_2 \delta(p(\theta)+p_1-p_2) e^{-i(-E_1+E_2)t} v_a^{a_1,a_2}(\theta_1,\theta_2) A^\dagger_{\bar a_1}(\theta_1)^{(in)} A_{a_2}(\theta_2)^{(in)} + \\ &+ \int d\theta_1 d\theta_2 \delta(p(\theta)+p_1+p_2) e^{-i(-E_1-E_2)t} w_a^{a_1,a_2}(\theta_1,\theta_2) A^\dagger_{\bar a_1}(\theta_1)^{(in)} A^\dagger_{\bar a_2}(\theta_2)^{(in)} + \cdots \Big) + \\ &+ e^{-ip(\theta)x}\Big(e^{iE(\theta)t}A^\dagger_{\bar a}(\theta)^{(in)} + \cdots\Big)\Bigg]\end{aligned}$$

and a similar expression exists in terms of *out* operators (only in free theories are extrapolating fields equal to fundamental fields). Note that the particle types involved on the right-hand side must be such that the $U(1)$ charge is the same on both sides. Inverting, one would obtain

$$\tilde\Psi_a(x,t) = \Psi_a(x,t) + \int dx_1 dx_2 \; [K_a^{a_1,a_2}(x,x_1,x_2)\Psi_{a_1}(x_1,t)\Psi_{a_2}(x_2,t) + \cdots]$$

for some $K_a^{a_1,a_2}(x,x_1,x_2)$, where the $\cdots$ mean terms of similar type involving one time-derivative of the fields, complex conjugate fields, and more and more factors. Again, on the right-hand side, the $U(1)$ charge is the same as that of the left-hand side. The important point is that the quantities $K_a^{a_1,a_2}(x,x_1,x_2)$ vanish exponentially fast as x_1 or x_2 are brought far from x:

$$K_a^{a_1,a_2}(x,x_1,x_2) \propto e^{-m|x-x_1|} \quad (|x-x_1|\to\infty)\ , \quad K_a^{a_1,a_2}(x,x_1,x_2) \propto e^{-m|x-x_2|} \quad (|x-x_2|\to\infty)\ .$$

That is, although the extrapolating fields are not strictly local, they are "of finite extent", in the sense that they have tails that decrease exponentially.

Now, the behavior near and at $\theta_1=\theta_2$, for finite p_1+p_2, is obtained from (5.28) by looking at the region $|x_1+x_2|\to\infty$ with $|x_1-x_2|$ finite. Since the extrapolating fields are then very far from the field $\mathcal{O}$ and since they are of finite extent, the resulting correlation function factorizes:

$$\begin{aligned} g_{\bar a_2,a_1}(\theta_2,\theta_1) &\sim \langle \mathrm{vac}| \int dx_2\, e^{-ip_2x_2} e^{2\pi i\omega\Theta(-x_1)} \left(\partial_t \tilde\Psi_{\bar a_2}(x_2,0)^{(out)} - iE_2\tilde\Psi_{\bar a_2}(x_2,0)^{(out)}\right) \times \\ &\quad \times \int dx_1\, e^{ip_1x_1}\left(\partial_t\tilde\Psi^\dagger_{a_1}(x_1,0)^{(in)} + iE_1\tilde\Psi^\dagger_{a_1}(x_1,0)^{(in)}\right)|\mathrm{vac}\rangle \times \\ &\quad \langle \mathrm{vac}|\mathcal{O}|\mathrm{vac}\rangle\ . \end{aligned}$$

The operations involved in factorizing are as follows: we first bring the operator at x_1 just to the left of $\mathcal{O}$ inside the vacuum expectation value in (5.28), which brings a factor of semi-locality since the extrapolating fields are for from the point $x=0$, are of finite extent and have a well-defined $U(1)$ charge; then we factorize the correlation function as above, using the fact that the extrapolating fields have finite extent. Then, we can use translation invariance and the change of variable $x_2\mapsto x_2+x_1$ in order to find

$$\begin{aligned} g_{\bar a_2,a_1}(\theta_2,\theta_1) &\sim \int dx_1 e^{ix_1(p_1-p_2)} e^{2\pi i\omega\Theta(-x_1)} \int dx_2\, e^{-ip_2x_2}\langle \mathrm{vac}|\left(\partial_t\tilde\Psi_{\bar a_2}(x_2,0)^{(out)} - iE_2\tilde\Psi_{\bar a_2}(x_2,0)^{(out)}\right)\times \\ &\quad \times\left(\partial_t\tilde\Psi^\dagger_{a_1}(0,0)^{(in)} + iE_1\tilde\Psi^\dagger_{a_1}(0,0)^{(in)}\right)|\mathrm{vac}\rangle\ \langle\mathrm{vac}|\mathcal{O}|\mathrm{vac}\rangle\ . \end{aligned}$$

The integral over x_1 can be evaluated using the distributional identities

$$\int dx\, e^{ipx}\,\mathrm{sign}(x) = 2i\underline{\mathrm{P}}\left(\frac{1}{p}\right)\ , \quad \int dx\, e^{ipx} = 2\pi\delta(p) \tag{5.29}$$

where $\underline{\mathrm{P}}$ means *principal value* (that is, under integration, we cut a region of length ϵ symmetricaly distributed around the pole at $p=0$ and evaluate the limit $\epsilon\to 0$ after integrating). This gives

$$\int dx_1 e^{ix_1(p_1-p_2)} e^{2\pi i\omega\Theta(-x_1)} = \left(1-e^{2\pi i\omega}\right) i\underline{\mathrm{P}}\left(\frac{1}{p_1-p_2}\right) + \left(1+e^{2\pi i\omega}\right)\pi\delta(p_1-p_2)\ .$$

The integral over x_2 can be evaluated by using the explicit expression for the extrapolating fields. A shorter way to evaluate it at $\theta_1=\theta_2$ (which is the only point we need) is to realize that we could have done the same calculation with $\mathcal{O}$ being just the identity operator $\mathbf{1}$, with $\omega=0$, and that this should give $2\pi\delta(\theta_1-\theta_2)\delta_{a_1,\bar{a}_2}$, hence

$$\int dx_2\, e^{-ip_2x_2}\langle\mathrm{vac}|\left(\partial_t\tilde{\Psi}_{\bar{a}_2}(x_2,0)^{(out)} - iE_2\tilde{\Psi}_{\bar{a}_2}(x_2,0)^{(out)}\right)\left(\partial_t\tilde{\Psi}^\dagger_{a_1}(0,0)^{(in)} + iE_2\tilde{\Psi}^\dagger_{a_1}(0,0)^{(in)}\right)|\mathrm{vac}\rangle = E_2\delta_{a_1,\bar{a}_2}\ .$$

Puting all that together, we find

$$g_{\bar{a}_2,a_1}(\theta_2,\theta_1) \sim \left[\left(1-e^{2\pi i\omega}\right) i\underline{\mathrm{P}}\left(\frac{1}{\theta_1-\theta_2}\right) + \left(1+e^{2\pi i\omega}\right)\pi\delta(\theta_1-\theta_2)\right] C_{a_1,a_2}$$

where we used the fact that $\delta_{a_1,\bar{a}_2}$ just gives the matrix element C_{a_1,a_2} of the conjugation matrix.

Now, the function $g_{\bar{a}_2,a_1}(\theta_2,\theta_1)$ is simply related to the form factor $F^{\mathcal{O}}_{a_1,a_2}(\theta_1,\theta_2)$ by crossing symmetry: $F^{\mathcal{O}}_{a_1,a_2}(\theta_1,\theta_2) = g_{\bar{a}_2,a_1}(\theta_2-i\pi,\theta_1)$ where we analytically continue only the part $\left(1-e^{2\pi i\omega}\right) i\underline{\mathrm{P}}\left(\frac{1}{\theta_1-\theta_2}\right)$. This gives the kinematical residue equation in the 2-particle case.

One can try to do the many-particle case in a similar way still for non-integrable models, by considering the matrix element

$$g_{\bar{a}_n,a_1,a_2,\ldots,a_{n-1}}(\theta_n,\theta_1,\theta_2,\ldots,\theta_{n-1}) = \langle\mathrm{vac}|A_{\bar{a}_n}(\theta_n)^{(out)}\mathcal{O}A^\dagger_{a_1}(\theta_1)^{(in)}A^\dagger_{a_2}(\theta_2)^{(in)}\cdots A^\dagger_{a_{n-1}}(\theta_{n-1})^{(in)}|\mathrm{vac}\rangle \tag{5.30}$$

with $\theta_1>\cdots>\theta_{n-1}$ and writing the asymptotic state operators in terms of extrapolating fields integrated over their positions $x_1,\ldots,x_n$. There is a subtlety, however, that lies in using the identities (5.29). They should really be understood as conditionally convergent integral, made convergent by giving a small x-dependent imaginary part to p, positive for $x>0$ and negative for $x<0$. This means that when we look at the region where x_n is very positive, we will really be considering the matrix element

$$x_n \gg m^{-1} \quad : \quad g_{\bar{a}_n,a_1,a_2,\ldots,a_{n-1}}(\theta_n-i0^+,\theta_1,\theta_2,\theta_{n-1})$$

whereas when we look at the region where both x_n is very negative, we will be considering the matrix element

$$x_n \ll -m^{-1} \quad : \quad g_{\bar{a}_n,a_1,a_2,\ldots,a_{n-1}}(\theta_n+i0^+,\theta_1,\theta_2,\theta_{n-1})\ .$$

The first matrix element is related by crossing symmetry to the form factor $F^{\mathcal{O}}_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n)$ for $\theta_n>\theta_{n-1}$ and the second is related to $F^{\mathcal{O}}_{a_n,a_1,\ldots,a_{n-1}}(\theta_n,\theta_1,\ldots,\theta_{n-1})$ for $\theta_n<\theta_1$. Hence, the two regions of x_n will give us contributions to poles of these two different functions of θ_n at different points: the first at $\theta_{n-1}+i\pi$, the second at $\theta_1-i\pi$. The second can also be seen as a pole at $\theta_1+i\pi$ of the analytic continuation of the form factor $F^{\mathcal{O}}_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n)$ from the region $\theta_n>\theta_{n-1}$, under the appropriate analyticity assumption (included in the maximal analyticity assumption). It is really from this and the fact that $\theta_{n-1}=\theta_1$ in the two-particle case that we obtained the two-particle result above.

The residue of these two contributions to poles can be evaluated as above for the two-particle case. We get a $n-2$-particle form factor, times an appropriate sign and semi-locality factor. These can be evaluated

as follows. In the first case (x_n positive), we will have to bring the right-most operators $\tilde{\Psi}^{in}_{a_{n-1}}(x_{n-1})$ almost all the way to the left, just next to $\tilde{\Psi}^{out}_{\bar{a}_n}(x_n)$, on its right, before doing the factorisation. If both $\mathcal{O}$ and the particle $\bar{a}_n$ are fermionic, then the particle a_{n-1} will have to be fermionic as well and the operators $\tilde{\Psi}_{a_{n-1}}(x_{n-1})$ will cross an odd number of fermionic extrapolating fields as well as the operator $\mathcal{O}$, so that no sign will be gained in this process. A little thought on similar lines shows that no matter if $\mathcal{O}$ or a_n are bosonic or fermionic, no sign will be gained. In the second case, (x_n negative) the extrapolating field associated to a_1, which will have to be of the same statistics as $\bar{a}_n$, will only have to cross the operator $\mathcal{O}$, so that we will have a factor $(-1)^{f_{\mathcal{O}} f_{\Psi_{a_n}}}$. Moreover, by arguments similar to those above for the two-particle case, we will have a factor of semi-locality only for x_n negative, that is, in the second case.

We still do not have the full information about the poles, because the two contributions are for poles at different points. But in integrable models, we can also calculate the part of the x_n and x_{n-1} integrals for very negative x_n and x_{n-1} in the matrix element $g_{\bar{a}_n,a_1,a_2,\ldots,a_{n-1}}(\theta_n,\theta_1,\theta_2,\ldots,\theta_{n-1})$ with $\theta_1 > \ldots > \theta_{n-1} > \theta_n$ written as integrations over extrapolating fields from (5.30). Let us divide this matrix element into two part: one that contains the x_{n-1}-integral from $-\infty$ up to some value X, the other that goes from X to ∞:

$$g_{\bar{a}_n,a_1,\ldots,a_{n-1}}(\theta_n,\theta_1,\ldots,\theta_{n-1}) = g^{(-)}_{\bar{a}_n,a_1,\ldots,a_{n-1}}(\theta_n,\theta_1,\ldots,\theta_{n-1}) + g^{(+)}_{\bar{a}_n,a_1,\ldots,a_{n-1}}(\theta_n,\theta_1,\ldots,\theta_{n-1}) \ .$$

The latter part gives rise to a contribution to the pole at $\theta_n = \theta_{n-1}$ that we already know how to calculate, but the former part also gives a contribution, which we do not know yet how to calculate. However, we know how to write this matrix element as an analytic continuation of the matrix element $g_{\bar{a}_n,a_{n-1},a_1,\ldots,a_{n-2}}(\theta_n,\theta_{n-1},\theta_1,\ldots,\theta_{n-2})$ with $\theta_{n-1} > \theta_1 > \cdots > \theta_{n-2}$, using multiple applications of generalized Watson's theorem, symbolically:

$$g_{\bar{a}_n,a_1,\ldots,a_{n-1}}(\theta_n,\theta_1,\ldots,\theta_{n-1}) = \Big(\prod S\Big)\, g_{\bar{a}_n,a_{n-1},a_1,\ldots,a_{n-2}}(\theta_n,\theta_{n-1},\theta_1,\ldots,\theta_{n-2}) \ .$$

This analytic continuation does not exactly hold separately for the positive-x_{n-1} and the negative-x_{n-1} parts; for instance:

$$g^{(-)}_{\bar{a}_n,a_1,\ldots,a_{n-1}}(\theta_n,\theta_1,\ldots,\theta_{n-1}) = \Big(\prod S\Big)\, g^{(-)}_{\bar{a}_n,a_{n-1},a_1,\ldots,a_{n-2}}(\theta_n,\theta_{n-1},\theta_1,\ldots,\theta_{n-2}) + \Delta$$

where Δ is a correction. However, we can evaluate the residue of the pole of the first term on the right-hand side at $\theta_n = \theta_{n-1}$ and we find that it is independent of X. Hence, since the correction Δ vanishes as $X \to \infty$, this gives the full residue of that pole of the left-hand side for all finite X. The contributions at x_n, x_{n-1} very positive and at x_n, x_{n-1} very negative are the only two contributions to the pole at $\theta_n = \theta_{n-1}$ of $g_{\bar{a}_n,a_1,a_2,\ldots,a_{n-1}}(\theta_n,\theta_1,\theta_2,\ldots,\theta_{n-1})$, hence we have the full residue of the pole at $\theta_n = \theta_{n-1} + i\pi$ of the form factor $F^{\mathcal{O}}_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n)$, and we have shown point 5.

Note that a similar argument also holds for the part of $g_{\bar{a}_n,a_1,a_2,\ldots,a_{n-1}}(\theta_n,\theta_1,\theta_2,\ldots,\theta_{n-1})$ supported at $\theta_n = \theta_{n-1}$. In general, we can write

$$g_{\bar{a}_n,a_1,a_2,\ldots,a_{n-1}}(\theta_n,\theta_1,\theta_2,\ldots,\theta_{n-1}) = \left(\frac{-iR}{\theta_n - \theta_{n-1} + i0^+} + 2\pi\delta(\theta_n - \theta_{n-1})\right) F^{\mathcal{O}}_{a_1,\ldots,a_{n-2}}(\theta_1,\ldots,\theta_{n-2}) + \ldots$$

where $\ldots$ are convergent at $\theta_n = \theta_{n-1}$ and R is as calculated above. This should be understood using

$$\frac{1}{\theta + i0^+} = \underline{\mathrm{P}}\left(\frac{1}{\theta}\right) - i\pi\delta(\theta) \ .$$

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