Integrability - Assessment, due 17 December 2012

Benjamin Doyon
King's College London

Fall 2012
London Taught Centre

Problem 1

a) Derive the fundamental Sklyanin relation

$$\{L_1(\lambda), L_2(\mu)\} = [r_{12}(\lambda,\mu), L_1(\lambda)L_2(\mu)]$$

from the local Poisson brackets

$$\{U_1(\lambda;x), U_2(\mu;y)\} = [r_{12}(\lambda,\mu), U_1(\lambda;x) + U_2(\mu;y)]\delta(x-y)$$

for the connection $U(\lambda;x)$ of the general zero-curvature formulation of integrable field theory. Recall that

$$L(\lambda) = \mathcal{P}\exp\left[\int_{-\infty}^{\infty} ds\, U(\lambda;s)\right].$$

b) Show that the fundamental Sklyanin relation implies that $\mathrm{Tr}(L(\lambda))$ is a generating function for Poisson commuting quantities (you do not need to discuss locality!).

Problem 2

a) Calculate the conserved quantity

$$Q_2 = \left(\frac{d^2F(\lambda)}{d\lambda}F(\lambda)^{-1}\right)_{\lambda=i/2} - Q_1^2$$

where

$$F(\lambda) = \mathrm{Tr}_a T_a(\lambda)$$

and $T_a(\lambda)$ is the transfer matrix for the Heisenberg chain. You only need to bring Q_2 into a form that makes it obvious that it is a local conserved quantity. In addition to the relations from the lectures / notes, you can use the following relations:

- The conserved quantity Q_1 is
$$Q_1 = -i\sum_j P_{j,j+1}$$
where $P_{j,k}$ is the permutation operator between the sites j and k.
- The relations
$$\mathrm{Tr}_a\left(P_{N,a}\cdots P_{1,a}\right) = U, \quad \mathrm{Tr}_a\left(P_{N,a}\cdots \widehat{P_{j,a}}\cdots\widehat{P_{k,a}}\cdots P_{1,a}\right) = U_{j,k}$$
where U is the "clockwise" cyclic permutation operator, and $U_{j,k}$ is the clockwise cyclic permutation operator which skips the sites j and k.

- The relation

$$U_{j,k} = :P_{j,j+1}P_{k,k+1}: U$$

where the "normal ordering" of permutation operators $:P_{j,j+1}P_{k,k+1}:$ orders the factors in such a way that the second index of the first factor is not equal to the first index of the second factor.

b) Calculate the exchange relations for the pairs $B(\lambda), A(\mu)$; $B(\lambda), B(\mu)$ and $B(\lambda), D(\mu)$ of auxiliary-space matrix elements of the transfer matrix, from the RTT relations with the XXZ-chain R matrix given by

$$R(\lambda,\mu) = \begin{pmatrix} \varphi(\lambda-\mu+\eta) & 0 & 0 & 0 \\ 0 & \varphi(\lambda-\mu) & \varphi(\eta) & 0 \\ 0 & \varphi(\eta) & \varphi(\lambda-\mu) & 0 \\ 0 & 0 & 0 & \varphi(\lambda-\mu+\eta) \end{pmatrix}$$

where

$$\varphi(\lambda) = \sinh(\lambda)$$

Problem 3

a) Write down the Yang-Baxter equations for the two-particle scattering matrix $S_{a_1,a_2}^{b_1,b_2}(\theta_1-\theta_2)$ of factorized scattering theory. Explain in your words, with drawings, how these Yang-Baxter equations are derived from the presence of higher-spin local conserved quantities Q_s.

b) Consider a QFT with only one particle type. By using the resolution of the identity into asymptotic states and by using space-time translation and relativistic invariances, show that the two-point function of a hermitian, spinless field $\mathcal{O}$ can be written, with $x^2 > t^2$, as

$$\langle \text{vac}|\mathcal{O}(x,t)\mathcal{O}(0,0)|\text{vac}\rangle = \sum_{n=0}^{\infty} \int_{\theta_1>\ldots>\theta_n} d\theta_1 \cdots d\theta_n |f(\theta_1,\ldots,\theta_n)|^2 e^{-\sqrt{x^2-t^2}\sum_j m\cosh\theta_j}.$$

Here

$$f(\theta_1,\ldots,\theta_n) := \langle \text{vac}|\mathcal{O}(0,0)|\theta_1,\ldots,\theta_n\rangle$$

are the the form factors of the field positioned at the space-time origin. **Hint:** use relativistic invariance of the form factors to make a shift of the rapidities by $i\pi/2$, and recall that $\sinh(\theta+i\pi/2) = i\cosh\theta$ (or, if you do the problem differently from what I have in mind, make a shift by another appropriate pure imaginary value).

References

[1] "Introduction to classical integrable systems", O. Babelon, D. Bernard and M. Talon, Cambridge University Press, 2003

[2] "How algebraic Bethe ansatz works for integrable models", L.D. Faddeev, published in Les Houches 1995, Relativistic gravitation and gravitational radiation pp. 149-219, hep-th/9605187, 1996

[3] "Algebraic analysis of solvable lattice models", M. Jimbo and T. Miwa, Conference Board of the Mathematical Sciences 85, American Mathematical Society ,1993

[4] “On the quantum inverse scattering problem”, J. M. Maillet and V. Terras, Nucl. Phys. B575, 627-644, hep-th/9911030, 2000

[5] “Quantum inverse scattering method and correlation functions”, V.E. Korepin, N.M. Bogoliubov, A.G. Izergin, Cambridge University Press, 1993

[6] “Form factors in completely integrable models of quantum field theory”, F. A. Smirnov, World Scientific, 1992