Quantum Mechanics III — Michaelmas Term 2009

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Homework 1 – due 5 November 2009

1. Let $\hat{A}$ and $\hat{B}$ be linear operators on an inner-product vector space over $\mathbb{C}$, and $\hat{A}^\dagger$, $\hat{B}^\dagger$ the usual hermitian conjugates. Show that
$$(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger \hat{A}^\dagger. \tag{1}$$

2. Consider a vector $|\psi\rangle = \int_{-\infty}^{\infty} dx\, \psi(x)|x\rangle$ written here as a decomposition into the basis of normalised position eigenstates $\{|x\rangle,\, x \in \mathbb{R}\}$. The wave function $\langle x|\psi\rangle = \psi(x)$ is
$$\psi(x) = \frac{a}{x + iL} \tag{2}$$
for some positive length $L > 0$ and complex a.
   (a) Normalise the state (i.e. choose a such that $\langle\psi|\psi\rangle = 1$).
   (b) What is the probability that the particle is found in a position between 0 and L?
   (c) Write down the state as a decomposition into the basis of normalised *momentum* eigenstates $\{|p\rangle,\, p \in \mathbb{R}\}$ [hint: to do the integral, shift the contour towards $\pm i\infty$ as appropriate, getting residues].
   (d) What is the probability that the momentum of the particle is found to be positive? That it is found negative? What is the average momentum?

3. Consider a 2-dimensional vector space with orthonormal basis $\{|1\rangle,\, |2\rangle\}$. Consider a linear operator $\hat{A}$ acting on it as follows:
$$\begin{aligned} \hat{A}|1\rangle &= |1\rangle + i|2\rangle \\ \hat{A}|2\rangle &= -i|1\rangle + |2\rangle \end{aligned} \tag{3}$$
   (a) Is $\hat{A}$ an observable?
   (b) If yes, find the possible results of a measurement of the associated physical quantity. What are the probabilities of these measurement results on the state $|\psi\rangle = |1\rangle$? What is the normalised state just after the measurement if it is the smallest value that was observed?

Answers

1. By definition, $(|v\rangle, \hat{A}^\dagger|w\rangle) = (\hat{A}|v\rangle, |w\rangle)$. Then, $(|v\rangle, (\hat{A}\hat{B})^\dagger|w\rangle) = (\hat{A}\hat{B}|v\rangle, |w\rangle) = (\hat{B}|v\rangle, \hat{A}^\dagger|w\rangle) = (|v\rangle, \hat{B}^\dagger\hat{A}^\dagger|w\rangle)$, which shows (1).

2. (a) The condition of normalisation is

$$\begin{aligned}
1 &= \langle\psi|\psi\rangle \\
&= \int_{-\infty}^{\infty} dx\, \overline{\psi(x)}\psi(x) \\
&= \int_{-\infty}^{\infty} dx\, \frac{|a|^2}{x^2+L^2} \\
&= \frac{\pi|a|^2}{L}.
\end{aligned} \tag{4}$$

For the integral, use, for instance, the change of variable $x = L\tan\theta$. Hence we must have $|a| = \sqrt{L/\pi}$. We may choose $a = \sqrt{L/\pi}$ (i.e. we may set the phase factor to 1).

(b) The probability of finding the particle in a position between 0 and L is given by

$$\begin{aligned}
p(X\in[0,L]) &= \int_0^L dx\, |\langle x|\psi\rangle|^2 \\
&= \int_0^L dx\, \overline{\psi(x)}\psi(x) \\
&= \int_0^L dx\, \frac{L}{\pi(x^2+L^2)} \\
&= \frac{1}{4}
\end{aligned} \tag{5}$$

(c) We want to find $\tilde{\psi}(p)$ such that $|\psi\rangle = \int_{-\infty}^{\infty} \tilde{\psi}(p)|p\rangle$. Hence, we have

$$\begin{aligned}
\tilde{\psi}(p) &= \langle p|\psi\rangle \\
&= \int_{-\infty}^{\infty} dx\, \langle p|x\rangle\langle x|\psi\rangle \\
&= \sqrt{\frac{L}{2\pi^2\hbar}} \int_{-\infty}^{\infty} dx\, \frac{e^{-ipx/\hbar}}{x+iL} \\
&= \sqrt{\frac{L}{2\pi^2\hbar}} \times \begin{cases} -2\pi i e^{-pL/\hbar} & p>0 \\ 0 & p<0 \end{cases} \\
&= -i\sqrt{\frac{2L}{\hbar}} e^{-pL/\hbar}\Theta(p)
\end{aligned} \tag{6}$$

where $\Theta(p)$ is Heaviside's step function. In order to evaluate the integral, we shifted the x contour all the way towards the positive imaginary direction for $p < 0$, and the negative imaginary direction for $p > 0$, picking up the residues on the way. The completely shifted contour itself has zero contribution because our choice of direction made the exponential factor vanish. Also, we used the wave function of momentum eigenstates

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \tag{7}$$

and the value of a found in 2(a). We can check that the resulting function $\tilde{\psi}(p)$ gives the correct normalisation:

$$\begin{aligned}
\langle\psi|\psi\rangle &= \int_{-\infty}^{\infty} dp\,\langle\psi|p\rangle\langle p|\psi\rangle \\
&= \int_{-\infty}^{\infty} dp\,\overline{\tilde{\psi}(p)}\tilde{\psi}(p) \\
&= \frac{2L}{\hbar}\int_{0}^{\infty} dp\,e^{-2pL/\hbar} \\
&= 1
\end{aligned} \tag{8}$$

(d) The probability of finding the momentum to be positive is 1, and to be negative is 0, since $\langle p|\psi\rangle = 0$ for any $p < 0$. The average momentum is

$$\begin{aligned}
\langle\psi|\hat{P}|\psi\rangle &= \int_{-\infty}^{\infty} dp\,\overline{\tilde{\psi}(p)}\,p\,\tilde{\psi}(p) \\
&= \frac{2L}{\hbar}\int_{0}^{\infty} dp\,p\,e^{-2pL/\hbar} \\
&= \frac{\hbar}{2L}
\end{aligned} \tag{9}$$

Note that we can evaluate the same average momentum using the representation of $\hat{P}$ on wave functions:

$$\begin{aligned}
\langle\psi|\hat{P}|\psi\rangle &= \int_{-\infty}^{\infty} dx\,\overline{\psi(x)}\left(-i\hbar\frac{d}{dx}\right)\psi(x) \\
&= \frac{iL\hbar}{\pi}\int_{-\infty}^{\infty} dx\,\frac{1}{(x-iL)(x+iL)^2}
\end{aligned} \tag{10}$$

which indeed gives the same result.

3. (a) The most convenient way to do things, in this finite-dimensional case, is to use matrices:

$$|1\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |2\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{11}$$

with

$$\hat{A} = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \tag{12}$$

(note: the signs are all correct!). Then, the hermitian conjugation † is just the combination of *complex conjugation and transpose.* Hence, we see immediately that

$$\hat{A}^\dagger = \hat{A} \tag{13}$$

so that this is an observable (completeness follows from hermiticity in the finite-dimensional case).

(b) We need to find the eigenvalues of $\hat{A}$:

$$\det \begin{pmatrix} 1-\lambda & -i \\ i & 1-\lambda \end{pmatrix} = 0 \tag{14}$$

so that

$$(1-\lambda)^2 - 1 = 0 \tag{15}$$

which gives $\lambda = 0, 2$. These are the possible values of a measurement of the physical quantity A associated to $\hat{A}$. The assocaited probabilities on the state $|1\rangle$ are $|\langle A = 0|1\rangle|^2$ and $|\langle A = 2|1\rangle|^2$ where $|A = 0\rangle$ and $|A = 2\rangle$ are the normalised eigenvectors associated to eigenvalues 0 and 2. Hence, we need to diagonalise $\hat{A}$:

$$\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{16}$$

Consider $\lambda = 0$. The first line gives $v_1 - iv_2 = 0$, hence

$$|A = 0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} \tag{17}$$

Now consider $\lambda = 2$. The first line gives $v_1 - iv_2 = 2v_1$, hence

$$|A = 2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix} \tag{18}$$

Hence we find

$$p(A = 0) = |\langle A = 0|1\rangle|^2 = \frac{1}{2}, \quad p(A = 2) = |\langle A = 2|1\rangle|^2 = \frac{1}{2} \tag{19}$$

Note that these probabilities add up to 1 as it should be. If the smallest value was observed, the value $A = 0$, then the normalised state vector just after the measurement is $|A = 0\rangle$.