University of Durham
EXAMINATION PAPER

date

May/June 2006

exam code

MATH3111/01

description

QUANTUM MECHANICS III

Time allowed:

3 hours

Examination material provided:

None

Instructions:

Credit will be given for the best **FOUR** answers from Section A
and the best **THREE** answers from Section B.
Questions in Section B carry **TWICE** as many marks as those in Section A.
Use of electronic calculators is forbidden.

continued

SECTION A

1. (a) Calculate the commutator

$$[(x^a + 1)\frac{d}{dx}, \frac{d^2}{dx^2}]$$

Show that when $a = 2$ its value is $-2\frac{d}{dx} - 4x\frac{d^2}{dx^2}$.

(b) Is the operator $\widehat{A} = \frac{d^2}{dx^2}$ hermitian in the space of functions $f(x)$; $-\pi \le x \le \pi$, $f(\pi) = f(-\pi) = 0$?

Show that any eigenfunction of $\widehat{A}$ with eigenvalue λ is automatically an eigenfunction of $\widehat{B} = \widehat{A}^2$ and find its corresponding eigenvalue.

2. (a) Two hermitian operators $\widehat{A}$ and $\widehat{B}$ satisfy the following commutation relation

$$[\widehat{A}, \widehat{B}] = 2\widehat{A}.$$

If $|b\rangle$ is an eigenstate of $\widehat{B}$ with eigenvalue b show that $\widehat{A}|b\rangle$ is also its eigenstate. What is its eigenvalue?

(b) Represent $\widehat{B}$ by an operator $\frac{d}{dx}$ acting on suitable wavefunctions $\psi(x)$ and $\widehat{A}$ by $f(x)$. What is the general form of $f(x)$ which is compatible with the commutation relation between $\widehat{A}$ and $\widehat{B}$?

3. A quantum mechanical system has two orthonormal energy eigenstates $|a\rangle$ and $|b\rangle$ with energies $E_a > 0$ and $E_b = 2E_a$. A self-adjoint operator $\widehat{A}$ acts on these states as

$$\widehat{A}|a\rangle = |b\rangle, \quad \widehat{A}|b\rangle = |a\rangle.$$

(a) Determine the eigenstates of $\widehat{A}$ and their corresponding eigenvalues.

(b) At $t = 0$ a measurement of $\widehat{A}$ is performed and a value of 1 is found. What (normalised) state is the system in immediately after this measurement?

(c) What is the earliest time when another measurement of $\widehat{A}$ would definitively give again $+1$?

4. Consider a simple quantum mechanical system in which states are represented by three component vectors and operators by 3×3 matrices. State the necessary and sufficient conditions on the complex constants a, b and c so that

$$\widehat{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & b & c \end{pmatrix}$$

is self-adjoint and so can represent an observable.

What can be possible results of the measurements of M?

Calculate the expectation value of $\widehat{M}$ when the system is in the state $\begin{pmatrix} 1 \\ 1 \\ i \end{pmatrix}$.

continued

5. (a) State the uncertainty principle for position and momentum in a one dimensional system.

   (b) A one dimensional system has its Schrödinger wavefunction given by:

$$\psi(x) = \begin{cases} A \cos\{\frac{\pi x}{2L}\} & , \quad -L \leq x \leq L \\ 0 & , \quad |x| > L \end{cases}$$

   where A is constant.

   Normalise this wavefunction and calculate $<\widehat{x}>$, $<\widehat{x}^2>$, $<\widehat{p}>$ and $<\widehat{p}^2>$.

   Hint: You can use the integral

$$\frac{1}{2L}\int_{-L}^{L} dx\, x^2 \cos\frac{\pi x}{L} = -\frac{2L^2}{\pi^2}$$

6. In a quantum mechanical system an observable A is represented by an operator $\widehat{A}$ which has two normalised eigenvectors $|1\rangle$ and $|2\rangle$.

   Calculate $\langle\psi|\widehat{A}|\psi\rangle$ and $\langle\psi|\widehat{A}^2|\psi\rangle$ for a general normalised state $|\psi\rangle$. Which of the two quantities:

$$(\langle\psi|\widehat{A}|\psi\rangle)^2, \qquad \text{or} \qquad (\langle\psi|\widehat{A}^2|\psi\rangle)$$

   is larger. What is the condition on $|\psi\rangle$ for these two quantities to be equal?

continued

SECTION B

7. A one dimensional system involves a particle of mass m in a potential

$$V(x) = \begin{cases} V_1 & , \quad -L > x \\ 0 & , \quad -L < x < L \\ V_2 & , \quad x > L \end{cases}$$

(a) Show that the allowed bound state energies, E, must satisfy

$$\tan(2kL) = \frac{k(a_1 + a_2)}{k^2 - a_1 a_2},$$

where

$$k = \sqrt{\frac{2mE}{\hbar^2}} \quad \text{and} \quad a_i = \sqrt{\frac{2m(V_i - E)}{\hbar^2}}.$$

(b) Consider the case of $V_1 = V_2 = V_0$. Describe how the spectrum of this system is related to the spectrum of the system with the potential

$$V(x) = \begin{cases} \infty & , \quad x < 0 \\ 0 & , \quad 0 < x < L \\ V_0 & , \quad x > L \end{cases} .$$

8. Using $\widehat{L}_k = \sum_{mn} \epsilon_{kmn} \widehat{r}_m \widehat{p}_n$ and $[\widehat{r}_i, \widehat{p}_j] = i\hbar\delta_{ij}$ show that

$$[\widehat{r}_m, \widehat{L}_n] = -i\hbar \sum_p \epsilon_{nmp} \widehat{r}_p.$$

Consider $\widehat{M} = \widehat{r}_1 - i\widehat{r}_2$ and show that $\widehat{M}$ satisfies

$$[\widehat{L}_3, \widehat{M}] = -\hbar\widehat{M}, \quad [\widehat{L}^2, \widehat{M}] = 2\hbar^2\widehat{M} + 2\hbar\widehat{r}_3\widehat{L}_- - 2\hbar\widehat{M}\widehat{L}_3,$$

where, as usual,

$$\widehat{L}^2 = \widehat{L}_1^2 + \widehat{L}_2^2 + \widehat{L}_3^2 \quad \text{and} \quad \widehat{L}_- = \widehat{L}_1 - i\widehat{L}_2.$$

Hence show that, for some number λ

$$\widehat{M}|l, -l\rangle = \lambda|l+1, -l-1\rangle,$$

where $|l, m\rangle$ are conventional eigenstates of $\widehat{L}^2$ and $\widehat{L}_3$ *i.e.* they satisfy $\widehat{L}_3|l, m\rangle = \hbar m|l, m\rangle$ and $\widehat{L}^2|l, m\rangle = \hbar^2 l(l+1)|l, m\rangle$.

continued

9. A particle is moving in the following one-dimensional square-well potential:

$$V(x) \;=\; \begin{cases} 0 & , \quad |x| < A \\ \infty & , \quad |x| > A \end{cases} .$$

Show that the two lowest energy eigenfunctions of the Hamiltonian are given by

$$\psi_1 = \frac{1}{\sqrt{A}} \cos \frac{\pi x}{2A}, \quad \psi_2 = \frac{1}{\sqrt{A}} \sin \frac{\pi x}{A}$$

and determine these energies.

At time $t = 0$ the wave function of the moving particle is given by

$$\Psi(x) \;=\; \frac{\psi_1(x) + \psi_2(x)}{\sqrt{2}}.$$

What is this wavefunction at time t?

What is the probability that, at time t, the particle is in the interval $0 < x < A$?

10. A particle is moving in a 3-dimensional space in the potential

$$V(\boldsymbol{r}) \;=\; -V_0 \exp\!\left(-\frac{r}{\alpha}\right).$$

Separate the time independent Schrödinger equation in spherical polar coordinates

$$\psi(r, \theta, \varphi) \;=\; \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \frac{\chi_l(r)}{r} Y_{lm}(\theta, \varphi),$$

where $Y_{lm}(\theta, \varphi)$ are spherical harmonics and show that $\chi_l(r)$ satisfies

$$-\frac{\hbar^2}{2m}\frac{d^2\chi_l}{dr^2} + \left[\frac{l(l+1)\hbar^2}{2mr^2} - V_0 \exp\!\left(-\frac{r}{\alpha}\right)\right] \chi_l \;=\; E\chi_l$$

Consider $l = 0$, change variables $r \to \xi = \exp(-\frac{r}{2\alpha})$ and show that $\chi = \chi_0$ satisfies

$$\frac{d^2\chi}{d\xi^2} + \frac{1}{\xi}\frac{d\chi}{d\xi} + \left(B - \frac{C}{\xi^2}\right)\chi \;=\; 0$$

for certain constants B, C. What conditions are to be imposed on χ as a function of ξ and how can these be used to determine energy levels?

Note: $\nabla^2 = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\right) - \frac{\hat{L}^2}{r^2}$

Note also that Bessel's equation, *i.e.* the equation

$$\frac{1}{z}\frac{d}{dz}\left(z\frac{d\chi}{dz}\right) + \left(1 - \frac{\nu^2}{z^2}\right)\chi \;=\; 0$$

has a regular solution $J_\nu(z)$. Moreover, $J_\nu(z)$, as a function of z, possesses many zeros.

