Answers to mock exam, section B

Lie Group and Lie Algebras, 2010.

B4

(i) The Heisenberg group is the subgroup of $GL(3,\mathbb{R})$ formed by the matrices

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

Its center is the set of matrices in H that commute with everything in H:

$$Z(H) = \{A \in H : AB = BA \ \forall\ B \in H\}.$$

The product of two elements of H is given by

$$AA' = \begin{pmatrix} 1 & a + a' & b' + ac' + b \\ 0 & 1 & c + c' \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, requiring $AA' = A'A$ for all a', b', c' gives us the requirement $ac' = a'c$ so that $a = 0$ (e.g. taking $a' = 0$ and $c' = 1$) and $c = 0$ (e.g. taking $a = 1$ and $c' = 0$). The center is then

$$Z(H) = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}.$$

The product of two central elements A and A' gives

$$AA' = \begin{pmatrix} 1 & 0 & b + b' \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so that we see that the map $\phi : \mathbb{R} \to Z(H) : b \mapsto \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an isomorphism (bijective, and preserves products: $\phi(b)\phi(b') = \phi(b + b')$). Hence, $Z(H)$ is isomorphic to $\mathbb{R}$.

In general, the center of a group is a normal subgroup (hence in particular a subgroup). We could prove this in general, but here let us show it in the particular case of the group H. $Z(H)$ is a subgroup: it contains the identity ($b = 0$), every element has an inverse ($b \mapsto -b$), and the product is closed (obvious by the formula above). It is also a normal subgroup, since by construction $AA' = A'A$ for all $A \in H$ and for any $A' \in Z(H)$, so that $A^{-1}A'A = A' \in Z(H)$.

$$A^{-1} = \begin{pmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}.$$

The quotient group $Q = H/Z(H)$ is the group of left-cosets, $\{hZ(H) : h \in H\}$. Using the multiplication law, taking

$$Z(H) = \begin{pmatrix} 1 & 0 & \mathbb{R} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we see that the quotien group is

$$Q = \left\{ \begin{pmatrix} 1 & a & \mathbb{R} \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, c \in \mathbb{R} \right\}.$$

For $A, A' \in Q$, the product is

$$AA' = \begin{pmatrix} 1 & a'+a & \mathbb{R} \\ 0 & 1 & c'+c \\ 0 & 0 & 1 \end{pmatrix}.$$

This is clearly abelian, $AA' = A'A$. Also, the map $\phi : \mathbb{R}^2 \to Q : (a,c) \mapsto \begin{pmatrix} 1 & a & \mathbb{R} \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ is an isomorphism (bijective, and preserves products: $\phi(a,c)\phi(a',c') = \phi(a+a', c+c')$), so that Q is isomorphic to $\mathbb{R}^2$.

Finally, let us see if we could have a semi-direct product $Q \ltimes Z(H)$ isomorphic to H. Since we know that Q is isomorphic to $\mathbb{R}^2$ and that $Z(H)$ is isomorphic to $\mathbb{R}$, we are looking for an isomorphism between $\mathbb{R}^2 \ltimes \mathbb{R}$ and H. Consider in general the multiplication law for such a semidirect product:

$$((a,c),b)((a',c'),b') = ((a+a', c+c'), b + \phi_{(a,c)}(b'))$$

where $\phi_{(a,c)}$ is an element of the automorphism group of $\mathbb{R}$ and depends on (a,c). We know that the multiplication law for H gives

$$((a,c),b)((a',c'),b') = ((a+a', c+c'), b + b' + ac')$$

so that we would require

$$\phi_{(a,c)}(b') = b' + ac'.$$

This is clearly not an automorphism of $\mathbb{R}$ depending (a,c): it is not an automorphism of $\mathbb{R}$, and there is a c' dependence. Hence, there is no semi-direct product $Q \ltimes Z(H)$ isomorphic to H.

(ii) The Lie algebra of the Heisenberg group is the set of matrices of the form

$$\left\{ X = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : (a,b,c) \in \mathbb{R}^3 \right\}.$$

Explicitly exponentiating gives

$$e^X = \mathbf{1} + \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a & b + ac/2 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

On the other hand, the matrices in the Heisenberg group are

$$\left\{ M = \begin{pmatrix} 1 & A & B \\ 0 & 1 & C \\ 0 & 0 & 1 \end{pmatrix} : (A,B,C) \in \mathbb{R}^3 \right\}.$$

Hence, we find that the exponential map is

$$(a,b,c) \mapsto (A,B,C) = (a, b + ac/2, c).$$

The inverse map can easily be obtained:

$$(A,B,C) \mapsto (a,b,c) = (A, B - AC/2, C).$$

This makes it clear that the exponential map is onto (surjective: every (A,B,C) has a pre-image) and 1-1 (injective: if $(A,B,C) \neq (A',B',C')$, then $(A, B - AC/2, C) \neq (A', B' - A'C'/2, C')$ - clear if $A \neq A'$ or $C \neq C'$, and if $A = A'$ and $C = C'$, it becomes clear for $B' \neq B'$ as well).

(iii) For the group $SU(2)$, the three conditions on the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ coming from $A^\dagger A = \mathbf{1}$ are $|a|^2 + |c|^2 = 1$, $|b|^2 + |d|^2 = 1$ and $a^* b + c^* d = 0$, and the condition saying that A has unit determinant is $ad - bc = 1$. Given a and c satisfying the first equation, we can solve the third for b, giving $b = -c^* d / a^*$. The unit determinant condition then is $ad + |c|^2 d/a^* = 1$ so that $d = a^*$, which implies $b = -c^*$. Hence, we can write

$$A = \begin{pmatrix} a & -c^* \\ c & a^* \end{pmatrix}$$

with the condition that $|a|^2 + |c|^2 = 1$. Hence, all matrices are characterised by 2 complex numbers, a and c, with one condition, $|a|^2 + |c|^2 = 1$. Each complex number is 2 real numbers, $a = a_r + i a_i$ and $c = c_r + i c_i$, so what we have is a hyper-surface in $\mathbb{R}^4$ (i.e. a manifold of dimension 3 in $\mathbb{R}^4$). This manifold is a 3-sphere, defined by the equation $a_r^2 + a_i^2 + c_r^2 + c_i^2 = 1$.

B5

(i) Following the hint, we first do

$$e^A e^B = e^A e^B e^{-A} e^A = e^{e^A B e^{-A}} e^A = e^{e^{\mathrm{ad}A}(B)} e^A.$$

Now we have

$$e^{\mathrm{ad}A}(B) = B + [A,B] + \frac{1}{2}[A,[A,B]] + \ldots = B + uA + 0 + \ldots = B + uA.$$

Hence,

$$e^A e^B = e^{B+uA} e^A.$$

Next, we define $X = B + uA$ and $Y = A$ in order to use the BCH formula. We find

$$[X,Y] = [B + uA, A] = -uA = -uY.$$

Hence, $\mathrm{ad}X(Y) = -uY$ and $\mathrm{ad}Y(Y) = 0$, so that $\mathrm{ad}X$ and adY both preserve the vector space $\mathbb{C}Y$: they are just multiplication operators. Hence, we can just replace them by the number that they multiply by, i.e. $\mathrm{ad}X \mapsto -u$ and $\mathrm{ad}Y \mapsto 0$. We get

$$\log(e^A e^B) = B + uA + \int_0^1 dt \, \frac{\log(e^{-u})}{1 - (e^{-u})^{-1}} A = B + uA + \frac{-u}{1 - e^u} A = B + \frac{u}{1 - e^{-u}} A$$

(ii) Since $[P,H] = 0$, we immediately can write

$$e^{ixP} e^{ixH} e^{\theta B} = e^{ix(P+H)} e^{\theta B}.$$

Then, we see that $[ix(P+H), \theta B] = -ix\theta(P+H)$, so in the formula of (i) we can set $A \mapsto ix(P+H)$, $B \mapsto \theta B$ and $u \mapsto -\theta$. We directly obtain

$$e^{ixP} e^{ixH} e^{\theta B} = e^{\theta\left(\frac{1}{e^{\theta}-1} ix(P+H) + B\right)}.$$

B6

(i) We only have to verify the derivation property on the commutator $[D_1, D_2]$. We have

$$\begin{aligned}
[D_1,D_2]([xy]) &= D_1(D_2([xy])) - D_2(D_1([xy])) \\
&= D_1([D_2(x)y] + [xD_2(y)]) - D_2([D_1(x)y] + [xD_1(y)]) \\
&= [D_1(D_2(x))y] + [D_2(x)D_1(y)] + [D_1(x)D_2(y)] + [xD_1(D_2(y))] \\
&\quad -[D_2(D_1(x))y] - [D_1(x)D_2(y)] - [D_2(x)D_1(y)] - [xD_2(D_1(y))] \\
&= [D_1(D_2(x))y] - [D_2(D_1(x))y] + [xD_1(D_2(y))] - x[D_2(D_1(y))] \\
&= [[D_1,D_2](x)y] + [x[D_1,D_2](y)]
\end{aligned}$$

hence $[D_1, D_2]$ is a derivation.

(ii) In order for the kernel to be a subalgebra, it needs to be a subspace, and the Lie bracket needs to be closed in it. The kernel is certainly a subspace, because if $x \in \ker(D)$ and $y \in \ker(D)$, then $D(ax + by) = aD(x) + bD(y) = 0$ so that $ax + by \in \ker(D)$. Then, we need to check that for x and y as previously, we have $[xy] \in \ker(D)$. We calculate $D([xy]) = [D(x)y] + [xD(y)] = 0 + 0$ hence indeed this holds. So, the kernel is a subalgebra. In order for it to be an ideal, we would need additionally that $D([xy]) = 0$ for $x \in \ker(D)$ and for all $y \in L$. But we have $D([xy]) = [D(x)y] + [xD(y)] = [xD(y)] \neq 0$, hence $[xy] \notin \ker(D)$, so *no*, the kernel is not an ideal.

(iii) The Killing form is $\kappa(x, y) = \mathrm{Tr}(\mathrm{ad}\, x \mathrm{ad}\, y)$. Consider $\kappa(x, D(y))$. Since D is a derivation, it satisfies $D([xy]) = [D(x)y] + [xD(y)]$. Note that

$$\mathrm{ad}\,(D(y))(z) = [D(y)z] = D([yz]) - [yD(z)] = D(\mathrm{ad}\, y(z)) - \mathrm{ad}\, y(D(z)) = [D, \mathrm{ad}\, y](z).$$

As operators on L, this equation means $\mathrm{ad}\,(D(y)) = [D, \mathrm{ad}\, y]$. Hence, $\kappa(x, D(y)) = \mathrm{Tr}(\mathrm{ad}\, x\, [D, \mathrm{ad}\, y]) = \mathrm{Tr}([\mathrm{ad}\, x, D]\, \mathrm{ad}\, y)$ using cyclicity of the trace. But then,

$$[\mathrm{ad}\, x, D] = -[D, \mathrm{ad}\, x] = -\mathrm{ad}\,(D(x))$$

hence we find $\kappa(x, D(y)) = -\kappa(D(x), y)$. Using symmetry of the Killing form, this is $\kappa(x, D(y)) = -\kappa(y, D(x))$, which is anti-symmmetry of $\alpha(x, y) = \kappa(x, D(y))$.