King's College London

University Of London

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Candidate No: .... **Desk No:** ....

MSc Examination

7CCMMS01 (CMMS01) Lie groups and Lie algebras (mock exam)

Summer 2010

Time Allowed: Two Hours

This paper consists of two sections, Section A and Section B.
Section A contributes half the total marks for the paper.
Answer all questions in Section A.
All questions in Section B carry equal marks, but if more than two are attempted, then only the best two will count.

NO calculators are permitted.

TURN OVER WHEN INSTRUCTED

A 1. 20 points

(i) Define: a) Lie algebra, b) group, c) Lie algebra homomorphism, d) group homomorphism, e) subgroup, and f) normal subgroup.

(ii) Let G and H be groups. Show that the kernel of a homomorphism $\phi : G \to H$ is a normal subgroup of G (you can use without proof the main properties of homomorphisms).

(iii) State the definition of the property "semisimple" of a Lie algebra in terms of solvable ideals. Show that if $\mathfrak{g}$ is semisimple, then it does not have abelian ideals other than $\{0\}$.

(iv) Consider a vector space V spanned over $\mathbb{R}$ by three independent elements x, y, z. Suppose $[\cdot\cdot]$ is a bilinear product on V satisfying $[vw] = -[wv]\,\forall\, v, w \in V$ and $[xy] = z$, $[xz] = y$, $[yz] = 0$. Show that V with this product is a Lie algebra.

A 2. 15 points

(i) State the definition of a matrix Lie group.

(ii) Show that the set of matrices $O(n) = \{A \in \mathrm{Mat}(n;\mathbb{C}) \,|\, A^T A = \mathbf{1}\}$ is a matrix Lie group for any positive integer n (don't forget to show, in particular, that it is a group).

(iii) Is $O(n)$ compact? Prove your claim.

A 3. 15 points

(i) The group $SU(2)$ is the group of all 2×2 unitary matrices of determinant 1. Show that the Lie algebra of $SU(2)$ is

$$su(2) = \{A \in \mathrm{Mat}(2;\mathbb{C}) : A^\dagger = -A,\ \mathrm{Tr}(A) = 0\},$$

i.e. it is the set of all 2×2 anti-hermitian traceless matrices (you can use without proof the properties of the exponential).

(ii) Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. Suppose any element of G is of the form e^g for some $g \in \mathfrak{g}$. Suppose also that the Baker-Campbell-Hausdorf formula for $e^g e^{g'}$ converges for any $g, g' \in \mathfrak{g}$. Show that if $\mathfrak{g}$ has an ideal $\mathfrak{h}$, then G has a normal subgroup $H = \exp\mathfrak{h}$ (you can use without proof the relation between Ad and ad, and the basic properties of the exponential).

(iii) Consider $sl(2;\mathbb{C})$, the algebra of 2 by 2 traceless complex matrices. Let $\mathfrak{h} \in sl(2;\mathbb{C})$ be the subalgebra

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, \quad a \in \mathbb{C} \right\}.$$

Given that $\mathfrak{h}$ is a Cartan subalgebra of $sl(2;\mathbb{C})$, find the roots and the corresponding root space decomposition.

B 4. (i) Determine the center $Z(H)$ of the Heisenberg group H, and show that it is isomorphic to $\mathbb{R}$. Is $Z(H)$ a subgroup of H? Is it a normal subgroup? Prove your claims. Show that the quotient group $Q = H/Z(H)$ is abelian, and isomorphic to $\mathbb{R}\times\mathbb{R}$. Show that there *does not* exist any semi-direct product $Q \ltimes Z(H)$ isomorphic to H.

(ii) Show that the exponential mapping from the Lie algebra of the Heisenberg group to the Heisenberg group is bijective.

(iii) Describe the group $SU(2)$ as a subset of $\mathbb{R}^4$.

B 5. (i) Using the BCH formula

$$\log(e^X e^Y) = X + \int_0^1 dt \, \frac{\log(e^{\operatorname{ad} X} e^{t \operatorname{ad} Y})}{\mathbf{1} - (e^{\operatorname{ad} X} e^{t \operatorname{ad} Y})^{-1}}(Y)$$

and the properties of the exponential, show that

$$\log(e^A e^B) = \frac{u}{1 - e^{-u}} A + B$$

in the case where $[A, B] = uA$ for some complex number u. Discuss what happens in the case where $u \to 0$. **Hint:** first use the properties of the exponential to prove $e^A e^B = e^{B+uA} e^A$.

(ii) In two dimensions, the Poincaré algebra has three generators: the energy H, the momentum P and the boost operator B. They satisfy the commutation relations

$$[P, H] = 0, \quad [B, P] = H, \quad [B, H] = P.$$

Using the formula obtained in (i), evaluate the operator U in

$$e^{ixP} e^{ixH} e^{\theta B} = e^U$$

where x, θ are real numbers.

B 6. A derivation D on a Lie algebra L is a linear map $D \in gl(L)$ with the property that $D([xy]) = [D(x)y] + [xD(y)]$ for all $x, y \in L$.

(i) Prove that the commutator of two derivations, $[D_1, D_2]$, is a derivation.

(ii) Prove that the kernel of a derivation $\ker(D) = \{x \in L : D(x) = 0\}$ is a subalgebra. Is it also an ideal?

(iii) Given a derivation D on L, prove that the bilinear form $\alpha(x, y) = \kappa(x, D(y))$, where κ is the Killing form $\kappa(x, y) = \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y)$, is *anti-symmetric*, $\alpha(x, y) = -\alpha(y, x)$.