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Lie Groups and Lie Algebras

Lecture notes for 7CCMMS01/CMMS01/CM424Z

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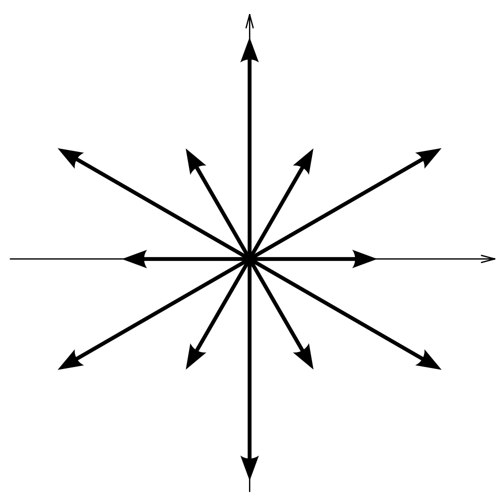
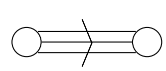




Contents

Exercise 0.1 :
I certainly did not manage to remove all errors from this script. So the first exercise is to find all errors and tell them to me.

1 Preamble

The topic of this course is Lie groups and Lie algebras, and their representations. As a preamble, let us have a quick look at the definitions. These can then again be forgotten, for they will be restated further on in the course.

Definition 1.1 :
A *Lie group* is a set G endowed with the structure of a smooth manifold and of a group, such that the multiplication $\cdot : G \times G \to G$ and the inverse $(\ )^{-1} : G \to G$ are smooth maps.

This definition is more general than what we will use in the course, where we will restrict ourselves to so-called matrix Lie groups. The manifold will then always be realised as a subset of some $\mathbb{R}^d$. For example the manifold S^3, the three-dimensional sphere, can be realised as a subset of $\mathbb{R}^4$ by taking all points of $\mathbb{R}^4$ that obey $x_1^2+x_2^2+x_3^2+x_4^2 = 1$. [You can look up 'Lie group' and 'manifold' on eom.springer.de, wikipedia.org, mathworld.wolfram.org, or planetmath.org.]

In fact, later in this course Lie algebras will be more central than Lie groups.

Definition 1.2 :
A *Lie algebra* is a vector space V together with a bilinear map $[\ ,\ ] : V \times V \to V$, called *Lie bracket*, satisfying
(i) $[X, X] = 0$ for all $X \in V$ (*skew-symmetry*),
(ii) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in V$ (*Jacobi identity*).

We will take the vector space V to be over the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$, but it could be over any field.

Exercise 1.1 :
Show that for a real or complex vector space V, a bilinear map $b(\cdot, \cdot) : V \times V \to V$ obeys $b(u, v) = -b(v, u)$ (for all u, v) if and only if $b(u, u) = 0$ (for all u). [If you want to know, the formulation $[X, X] = 0$ in the definition of a Lie algebra is preferable because it also works for the field $\mathbb{F}_2$. There, the above equivalence is not true because in $\mathbb{F}_2$ we have $1 + 1 = 0$.]

Notation 1.3 :
(i) "iff" is an abbreviation for "if and only if".
(ii) If an exercise features a "*" it is optional, but the result may be used in the

following. In particular, it will not be assumed in the exam that these exercises have been done. (This does not mean that material of these exercises cannot appear in the exam.)
(iii) If a whole section is marked by a "*", its material was not covered in the course, and it will not be assumed in the exam that you have seen it before. It will also not be assumed that you have done the exercises in a section marked by (*).
(iv) If a paragraph is marked as "Information", then as for sections marked by (*) it will not be assumed in the exam that you have seen it before.

2 Symmetry in Physics

The state of a physical system is given by a collection of particle positions and momenta in classical mechanics or by a wave function in quantum mechanics. A *symmetry* is then an invertible map f on the space of states which commutes with the time evolution (as given Newton's equation $m\ddot{x} = -\nabla V(x)$, or the Schrödinger equation $i\hbar\frac{\partial}{\partial t}\psi = H\psi$)

$$\begin{tikzcd} \text{state at time } 0 \arrow[r, "\text{evolve}"] \arrow[d, "f"] \arrow[dr, phantom, "="] & \text{state at time } t \arrow[d, "f"] \\ \text{state}' \text{ at time } 0 \arrow[r, "\text{evolve}"] & \text{state}' \text{ at time } t \end{tikzcd} \tag{2.1}$$

Symmetries are an important concept in physics. Recent theories are almost entirely constructed from symmetry considerations (e.g. gauge theories, supergravity theories, two-dimensional conformal field theories). In this approach one demands the existence of a certain symmetry and wonders what theories with this property one can construct. But let us not go into this any further.

2.1 Definition of a group

Symmetry transformations like translations and rotations can be composed and undone. Also 'doing nothing' is a symmmetry in the above sense. An appropriate mathematical notion with these properties is that of a group.

Definition 2.1:
A *group* is a set G together with a map $\cdot : G \times G \to G$ (*multiplication*) such that
(i) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ (*associativity*)
(ii) there exists $e \in G$ s.t. $e \cdot x = x = x \cdot e$ for all $x \in G$ (*unit law*)
(iii) for each $x \in G$ there exists an $x^{-1} \in G$ such that $x \cdot x^{-1} = e = x^{-1} \cdot x$ (*inverse*)

Exercise 2.1:
Prove the following consequences of the group axioms: The unit is unique. The inverse is unique. The map $x \mapsto x^{-1}$ is invertible as a map from G to G. $e^{-1} = e$. If $gg = g$ for some $g \in G$, then $g = e$. The set of integers together with addition $(\mathbb{Z}, +)$ forms a group. The set of integers together with multiplication $(\mathbb{Z}, \cdot)$ does not form a group.

Of particular relevance for us will be groups constructed from matrices. Denote by $\mathrm{Mat}(n, \mathbb{R})$ (resp. $\mathrm{Mat}(n, \mathbb{C})$) the $n \times n$-matrices with real (resp. complex) entries. Let

$$GL(n, \mathbb{R}) = \{M \in \mathrm{Mat}(n, \mathbb{R}) | \det(M) \neq 0\} \quad . \tag{2.2}$$

Together with matrix multiplication (and matrix inverses, and the identity matrix as unit) this forms a group, called *general linear group* of degree n over $\mathbb{R}$. This is the basic example of a Lie group.

Exercise 2.2:
Verify the group axioms for $GL(n, \mathbb{R})$. Show that $\mathrm{Mat}(n, \mathbb{R})$ (with matrix multiplication) is not a group.

Definition 2.2:
Given two groups G and H, a *group homomorphism* is a map $\varphi : G \to H$ such that $\varphi(g \cdot h) = \varphi(g) \cdot \varphi(h)$ for all $g, h \in G$.

Exercise 2.3:
Let $\varphi : G \to H$ be a group homomorphism. Show that $\varphi(e) = e$ (the units in G and H, respectively), and that $\varphi(g^{-1}) = \varphi(g)^{-1}$.

Here is some more vocabulary.

Definition 2.3:
A map $f : X \to Y$ between two sets X and Y is called *injective* iff $f(x) = f(x') \Rightarrow x = x'$ for all $x, x' \in X$, it is *surjective* iff for all $y \in Y$ there is a $x \in X$ such that $f(x) = y$. The map f is *bijective* iff it is surjective and injective.

Definition 2.4:
An *automorphism* of a group G is a bijective group homomorphism from G to G. The set of all automorphisms of G is denoted by $\mathrm{Aut}(G)$.

Exercise 2.4:
Let G be a group. Show that $\mathrm{Aut}(G)$ is a group. Show that the map $\varphi_g : G \to G$, $\varphi_g(h) = ghg^{-1}$ is in $\mathrm{Aut}(G)$ for any choice of $g \in G$.

Definition 2.5:
Two groups G and H are *isomorphic* iff there exists a bijective group homomorphism from G to H. In this case we write $G \cong H$.

2.2 Rotations and the Euclidean group

A typical symmetry is invariance under rotations and translations, as e.g. in the Newtonian description of gravity. Let us start with rotations.

Take $\mathbb{R}^n$ (for physics probably $n = 3$) with the standard inner product

$$g(u,v) = \sum_{i=1}^{n} u_i v_i \quad \text{for} \quad u,v \in \mathbb{R}^n \quad . \tag{2.3}$$

A linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is an *orthogonal transformation* iff

$$g(Tu,Tv) = g(u,v) \quad \text{for all} \quad u,v \in \mathbb{R}^n \quad . \tag{2.4}$$

Denote by $e_i = (0,0,\dots,1,\dots,0)$ the i'th basis vector of $\mathbb{R}^n$, so that in component notation $(Te_i)_k = T_{ki}$. Evaluating the above condition in this basis gives

$$\text{lhs} = g(Te_i,Te_j) = \sum_k T_{ki}T_{kj} = (T^tT)_{ij} \ , \quad \text{rhs} = g(e_i,e_j) = \delta_{ij} \quad . \tag{2.5}$$

Hence T is an orthogonal transformation iff $T^tT = \mathbf{1}$ (here $\mathbf{1}$ denotes the unit matrix $(\mathbf{1})_{ij} = \delta_{ij}$).

What is the geometric meaning of an orthogonal transformation? The condition $g(Tu,Tv) = g(u,v)$ shows that it preserves the length $|u| = \sqrt{g(u,u)}$ of a vector as well as the angle $\cos(\theta) = g(u,v)/(|u|\,|v|)$ between two vectors.

However, T does not need to preserve the orientation. Note that

$$T^tT = \mathbf{1} \ \Rightarrow \ \det(T^tT) = \det(\mathbf{1}) \ \Rightarrow \ \det(T)^2 = 1 \ \Rightarrow \ \det(T) \in \{\pm 1\} \tag{2.6}$$

The orthogonal transformations T with $\det(T) = 1$ preserve orientation. These are rotations.

Definition 2.6:
(i) The *orthogonal group* $O(n)$ is the set

$$O(n) = \{M \in \mathrm{Mat}(n,\mathbb{R}) | M^tM = \mathbf{1}\} \tag{2.7}$$

with group multiplication given by matrix multiplication.
(ii) The *special orthogonal group* $SO(n)$ is given by those elements M of $O(n)$ with $\det(M) = 1$.

For example, $SO(3)$ is the group of rotations of $\mathbb{R}^3$.

Let us check that $O(n)$ is indeed a group.
(a) Is the multiplication well-defined?
Given $T,U \in O(n)$ we have to check that also $TU \in O(n)$. This follows from $(TU)^tTU = U^tT^tTU = U^t\mathbf{1}U = \mathbf{1}$.
(b) Is the multiplication associative?

The multipication is that of $\mathrm{Mat}(n, \mathbb{R})$, which is associative.
(c) Is there a unit element?
The obvious candidate is $\mathbf{1}$, all we have to check is if it is an element of $O(n)$. But this is clear since $\mathbf{1}^t \mathbf{1} = \mathbf{1}$.
(d) Is there an inverse for every element?
For an element $T \in O(n)$, the inverse should be the inverse matrix T^{-1}. It exists because $\det(T) \neq 0$. It remains to check that it is also in $O(n)$. To this end note that $T^t T = \mathbf{1}$ implies $T^t = T^{-1}$ and hence $(T^{-1})^t T^{-1} = T T^{-1} = \mathbf{1}$.

Definition 2.7:
A *subgroup* of a group G is a non-empty subset H of G, s.t. $g, h \in H \Rightarrow g \cdot h \in H$ and $g \in H \Rightarrow g^{-1} \in H$. We write $H \leq G$ for a subgroup H of G.

From the above we see that $O(n)$ is a subgroup of $GL(n, \mathbb{R})$.

Exercise 2.5:
(i) Show that a subgroup $H \leq G$ is in particular a group, and show that it has the same unit element as G.
(ii) Show that $SO(n)$ is a subgroup of $GL(n, \mathbb{R})$.

The transformations in $O(n)$ all leave the point zero fixed. If we are to describe the symmetries of euclidean space, there should be no such distinguished point, i.e. we should include *translations*. It is more natural to consider the euclidean group.

Definition 2.8:
The *euclidean group* $E(n)$ consists of all maps $\mathbb{R}^n \to \mathbb{R}^n$ that leave distances fixed, i.e. for all $f \in E(n)$ and $x, y \in \mathbb{R}^n$ we have $|x - y| = |f(x) - f(y)|$.

The euclidean group is in fact nothing but orthogonal transformations complemented by translations.

Exercise 2.6:
Prove that
(i*) for every $f \in E(n)$ there is a unique $T \in O(n)$ and $u \in \mathbb{R}^n$, s.t. $f(v) = Tv + u$ for all $v \in \mathbb{R}^n$.
(ii) for $T \in O(n)$ and $u \in \mathbb{R}^n$ the map $v \mapsto Tv + u$ is in $E(n)$.

The exercise shows that there is a bijection between $E(n)$ and $O(n) \times \mathbb{R}^n$ as *sets*. However, the multiplication is *not* $(T, x) \cdot (R, y) = (TR, y + x)$. Instead one finds the following. Writing $f_{T,u}(v) = Tv + u$, we have

$$f_{T,x}(f_{R,y}(v)) = f_{T,x}(Rv + y) = TRv + Ty + x = f_{TR,Ty+x}(v) \tag{2.8}$$

so that the group multiplication is

$$(T, x) \cdot (R, y) = (TR, Ty + x) \quad . \tag{2.9}$$

Definition 2.9 :
Let H and N be groups.
(i) The *direct product* $H \times N$ is the group given by all pairs (h,n), $h \in H$, $n \in N$ with multiplication and inverse

$$(h,n) \cdot (h',n') = (h \cdot h', n \cdot n') \quad , \quad (h,n)^{-1} = (h^{-1}, n^{-1}) \quad . \tag{2.10}$$

(ii) Let $\varphi : H \to \mathrm{Aut}(N)$, $h \mapsto \varphi_h$ be a group homomorphism. The *semidirect product* $H \ltimes_\varphi N$ (or just $H \ltimes N$ for short) is the group given by all pairs (h,n), $h \in H$, $n \in N$ with multiplication and inverse

$$(h,n) \cdot (h',n') = (h \cdot h', n \cdot \varphi_h(n')) \quad , \quad (h,n)^{-1} = (h^{-1}, \varphi_{h^{-1}}(n^{-1})) \quad . \tag{2.11}$$

Exercise 2.7 :
(i) Starting from the definition of the semidirect product, show that $H \ltimes_\varphi N$ is indeed a group. [To see why the notation H and N is used for the two groups, look up "semidirect product" on wikipedia.org or eom.springer.de.]
(ii) Show that the direct product is a special case of the semidirect product.
(iii) Show that the multiplication rule $(T,x)\cdot(R,y) = (TR, Ty+x)$ found in the study of $E(n)$ is that of the semidirect product $O(n) \ltimes_\varphi \mathbb{R}^n$, with $\varphi : O(n) \to \mathrm{Aut}(\mathbb{R}^n)$ given by $\varphi_T(u) = Tu$.

Altogether, one finds that the euclidean group is isomorphic to a semidirect product

$$E(n) \cong O(n) \ltimes \mathbb{R}^n \quad . \tag{2.12}$$

2.3 Lorentz and Poincaré transformations

The n-dimensional *Minkowski space* is $\mathbb{R}^n$ together with the non-degenenerate bilinear form

$$\eta(u,v) = u_0 v_0 - u_1 v_1 - \cdots - u_{n-1} v_{n-1} \quad . \tag{2.13}$$

Here we labelled the components of a vector u starting from zero, u_0 is the 'time' coordinate and $u_1, \dots, u_{n-1}$ are the 'space' coordinates.

The symmetries of Minkowski space are described by the Lorentz group, if one wants to keep the point zero fixed, or by the Poincaré group, if just distances w.r.t. η should remain fixed.

Definition 2.10 :
(i) The *Lorentz group* $O(1,n-1)$ is defined to be

$$O(1,n-1) = \{ M \in GL(n,\mathbb{R}) | \eta(Mu,Mv) = \eta(u,v) \text{ for all } u,v \in \mathbb{R}^n \} \quad . \tag{2.14}$$

(ii) The *Poincaré group* $P(1,n-1)$ [there does not seem to be a standard symbol; we will use P] is defined to be

$$P(1,n-1) = \{ f : \mathbb{R}^n \to \mathbb{R}^n \,|\, \eta(x-y,x-y) = \eta(f(x)-f(y), f(x)-f(y)) \text{ for all } x,y \in \mathbb{R}^n \} \quad . \tag{2.15}$$

Exercise 2.8:
Show that $O(1,n{-}1)$ can equivalently be written as

$$O(1,n{-}1) = \{M \in GL(n,\mathbb{R}) | M^t J M = J\}$$

where J is the diagonal matrix with entries $J = \mathrm{diag}(1,-1,\dots,-1)$.

Similar to the euclidean group $E(n)$, an element of the Poincaré group can be written as a composition of a Lorentz transformation $\Lambda \in O(1,n{-}1)$ and a translation.

Exercise 2.9:
(i*) Prove that for every $f \in P(1,n{-}1)$ there is a unique $\Lambda \in O(1,n{-}1)$ and $u \in \mathbb{R}^n$, s.t. $f(v) = \Lambda v + u$ for all $v \in \mathbb{R}^n$.
(ii) Show that the Poincaré group is isomorphic to the semidirect product $O(1,n{-}1) \ltimes \mathbb{R}^n$ with multiplication

$$(\Lambda, u) \cdot (\Lambda', u') = (\Lambda\Lambda', \Lambda u' + u) \quad . \tag{2.16}$$

2.4 (*) Symmetries in quantum mechanics

In quantum mechanics, symmetries are at their best [attention: personal opinion]. In particular, the *representations* of symmetries on vector spaces play an important role. We will get to that in section 4.1.

Definition 2.11:
Given a vector space E and two linear maps $A, B \in \mathrm{End}(E)$ [the endomorphisms of a vector space E are linear maps from E to E], the *commutator* $[A,B]$ is

$$[A,B] = AB - BA \in \mathrm{End}(E) \quad . \tag{2.17}$$

Lemma 2.12:
Given a vector space E, the space of linear maps $\mathrm{End}(E)$ together with the commutator as Lie bracket is a Lie algebra. This Lie algebra will be called $gl(E)$, or also $\mathrm{End}(E)$.

The reason to call this Lie algebra $gl(E)$ will become clear later. Let us use the proof of this lemma to recall what a Lie algebra is.

Proof of lemma:
Abbreviate $V = \mathrm{End}(E)$.
(a) $[\ ,\ ]$ has to be a bilinear map from $V \times V$ to V.
Clear.
(b) $[\ ,\ ]$ has to obey $[A,A] = 0$ for all $A \in V$ (skew-symmetry).
Clear.
(c) $[\ ,\ ]$ has to satisfy the Jacobi identity $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$ for all $A, B, C \in V$.
This is the content of the next exercise. It follows that V is a Lie algebra. $\square$

Exercise 2.10:
Verify that the commutator $[A, B] = AB - BA$ obeys the Jacobi identity.

The states of a quantum system are collected in a Hilbert space $\mathcal{H}$ over $\mathbb{C}$. [Recall: Hilbert space = vector space with inner product $(\cdot, \cdot)$ which is complete w.r.t. the norm $|u| = \sqrt{(u, u)}$.] The time evolution is described by a self-adjoint operator H [i.e. $H^\dagger = H$] on $\mathcal{H}$. If $\psi(0) \in \mathcal{H}$ is the state of the system at time zero, then at time t the system is in the state

$$\psi(t) = \exp(\tfrac{t}{i\hbar}H)\psi(0) = \Big(1 + \tfrac{t}{i\hbar}H + \tfrac{1}{2!}\big(\tfrac{t}{i\hbar}H\big)^2 + \dots\Big)\psi(0) \quad . \tag{2.18}$$

[One should worry if the infinite sum converges. For finite-dimensional $\mathcal{H}$ it always does, see section 3.2.] Suppose we are given a self-adjoint operator A which commutes with the Hamiltonian,

$$[A, H] = 0 \quad . \tag{2.19}$$

Consider the family of operators $U_A(s) = \exp(isA)$ for $s \in \mathbb{R}$. The $U_A(s)$ are unitary (i.e. $U_A(s)^\dagger = U_A(s)^{-1}$) so they preserve probabilities (write $U = U_A(s)$)

$$|(U\psi, U\psi')|^2 = |(\psi, U^\dagger U\psi')|^2 = |(\psi, \psi')|^2 \quad . \tag{2.20}$$

Further, they commute with time-evolution

$$\begin{array}{ccc} \psi & \xrightarrow{\text{evolve}} & \exp(\tfrac{t}{i\hbar}H)\psi \\ \Big\downarrow{\scriptstyle U_A(s)} & = & \Big\downarrow{\scriptstyle U_A(s)} \\ U_A(s)\psi & \xrightarrow{\text{evolve}} & \begin{array}{c} U_A(s)\exp(\tfrac{t}{i\hbar}H)\psi \\ = \exp(\tfrac{t}{i\hbar}H)U_A(s)\psi \end{array} \end{array} \tag{2.21}$$

The last equality holds because A and H commute. Thus from A we obtain a continous one-parameter family of symmetries.
Some comments:

- The operator A is also called *generator of a symmetry*. If we take s to be very small we have $U_A(s) = \mathbf{1} + isA + O(s^2)$, and A can be thought of as an infinitesimal symmetry transformation.
- The infinitesimal symmetry transformations are easier to deal with than the whole family. Therefore one usually describes continuous symmetries in terms of their generators.
- The relation between a continuous family of symmetries and their generators will in essence be the relation between Lie groups and Lie algebras, the latter are an infinitesimal version of the former. It turns out that Lie algebras are much easier to work with and still capture most of the structure.

2.5 (*) Angular momentum in quantum mechanics

Consider a quantum mechanical state ψ in the position representation, i.e. a wave function $\psi(q)$. It is easy to see how to act on this with *translations*,

$$(U_{\text{trans}}(s)\psi)(x) = \psi(q+s) \quad . \tag{2.22}$$

So what is the infinitesimal generator of translations? Take s to be small to find

$$(U_{\text{trans}}(s)\psi)(q) = \psi(q) + s\frac{\partial}{\partial q}\psi(q) + O(s^2) \quad , \tag{2.23}$$

so that (the convention $\hbar = 1$ is used)

$$U_{\text{trans}}(s) = \mathbf{1} + s\frac{\partial}{\partial q} + O(s^2) = \mathbf{1} + isp + O(s^2) \quad \text{with} \quad p = -i\frac{\partial}{\partial q} \quad . \tag{2.24}$$

The infinitesimal generators of *rotations* in three dimensions are

$$L_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} q_j p_k \quad , \qquad \text{with} \quad i = 1,2,3 \quad , \tag{2.25}$$

and where ε_{ijk} is antisymmetric in all indices and $\varepsilon_{123} = 1$.

Exercise 2.11:
(i) Consider a rotation around the 3-axis,

$$(U_{\text{rot}}(\theta)\psi)(q_1,q_2,q_3) = \psi(q_1\cos\theta - q_2\sin\theta, q_2\cos\theta + q_1\sin\theta, q_3) \tag{2.26}$$

and check that infinitesimally

$$U_{\text{rot}}(\theta) = \mathbf{1} + i\theta L_3 + O(\theta^2) \quad . \tag{2.27}$$

(ii) Using $[q_r, p_s] = i\delta_{rs}$ (check!) verify the commutator

$$[L_r, L_s] = i\sum_{t=1}^{3} \varepsilon_{rst} L_t \quad . \tag{2.28}$$

(You might need the relation $\sum_{k=1}^{3} \varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ (check!).)

The last relation can be used to define a three-dimensional Lie algebra: Let V be the complex vector space spanned by three generators ℓ_1, ℓ_2, ℓ_3. Define the bilinear map $[\ ,\ ]$ on generators as

$$[\ell_r, \ell_s] = i\sum_{t=1}^{3} \varepsilon_{rst} \ell_t \quad . \tag{2.29}$$

This turns V into a complex Lie algebra:
- skew-symmetry $[x,x] = 0$: ok.

- Jacobi identity : turns out ok.
We will later call this Lie algebra $sl(2,\mathbb{C})$.

This Lie algebra is particularly important for atomic spectra, e.g. for the hydrogen atom, because the electrons move in a rotationally symmetric potential. This implies $[L_i, H] = 0$ and acting with one of the L_i on an energy eigenstate gives results in an energy eigenstate of the same energy. We say: The states at a given energy have to form a *representation* of $sl(2,\mathbb{C})$. Representations of $sl(2,\mathbb{C})$ are treated in section 4.2.

3 Matrix Lie Groups and their Lie Algebras

3.1 Matrix Lie groups

Definition 3.1 :
A *matrix Lie group* is a closed subgroup of $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$ for some $n \geq 1$.

Comments:
■ 'closed' in this definition stands for 'closed as a subset of the topological space $GL(n,\mathbb{R})$ (resp. $GL(n,\mathbb{C})$)'. It is equivalent to demanding that given a sequence A_n of matrices belonging to a matrix subgroup H s.t. $A = \lim_{n\to\infty} A_n$ exists and is in $GL(n,\mathbb{R})$ (resp. $GL(n,\mathbb{C})$), then already $A \in H$.
■ A matrix Lie group is a Lie group. However, not every Lie group is isomorphic to a matrix Lie group. We will not prove this. If you are interested in more details, consult e.g. [Baker, Theorem 7.24] and [Baker, Section 7.7].

So far we have met the groups
■ invertible linear maps $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$. In general we set $GL(V) = \{$ invertible linear maps $V \to V\}$, such that $GL(n,\mathbb{R}) = GL(\mathbb{R}^n)$, etc.
■ Some subgroups of $GL(n,\mathbb{R})$, namely $O(n) = \{M \in \mathrm{Mat}(n,\mathbb{R}) | M^t M = \mathbf{1}\}$ and $SO(n) = \{M \in O(n) | \det(M) = 1\}$.
■ Some semidirect products, $E(n) \cong O(n) \ltimes \mathbb{R}^n$ and $P(1,n{-}1) \cong O(1,n{-}1) \ltimes \mathbb{R}^n$.

All of these are matrix Lie groups, or isomorphic to matrix Lie groups:
■ For $O(n)$ and $SO(n)$ we already know that they are subgroups of $GL(n,\mathbb{R})$. It remains to check that they are closed as subsets of $GL(n,\mathbb{R})$. This follows since for a continuous function f and any sequence a_n with limit $\lim_{n\to\infty} a_n = a$ we have $\lim_{n\to\infty} f(a_n) = f(a)$. The defining relations $M \mapsto M^t M$ and $M \mapsto \det(M)$ are continuous functions.
Alternatively one can argue as follows: The preimage of a closed set under a continuous map is closed. The one-point sets $\{\mathbf{1}\} \subset \mathrm{Mat}(n,\mathbb{R})$ and $\{1\} \subset \mathbb{R}$ are closed.
■ For the groups $E(n)$ and $P(1,n{-}1)$ we use the following lemma.

Lemma 3.2:
Let $\varphi : \mathrm{Mat}(n,\mathbb{R}) \times \mathbb{R}^n \to \mathrm{Mat}(n{+}1,\mathbb{R})$ be the map

$$\varphi(M,v) = \left(\begin{array}{c|c} M & v \\ \hline 0 & 1 \end{array}\right) \quad . \tag{3.1}$$

(i) φ restricts to an injective group homomorphism from $O(n) \ltimes \mathbb{R}^n$ to $GL(n{+}1,\mathbb{R})$, and from $O(1,n{-}1) \ltimes \mathbb{R}^n$ to $GL(n{+}1,\mathbb{R})$.
(ii) The images $\varphi(O(n) \ltimes \mathbb{R}^n)$ and $\varphi(O(1,n{-}1) \ltimes \mathbb{R}^n)$ are closed subsets of $GL(n{+}1,\mathbb{R})$.

Proof:
(i) We need to check that

$$\varphi((R,u)\cdot(S,v)) = \varphi(R,u)\cdot\varphi(S,v) \quad . \tag{3.2}$$

The lhs is equal to

$$\varphi((R,u)\cdot(S,v)) = \varphi(RS, Rv+u) = \left(\begin{array}{c|c} RS & Rv+u \\ \hline 0 & 1 \end{array}\right) \tag{3.3}$$

while the rhs gives

$$\varphi(R,u)\cdot\varphi(S,v) = \left(\begin{array}{c|c} R & u \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} S & v \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} RS & Rv+u \\ \hline 0 & 1 \end{array}\right) \quad , \tag{3.4}$$

so φ is a group homomorphism. Further, it is clearly injective.
(ii) The images of $O(n) \ltimes \mathbb{R}^n$ and $O(1,n{-}1) \ltimes \mathbb{R}^n$ under φ consist of all matrices

$$\left(\begin{array}{c|c} R & u \\ \hline 0 & 1 \end{array}\right) \tag{3.5}$$

with $u \in \mathbb{R}^n$ and R an element of $O(n)$ and $O(1,n{-}1)$, respectively. This is a closed subset of $GL(n{+}1,\mathbb{R})$ since $O(n)$ (resp. $O(1,n{-}1)$) and $\mathbb{R}^n$ are closed. $\square$

Here are some matrix Lie groups which are subgroups of $GL(n,\mathbb{C})$.

Definition 3.3:
On $\mathbb{C}^n$ define the inner product

$$(u,v) = \sum_{k=1}^{n} (u_k)^* v_k \quad . \tag{3.6}$$

Then the *unitary group* $U(n)$ is given by

$$U(n) = \{A \in \mathrm{Mat}(n,\mathbb{C}) | (Au,Av) = (u,v) \text{ for all } u,v \in \mathbb{C}^n\} \tag{3.7}$$

and the *special unitary group* $SU(n)$ is given by

$$SU(n) = \{A \in U(n) | \det(A) = 1\} \quad . \tag{3.8}$$

Exercise 3.1 :
(i) Show that $U(n)$ and $SU(n)$ are indeed groups.
(ii) Let $(A^\dagger)_{ij} = (A_{ji})^*$ be the hermitian conjugate. Show that the condition $(Au, Av) = (u, v)$ for all $u, v \in \mathbb{C}^n$ is equivalent to $A^\dagger A = \mathbf{1}$, i.e.

$$U(n) = \{A \in \mathrm{Mat}(n, \mathbb{C}) \,|\, A^\dagger A = \mathbf{1}\} \ .$$

(iii) Show that $U(n)$ and $SU(n)$ are matrix Lie groups.

Definition 3.4 :
For $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, the *special linear group* $SL(n, \mathbb{K})$ is given by

$$SL(n, \mathbb{K}) = \{A \in \mathrm{Mat}(n, \mathbb{K}) | \det(A) = 1\} \quad . \tag{3.9}$$

3.2 The exponential map

Definition 3.5 :
The *exponential* of a matrix $X \in \mathrm{Mat}(n, \mathbb{K})$, for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, is

$$\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k = 1 + X + \tfrac{1}{2} X^2 + \dots \tag{3.10}$$

Lemma 3.6 :
The series defining $\exp(X)$ converges absolutely for all $X \in \mathrm{Mat}(n, \mathbb{K})$.

Proof:
Choose your favorite norm on $\mathrm{Mat}(n, \mathbb{K})$, say

$$\|X\| = \sum_{k,l=1}^{n} |A_{kl}| \quad . \tag{3.11}$$

The series $\exp(X)$ converges absolutely if the series of norms $\sum_{k=0}^{\infty} \frac{1}{k!} \|X^k\|$ converges. This in turn follows since $\|XY\| \leq \|X\| \, \|Y\|$ and since the series e^a converges for all $a \in \mathbb{R}$. $\square$

The following exercise shows a convenient way to compute the exponential of a matrix via its Jordan normal form ($\rightarrow$ wikipedia.org, eom.springer.de).

Exercise 3.2 :
(i) Show that for $\lambda \in \mathbb{C}$,

$$\exp \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = e^\lambda \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad .$$

(ii) Let $A \in \mathrm{Mat}(n, \mathbb{C})$. Show that for any $U \in GL(n, \mathbb{C})$

$$U^{-1} \exp(A) U = \exp(U^{-1} A U) \quad .$$

(iii) Recall that a complex $n \times n$ matrix A can always be brought to Jordan normal form, i.e. there exists an $U \in GL(n, \mathbb{C})$ s.t.

$$U^{-1} A U = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_r \end{pmatrix} \quad ,$$

where each Jordan block is of the form

$$J_k = \begin{pmatrix} \lambda_k & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_k \end{pmatrix} \quad , \quad \lambda_k \in \mathbb{C} \quad .$$

In particular, if all Jordan blocks have size 1, the matrix A is diagonalisable. Compute

$$\exp \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} \quad \text{and} \quad \exp \begin{pmatrix} 5 & 9 \\ -1 & -1 \end{pmatrix} \quad .$$

Exercise 3.3:
Let $A \in \mathrm{Mat(n, \mathbb{C})}$.
(i) Let $f(t) = \det(\exp(tA))$ and $g(t) = \exp(t \, \mathrm{tr}(A))$. Show that $f(t)$ and $g(t)$ both solve the first order DEQ $u' = \mathrm{tr}(A)\, u$.
(ii) Using (i), show that

$$\det(\exp(A)) = \exp(\mathrm{tr}(A)) \quad .$$

Exercise 3.4:
Show that if A and B commute (i.e. if $AB = BA$), then $\exp(A)\exp(B) = \exp(A + B)$.

3.3 The Lie algebra of a matrix Lie group

In this section we will look at the relation between matrix Lie groups and Lie algebras. As the emphasis on this course will be on Lie algebras, in this section we will state some results without proof.

Defintion 1.1:
A *Lie algebra* is a vector space V together with a bilinear map $[\ ,\ ] : V \times V \to V$, called *Lie bracket*, satisfying

(i) $[X, X] = 0$ for all $X \in V$ (*skew-symmetry*),
(ii) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in V$ (*Jacobi identity*).

If you have read through sections 2.4 and 2.5 you may jump directly to definition 3.7 below. If not, here are the definition of a commmutator and of the Lie algebra $gl(E) \equiv \mathrm{End}(E)$ restated.

Defintion 2.11:
Given a vector space E and two linear maps $A, B \in \mathrm{End}(E)$ [the endomorphisms of a vector space E are the linear maps from E to E], the *commutator* $[A, B]$ is

$$[A, B] = AB - BA \in \mathrm{End}(E) \quad . \tag{3.12}$$

Lemma 2.12:
Given a vector space E, the space of linear maps $\mathrm{End}(E)$ together with the commutator as Lie bracket is a Lie algebra. This Lie algebra will be called $gl(E)$, or also $\mathrm{End}(E)$.

Proof:
Abbreviate $V = \mathrm{End}(E)$.
(a) $[\ ,\ ]$ has to be a bilinear map from $V \times V$ to V.
Clear.
(b) $[\ ,\ ]$ has to obey $[A, A] = 0$ for all $A \in V$ (skew-symmetry).
Clear.
(c) $[\ ,\ ]$ has to satisfy the Jacobi identity $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$ for all $A, B, C \in V$.
This is the content of the next exercise. It follows that V is a Lie algebra. □

Exercise 2.10:
Verify that the commutator $[A, B] = AB - BA$ obeys the Jacobi identity.

The above exercise also shows that the $n \times n$ matrices $\mathrm{Mat}(n, \mathbb{K})$, for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, form a Lie algebra with the commutator as Lie bracket. This Lie algebra is called $gl(n, \mathbb{K})$. (In the notation of lemma 2.12, $gl(n, \mathbb{K})$ is the same as $gl(\mathbb{K}^n) \equiv \mathrm{End}(\mathbb{K}^n)$.)

Definition 3.7:
A *Lie subalgebra* h of a Lie algebra g is a a sub-vector space h of g such that whenever $A, B \in h$ then also $[A, B] \in h$.

Definition 3.8:
Let G be a matrix Lie group in $\mathrm{Mat}(n, \mathbb{K})$, for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.
(i) The *Lie algebra of* G is

$$g = \{A \in \mathrm{Mat}(n, \mathbb{K}) | \exp(tA) \in G \text{ for all } t \in \mathbb{R}\} \quad . \tag{3.13}$$

(ii) The *dimension of* G is the dimension of its Lie algebra (which is a vector space over $\mathbb{R}$).

The following theorem justifies the name 'Lie algebra of a matrix Lie group'. We will not prove it, but rather verify it in some examples.

Theorem 3.9:
The Lie algebra of a matrix Lie group, with commutator as Lie bracket, is a Lie algebra over $\mathbb{R}$ (in fact, a Lie subalgebra of $gl(n,\mathbb{R})$).

What one needs to show is that first, g is a vector space, and second, that for $A, B \in g$ also $[A, B] \in g$. The following exercise indicates how this can be done.

Exercise* 3.5:
Let G be a matrix Lie group and let g be the Lie algebra of G.
(i) Show that if $A \in g$, then also $sA \in g$ for all $s \in \mathbb{R}$.
(ii) The following formulae hold for $A, B \in \mathrm{Mat}(n,\mathbb{K})$: the *Trotter Product Formula*,

$$\exp(A+B) = \lim_{n\to\infty} \Big(\exp(A/n)\exp(B/n)\Big)^n \quad ,$$

and the *Commutator Formula*,

$$\exp([A,B]) = \lim_{n\to\infty} \Big(\exp(A/n)\exp(B/n)\exp(-A/n)\exp(-B/n)\Big)^{n^2} \quad .$$

(For a proof see [Baker, Theorem 7.26]). Use these to show that if $A, B \in g$, then also $A + B \in g$ and $[A, B] \in g$. (You will need that a matrix Lie group is closed.) Note that part (i) and (ii) combined prove Theorem 3.9.

3.4 A little zoo of matrix Lie groups and their Lie algebras

Here we collect the 'standard' matrix Lie groups (i.e. those which are typically mentioned without further explanation in text books). Before getting to the table, we need to define one more matrix Lie algebra.

Definition 3.10:
Let $\mathbf{1}_{n\times n}$ be the $n \times n$ unit matrix, and let

$$J_{\mathrm{sp}} = \begin{pmatrix} 0 & \mathbf{1}_{n\times n} \\ -\mathbf{1}_{n\times n} & 0 \end{pmatrix} \in \mathrm{Mat}(2n,\mathbb{R}) \quad .$$

The set

$$SP(2n) = \{M \in \mathrm{Mat}(2n,\mathbb{R}) | M^t J_{\mathrm{sp}} M = J_{\mathrm{sp}}\}$$

is called the $2n \times 2n$ *(real) symplectic group.*

Exercise 3.6:
Prove that $SP(2n)$ is a matrix Lie group.

Mat. Lie gr.	Lie algebra of the matrix Lie group	Dim. (over $\mathbb{R}$)
$GL(n,\mathbb{R})$	$gl(n,\mathbb{R}) = \mathrm{Mat}(n,\mathbb{R})$	n^2
$GL(n,\mathbb{C})$	$gl(n,\mathbb{C}) = \mathrm{Mat}(n,\mathbb{C})$	$2n^2$
$SL(n,\mathbb{R})$	$sl(n,\mathbb{R}) = \{A \in \mathrm{Mat}(n,\mathbb{R}) \vert \mathrm{tr}(A) = 0\}$	$n^2 - 1$
$SL(n,\mathbb{C})$	$sl(n,\mathbb{C}) = \{A \in \mathrm{Mat}(n,\mathbb{C}) \vert \mathrm{tr}(A) = 0\}$	$2n^2 - 2$
$O(n)$	$o(n) = \{A \in \mathrm{Mat}(n,\mathbb{R}) \vert A + A^t = 0\}$	$\frac{1}{2}n(n-1)$
$SO(n)$	$so(n) = o(n)$	
$SP(2n)$	$sp(2n) = \{A \in \mathrm{Mat}(2n,\mathbb{R}) \vert J_{\mathrm{sp}}A + A^t J_{\mathrm{sp}} = 0\}$	$n(2n+1)$
$U(n)$	$u(n) = \{A \in \mathrm{Mat}(n,\mathbb{C}) \vert A + A^\dagger = 0\}$	n^2
$SU(n)$	$su(n) = \{A \in u(n) \vert \mathrm{tr}(A) = 0\}$	$n^2 - 1$

Let us verify this list.

■ $GL(n,\mathbb{R})$:
We have to find all elements $A \in \mathrm{Mat}(n,\mathbb{R})$ such that $\exp(sA) \in GL(n,\mathbb{R})$ for all $s \in \mathbb{R}$. But $\exp(sA)$ is always invertible. Hence the Lie algebra of $GL(n,\mathbb{R})$ is just $\mathrm{Mat}(n,\mathbb{R})$ and its real dimension is n^2.

■ $GL(n,\mathbb{C})$:
Along the same lines as for $GL(n,\mathbb{R})$ we find that the Lie algebra of $GL(n,\mathbb{C})$ is $\mathrm{Mat}(n,\mathbb{C})$. As a vector space over $\mathbb{R}$ it has dimension $2n^2$.

■ $SL(n,\mathbb{R})$:
What are all $A \in \mathrm{Mat}(n,\mathbb{R})$ such that $\det(\exp(sA)) = 1$ for all $s \in \mathbb{R}$? Use

$$\det(\exp(sA)) = e^{s\,\mathrm{tr}(A)} \tag{3.14}$$

to see that $\mathrm{tr}(A) = 0$ is necessary and sufficient. The subspace of matrices with $\mathrm{tr}(A) = 0$ has dimension $n^2 - 1$.

■ $O(n)$:
What are all $A \in \mathrm{Mat}(n,\mathbb{R})$ s.t. $(\exp(sA))^t \exp(sA) = \mathbf{1}$ for all $s \in \mathbb{R}$? First, suppose that $M = \exp(sA)$ has the property $M^t M = \mathbf{1}$. Expanding this in s,

$$\mathbf{1} = (\mathbf{1} + sA^t)(\mathbf{1} + sA) + O(s^2) = \mathbf{1} + s(A^t + A) + O(s^2) \quad , \tag{3.15}$$

which shows that $A^t + A = 0$ is a necessary condition for A to be in the Lie algebra of $O(n)$. Further, it is also sufficient since $A + A^t = 0$ implies

$$(\exp(sA))^t \exp(sA) = \exp(sA^t)\exp(sA) = \exp(-sA)\exp(sA) = \mathbf{1} \quad . \tag{3.16}$$

In components, the condition $A^t + A = 0$ implies $A_{ii} = 0$ and $A_{ij} = -A_{ji}$. Thus only the entries A_{ij} with $1 \leq i < j \leq n$ can be choosen freely. The dimension of $o(n)$ is therefore $\frac{1}{2}n(n-1)$.

■ $SO(n)$:
What are all $A \in \mathrm{Mat}(n, \mathbb{R})$ s.t. $\exp(sA) \in O(n)$ and $\det(\exp(sA)) = 1$ for all $s \in \mathbb{R}$? First, $\exp(sA) \in O(n)$ (for all $s \in \mathbb{R}$) is equivalent to $A + A^t = 0$. Second, as for $SL(n, \mathbb{K})$ use

$$1 = \det(\exp(sA)) = e^{s\,\mathrm{tr}(A)} \tag{3.17}$$

to see that further $\mathrm{tr}(A) = 0$ is necessary and sufficient. However, $A + A^t = 0$ already implies $\mathrm{tr}(A) = 0$. Thus $SO(n)$ and $O(n)$ have the same Lie algebra.

■ $SU(n)$:
Here the calculation is the same as for $SO(n)$, except that now $A^\dagger + A = 0$ does *not* imply that $\mathrm{tr}(A) = 0$, so this is an extra condition.

Exercise 3.7:
In the table of matrix Lie algebras, verify the entries for $SL(n, \mathbb{C})$, $SP(2n)$, $U(n)$ and confirm the dimension of $SU(n)$.

A Lie algebra probes the structure of a Lie group close to the unit element. If the Lie algebras of two Lie groups agree, the two Lie groups look alike in a neighbourhood of the unit, but may still be different. For example, even though $o(n) = so(n)$ we still have $O(n) \not\cong SO(n)$.

Information 3.11:
This is easiest to see via topological considerations (which we will not treat in this course). The group $SO(n)$ is *path connected*, which means that for any $p, q \in SO(n)$ there is a continuous map $\gamma : [0,1] \to SO(n)$ such that $\gamma(0) = p$ and $\gamma(1) = q$ [Baker, section 9]. However, $O(n)$ cannot be path connected. To see this choose $p, q \in O(n)$ such that $\det(p) = 1$ and $\det(q) = -1$. The composition of a path γ with det is continuous, and on $O(n)$, det only takes values ± 1, so that it cannot change from 1 to -1 along γ. Thus there is no path from p to q. In fact, $O(n)$ has two connected components, and $SO(n)$ is the connected component containing the identity.

3.5 Examples: $SO(3)$ and $SU(2)$

SO(3)

We will need the following two notations. Let $\mathcal{E}(n)_{ij}$ denote the $n \times n$-matrix which has only one nonzero matrix element at position (i, j), and this matrix element is equal to one,

$$\big[\mathcal{E}(n)_{ij}\big]_{kl} = \delta_{ik}\delta_{jl} \quad . \tag{3.18}$$

For example,

$$\mathcal{E}(3)_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad . \tag{3.19}$$

If the value of n is clear we will usually abbreviate $\mathcal{E}_{ij} \equiv \mathcal{E}(n)_{ij}$.

Let ε_{ijk}, $i, j, k \in \{1, 2, 3\}$ be totally anti-symmetric in all indices, and let $\varepsilon_{123} = 1$.

Exercise 3.8:
(i) Show that $\mathcal{E}_{ab}\mathcal{E}_{cd} = \delta_{bc}\mathcal{E}_{ad}$.
(ii) Show that $\sum_{x=1}^{3} \varepsilon_{abx}\varepsilon_{cdx} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}$.

The Lie algebra $so(3)$ of the matrix Lie group $SO(3)$ consists of all real, anti-symmetric 3×3-matrices. The following three matrices form a basis of $so(3)$,

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad , \quad J_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad , \quad J_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad . \tag{3.20}$$

Exercise 3.9:
(i) Show that the generators J_1, J_2, J_3 can also be written as

$$J_a = \sum_{b,c=1}^{3} \varepsilon_{abc}\mathcal{E}_{bc} \qquad ; \quad a \in \{1, 2, 3\} \quad .$$

(ii) Show that $[J_a, J_b] = -\sum_{c=1}^{3} \varepsilon_{abc}J_c$
(iii) Check that $R_3(\theta) = \exp(-\theta J_3)$ is given by

$$R_3(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad .$$

This is a rotation by an angle θ around the 3-axis. Check explicitly that $R_3(\theta) \in SO(3)$.

SU(2)

The *Pauli matrices* are defined to be the following elements of $\mathrm{Mat}(2, \mathbb{C})$,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad . \tag{3.21}$$

Exercise 3.10:
Show that for $a, b \in \{1, 2, 3\}$, $[\sigma_a, \sigma_b] = 2i \sum_c \varepsilon_{abc}\sigma_c$.

The Lie algebra $su(2)$ consists of all anti-hermitian, trace-less complex 2×2 matrices.

Exercise 3.11:
(i) Show that the set $\{i\sigma_1, i\sigma_2, i\sigma_3\}$ is a basis of $su(2)$ as a real vector space. Convince yourself that the set $\{\sigma_1, \sigma_2, \sigma_3\}$ does *not* form a basis of $su(2)$ as a real vector space.
(ii) Show that $[i\sigma_a, i\sigma_b] = -2 \sum_{c=1}^{3} \varepsilon_{abc} i\sigma_c$.

$so(3)$ **and** $su(2)$ **are isomorphic**

Definition 3.12:
Let g, h be two Lie algebras.
(i) A linear map $\varphi : g \to h$ is a *Lie algebra homomorphism* iff

$$\varphi([a,b]) = [\varphi(a), \varphi(b)] \quad \text{for all} \quad a, b \in g \quad .$$

(ii) A Lie algebra homomorphism φ is a *Lie algebra isomorphism* iff it is invertible.

If we want to emphasise that g and h are Lie algebras over $\mathbb{R}$, we say that $\varphi : g \to h$ is a homomorphism (or isomorphism) of *real* Lie algebras. We also say *complex Lie algebra* for a Lie algebra whose underlying vector space is over $\mathbb{C}$.

Exercise 3.12:
Show that $so(3)$ and $su(2)$ are isomorphic as real Lie algebras.

Also in this case one finds that even though $so(3) \cong su(2)$, the Lie groups $SO(3)$ and $SU(2)$ are *not* isomorphic.

Information 3.13:
This is again easiest seen by topological arguments. One finds that $SU(2)$ is simply connected, i.e. every loop embedded in $SU(2)$ can be contracted to a point, while $SO(3)$ is not simply connected. In fact, $SU(2)$ is a two-fold covering of $SO(3)$.

3.6 Example: Lorentz group and Poincaré group

■ **Commutators of** $o(1, n-1)$**.**

Recall that the Lorentz group was given by

$$O(1,n-1) = \{M \in GL(n,\mathbb{R}) | M^t J M = J\} \tag{3.22}$$

where J is the diagonal matrix with entries $J = \mathrm{diag}(1, -1, \ldots, -1)$, and that these linear maps preserve the bilinear form

$$\eta(x,y) = x_0 y_0 - x_1 y_1 - \cdots - x_{n-1} y_{n-1} \tag{3.23}$$

on $\mathbb{R}^n$. Let $e_0, e_1, \ldots, e_{n-1}$ be the standard basis of $\mathbb{R}^n$ (i.e. $x = (x_0, \ldots, x_{n-1}) = \sum_k x_k e_k$). We will use the numbers

$$\eta_{kl} = \eta(e_k, e_l) = J_{kl} \quad . \tag{3.24}$$

Exercise 3.13:
Show that the Lie algebra of $O(1, n-1)$ is

$$o(1, n-1) = \{A \in \mathrm{Mat}(n, \mathbb{R}) | A^t J + JA = 0\} \quad .$$

If we write the matrices $A \in o(1, n-1)$ in block form, the condition $A^t J + JA = 0$ becomes

$$\begin{aligned} &\left(\begin{array}{c|c} a & c^t \\ \hline b^t & D^t \end{array}\right)\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -\mathbf{1} \end{array}\right) + \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -\mathbf{1} \end{array}\right)\left(\begin{array}{c|c} a & b \\ \hline c & D \end{array}\right) \\ &= \left(\begin{array}{c|c} a & -c^t \\ \hline b^t & -D^t \end{array}\right) + \left(\begin{array}{c|c} a & b \\ \hline -c & -D \end{array}\right) = 0 \end{aligned} \tag{3.25}$$

where $a \in \mathbb{C}$ and $D \in \mathrm{Mat}(n-1, \mathbb{R})$. Thus $a = 0$, $c = b^t$ and $D^t = -D$. Counting the free parameters gives the dimension to be

$$\dim(o(1, n-1)) = n-1 + \tfrac{1}{2}(n-1)(n-2) = \tfrac{1}{2}n(n-1) \quad . \tag{3.26}$$

Consider the following elements of $o(1, n-1))$,

$$M_{ab} = \eta_{bb}\mathcal{E}_{ab} - \eta_{aa}\mathcal{E}_{ba} \qquad a, b \in \{0, 1, \dots, n-1\} \quad . \tag{3.27}$$

These obey $M_{ab} = -M_{ba}$ and the set $\{M_{ab} | 0 \le a < b \le n-1\}$ forms a basis of $o(1, n-1))$.

Exercise 3.14:
Check that the commutator of the M_{ab}'s is

$$[M_{ab}, M_{cd}] = \eta_{ad}M_{bc} + \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac} \quad .$$

■ **Commutators of $p(1, n-1)$.**

In lemma 3.2 we found an embedding of the Poincaré group $P(1, n-1)$ into $\mathrm{Mat}(n+1, \mathbb{R})$. Let us denote the image in $\mathrm{Mat}(n+1, \mathbb{R})$ by $\tilde{P}(1, n-1)$. In the same lemma, we checked that $\tilde{P}(1, n-1)$ is a matrix Lie group. Let us compute its Lie algebra $p(1, n-1)$.

Exercise 3.15:
(i) Show that, for $A \in \mathrm{Mat}(n, \mathbb{R})$ and $u \in \mathbb{R}^n$,

$$\exp\left(\begin{array}{c|c} A & u \\ \hline 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} e^A & Bu \\ \hline 0 & 1 \end{array}\right) \qquad , \quad B = \sum_{n=1}^{\infty} \frac{1}{n!} A^{n-1} \quad .$$

[If A is invertible, then $B = A^{-1}(e^A - \mathbf{1})$.]
(ii) Show that the Lie algebra of $\tilde{P}(1, n-1)$ (the Poincaré group embedded in $\mathrm{Mat}(n+1, \mathbb{R})$) is

$$p(1, n-1) = \left\{ \left(\begin{array}{c|c} A & x \\ \hline 0 & 0 \end{array}\right) \middle| A \in o(1, n-1) \ , \ x \in \mathbb{R}^n \right\} \quad .$$

Let us define the generators M_{ab} for $a, b \in \{0, 1, \ldots, n{-}1\}$ as before and set in addition

$$P_a = \mathcal{E}_{an} \qquad , \quad a \in \{0, 1, \ldots, n{-}1\} \quad . \tag{3.28}$$

Exercise 3.16:
Show that, for $a, b, c \in \{0, 1, \ldots, n{-}1\}$,

$$[M_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b \quad , \quad [P_a, P_b] = 0 \quad .$$

We thus find that altogether the Poincaré algebra $p(1, n{-}1)$ has generators

$$\{M_{ab} | 0 \leq a < b \leq n{-}1\} \cup \{P_a | 0 \leq a \leq n{-}1\} \tag{3.29}$$

which obey the commutation relations

$$\begin{aligned} &[M_{ab}, M_{cd}] = \eta_{ad} M_{bc} + \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac} \quad , \\ &[M_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b \quad , \\ &[P_a, P_b] = 0 \quad . \end{aligned} \tag{3.30}$$

3.7 Final comments: Baker-Campbell-Hausdorff formula

Here are some final comments before we concentrate on the study of Lie algebras. Let g be the Lie algebra of a matrix Lie group G.
For $X, Y \in g$ close enough to zero, we have

$$\exp(X) \exp(Y) = \exp(X \star Y) \quad , \tag{3.31}$$

where

$$X \star Y = X + Y + \tfrac{1}{2}[X, Y] + \tfrac{1}{12}[X, [X, Y]] + \tfrac{1}{12}[Y, [Y, X]] + \ldots \tag{3.32}$$

can be expressed entirely in terms of commutators (which we will not prove). This is known as the *Baker-Campbell-Hausdorff identity.* For a proof, see [Bourbaki "Groupes et algeèbres de Lie" Ch. II § 6 n° 2 Thm. 1], and for an explicit formula [n° 4] of the same book.

Thus the Lie algebra g encodes all the information (group elements and their multiplication) of G in a neighbourhood of $\mathbf{1} \in G$.

Exercise 3.17:
There are some variants of the BCH identity which are also known as *Baker-Campbell-Hausdorff formulae.* Here we will prove some.
Let $\mathrm{ad}(A) : \mathrm{Mat}(n, \mathbb{C}) \to \mathrm{Mat}(n, \mathbb{C})$ be given by $\mathrm{ad}(A)B = [A, B]$. [This is called the *adjoint action.*]
(i) Show that for $A, B \in \mathrm{Mat}(n, \mathbb{C})$,

$$f(t) = e^{tA} B e^{-tA} \quad \text{and} \quad g(t) = e^{t\,\mathrm{ad}(A)} B$$

both solve the first order DEQ

$$\frac{d}{dt}u(t) = [A, u(t)] \quad .$$

(ii) Show that

$$e^A B e^{-A} = e^{\mathrm{ad}(A)} B = B + [A, B] + \tfrac{1}{2}[A, [A, B]] + \dots$$

(iii) Show that

$$e^A e^B e^{-A} = \exp(e^{\mathrm{ad}(A)} B) \quad .$$

(iv) Show that if $[A, B]$ commutes with A and B,

$$e^A e^B = e^{[A,B]} e^B e^A \quad .$$

(v) Suppose $[A, B]$ commutes with A and B. Show that $f(t) = e^{tA}e^{tB}$ and $g(t) = e^{tA+tB+\frac{1}{2}t^2[A,B]}$ both solve $\frac{d}{dt}u(t) = (A + B + t[A, B])u(t)$. Show further that

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]} \quad .$$

4 Lie algebras

In this course we will only be dealing with vector spaces over $\mathbb{R}$ or $\mathbb{C}$. When a definition or statement works for either of the two, we will write $\mathbb{K}$ instead of $\mathbb{R}$ or $\mathbb{C}$. (In fact, when we write $\mathbb{K}$ below, the statement or definition holds for every field.)

4.1 Representations of Lie algebras

Definition 4.1:
Let g be a Lie algebra over $\mathbb{K}$. A *representation* (V, R) of g is a $\mathbb{K}$-vector space V together with a Lie algebra homomorphism $R : g \to \mathrm{End}(V)$. The vector space V is called *representation space* and the linear map R the *action* or *representation map*. We will sometimes abbreviate $V \equiv (V, R)$.

In other words, (V, R) is a representation of g iff

$$R(x) \circ R(y) - R(y) \circ R(x) \;=\; R(\,[x, y]\,) \qquad \text{for all } x, y \in g \;. \tag{4.1}$$

Exercise 4.1:
It is also common to use 'modules' instead of representations. The two concepts are equivalent, as will be clear by the end of this exercise.
Let g be a Lie algebra over $\mathbb{K}$. A *g-module* V is a $\mathbb{K}$-vector space V together with a bilinear map $. : g \times V \to V$ such that

$$[x, y].w = x.(y.w) - y.(x.w) \qquad \text{for all} \quad x, y \in g, w \in V \quad . \tag{4.2}$$

(i) Show that given a g-module V, one gets a representation of g by setting $R(x)w = x.w$.
(ii) Given a representation (V, R) of g, show that setting $x.w = R(x)w$ defines a g-module on V.

Given a representation (V, R) of g and elements $x \in g$, $w \in V$, we will sometimes abbreviate $x.w \equiv R(x)w$.

Definition 4.2:
Let g be a Lie algebra.
(i) A representation (V, R) of g is *faithful* iff $R : g \to \mathrm{End}(V)$ is injective.
(ii) An *intertwiner* between two representations (V, R_V) and (W, R_W) is a linear map $f : V \to W$ such that

$$f \circ R_V(x) = R_W(x) \circ f \quad . \tag{4.3}$$

(iii) Two representations R_V and R_W are *isomorphic* if there exists an invertible intertwiner $f : V \to W$.

In particular, two representations whose representation spaces are of different dimension are never isomorphic. There are two representations one can construct for any Lie algebra g over $\mathbb{K}$.
■ The *trivial representation* is given by taking $\mathbb{K}$ as representation space (i.e. the one-dimensional $\mathbb{K}$-vector space $\mathbb{K}$ itself) and defining $R : g \to \mathrm{End}(\mathbb{K})$ to be $R(x) = 0$ for all $x \in g$. In short, the trivial representation is $(\mathbb{K}, 0)$.
■ The second representation is more interesting. For $x \in g$ define the map $\mathrm{ad}_x : g \to g$ as

$$\mathrm{ad}_x(y) = [x, y] \qquad \text{for all} \quad y \in g \quad . \tag{4.4}$$

Then $x \mapsto \mathrm{ad}_x$ defines a linear map $\mathrm{ad} : g \to \mathrm{End}(g)$. This can be used to define a representation of g on itself. In this way one obtains the *adjoint representation* (g, ad). This is indeed a representation of g because

$$\begin{aligned} &(\mathrm{ad}_x \circ \mathrm{ad}_y - \mathrm{ad}_y \circ \mathrm{ad}_x)(z) = [x, [y, z]] - [y, [x, z]] \\ &= [x, [y, z]] + [y, [z, x]] = -[z, [x, y]] = \mathrm{ad}_{[x,y]}(z) \quad . \end{aligned} \tag{4.5}$$

Exercise 4.2:
Show that for the Lie algebra $u(1)$, the trivial and the adjoint representation are isomorphic.

Given a representation R of g on $\mathbb{K}^n$ we define the *dual representation* R^+ via

$$R^+(x) = -R(x)^t \qquad \text{for all} \quad x \in g \quad . \tag{4.6}$$

That is, for the $n \times n$ matrix $R^+(x) \in \mathrm{End}(\mathbb{K}^n)$ we take minus the transpose of the matrix $R(x)$.

Exercise 4.3:
Show that if $(\mathbb{K}^n, R)$ is a representation of g, then so is $(\mathbb{K}^n, R^+)$ with $R^+(x) = -R(x)^t$.

The dual representation can also be defined for a representation R on a vector space V other then $\mathbb{K}^n$. One then takes R^+ to act on the dual vector space V^* and defines $R^+(x) = -R(x)^*$, i.e. $(V, R)^+ = (V^*, -R^*)$.

Definition 4.3:
Let g be a Lie-algebra and let (V, R) be a representation of g.
(i) A sub-vector space U of V is called *invariant subspace* iff $x.u \in U$ for all $x \in g$, $u \in U$. In this case we call (U, R) a *sub-representation* of (V, R).
(ii) (V, R) is called *irreducible* iff $V \neq \{0\}$ and the only invariant subspaces of (V, R) are $\{0\}$ and V.

Exercise 4.4:
Let $f : V \to W$ be an intertwiner of two representations V, W of g. Show that the *kernel* $\ker(f) = \{v \in V | f(v) = 0\}$ and the *image* $\mathrm{im}(f) = \{w \in W | w = f(v)$ for some $v \in V\}$ are invariant subspaces of V and W, respectively.

Recall the following result from linear algebra.

Lemma 4.4:
A matrix $A \in \mathrm{Mat}(n, \mathbb{C})$, $n > 0$, has at least one eigenvector.

This is the main reason why the treatment of *complex* Lie algebras is much simpler than that of real Lie algebras.

Lemma 4.5:
(*Schur's Lemma*) Let g be a Lie algebra and let U, V be two irreducible representations of g. Then an intertwiner $f : U \to V$ is either zero or an isomorphism.

Proof:
The kernel $\ker(f)$ is an invariant subspace of U. Since U is irreducible, either $\ker(f) = U$ or $\ker(f) = \{0\}$. Thus either $f = 0$ or f is injective. The image $\mathrm{im}(f)$ is an invariant subspace of V. Thus $\mathrm{im}(f) = \{0\}$ or $\mathrm{im}(f) = V$, i.e. either $f = 0$ or f is surjective. Altogether, either $f = 0$ or f is a bijection. $\square$

Corollary 4.6:
Let g be a Lie algebra over $\mathbb{C}$ and let U, V be two finite-dimensional, irreducible representations of g.
(i) If $f : U \to U$ is an intertwiner, then $f = \lambda \mathrm{id}_U$ for some $\lambda \in \mathbb{C}$.
(ii) If f_1 and f_2 are nonzero intertwiners from U to V, then $f_1 = \lambda f_2$ for some $\lambda \in \mathbb{C}^\times = \mathbb{C} - \{0\}$.

Proof:
(i) By lemma 4.4, f has an eigenvalue $\lambda \in \mathbb{C}$. Note that the linear map $h_\lambda = f - \lambda \mathrm{id}_U$ is an intertwiner from U to U since, for all $x \in g$, $u \in U$,

$$h_\lambda(x.u) = f(x.u) - \lambda x.u = x.f(u) - x.(\lambda u) = x.h_\lambda(u) \quad .$$

Let $u \neq 0$ be an eigenvector, $fu = \lambda u$. Then $h_\lambda(u) = 0$ so that h_λ is not an isomorphism. By Schur's lemma $h_\lambda = 0$ so that $f = \lambda \mathrm{id}_U$.
(ii) By Schur's Lemma, f_1 and f_2 are isomorphisms. $f_2^{-1} \circ f_1$ is an intertwiner from U to U. By part (i), $f_2^{-1} \circ f_1 = \lambda \mathrm{id}_U$, which implies $f_1 = \lambda f_2$. As $f_1 \neq 0$ we also have $\lambda \neq 0$. □

4.2 Irreducible representations of $sl(2,\mathbb{C})$

Recall that

$$sl(2,\mathbb{C}) = \{A \in \mathrm{Mat}(2,\mathbb{C}) | \mathrm{tr}(A) = 0\} \quad . \tag{4.7}$$

In section 3.4 we saw that this, understood as a *real* Lie algebra, is the Lie algebra of the matrix Lie group $SL(2,\mathbb{C})$. However, since $\mathrm{Mat}(2,\mathbb{C})$ is a complex vector space, we can also understand $sl(2,\mathbb{C})$ as a *complex* Lie algebra. We should really use a different symbol for the two, but by abuse of notation we (and everyone else) will not.

In this section, by $sl(2,\mathbb{C})$ we will always mean the *complex* Lie algebra. The aim of this section is to prove the following theorem.

Theorem 4.7:
The dimension gives a bijection

$$\dim : \left\{ \begin{array}{l} \text{finite dim. irreducible repns} \\ \text{of } sl(2,\mathbb{C}) \text{ up to isomorphism} \end{array} \right\} \longrightarrow \{1,2,3,\dots\} \quad . \tag{4.8}$$

All matrices A in $sl(2,\mathbb{C})$ are of the form

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \qquad \text{for} \quad a,b,c \in \mathbb{C} \quad . \tag{4.9}$$

A convenient basis will be

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad . \tag{4.10}$$

Exercise 4.5:
Check that for the basis elements of $sl(2,\mathbb{C})$ one has $[H,E] = 2E$, $[H,F] = -2F$ and $[E,F] = H$.

Exercise 4.6:
Let (V, R) be a representation of $sl(2,\mathbb{C})$. Show that if $R(H)$ has an eigenvector with non-integer eigenvalue, then V is infinite-dimensional.
Hint: Let $H.v = \lambda v$ with $\lambda \notin \mathbb{Z}$. Proceed as follows.

1) Set $w = E.v$. Show that either $w = 0$ or w is an eigenvector of $R(H)$ with eigenvalue $\lambda + 2$.

2) Show that either V is infinite-dimensional or there is an eigenvector v_0 of $R(H)$ of eigenvalue $\lambda_0 \notin \mathbb{Z}$ such that $E.v_0 = 0$.

3) Let $v_m = F^m.v_0$ and define $v_{-1} = 0$. Show by induction on m that

$$H.v_m = (\lambda_0 - 2m)v_m \quad \text{and} \quad E.v_m = m(\lambda_0 - m + 1)v_{m-1} \quad .$$

4) Conclude that if $\lambda_0 \notin \mathbb{Z}_{\geq 0}$ all v_m are nonzero.

Corollary 4.8:
(to exercise 4.6) In a finite-dimensional representation (V, R) of $sl(2,\mathbb{C})$ the eigenvalues of $R(H)$ are integers.

Exercise 4.7:
The Lie algebra $h = \mathbb{C}H$ is a subalgebra of $sl(2,\mathbb{C})$. Show that h has finite-dimensional representations where $R(H)$ has non-integer eigenvalues.

Next we construct a representation of $sl(2,\mathbb{C})$ for a given dimension.

Lemma 4.9:
Let $n \in \{1, 2, 3, \dots\}$ and let $e_0, \dots, e_{n-1}$ be the standard basis of $\mathbb{C}^n$. Set $e_{-1} = e_n = 0$. Then

$$\begin{aligned} H.e_m &= (n - 1 - 2m)e_m \\ E.e_m &= m(n - m)e_{m-1} \\ F.e_m &= e_{m+1} \end{aligned} \tag{4.11}$$

defines an irreducible representation V_n of $sl(2,\mathbb{C})$ on $\mathbb{C}^n$.

Proof: To see that this is a representation of $sl(2,\mathbb{C})$ we check the definition explicitly. For example

$$[E, F].e_m = H.e_m = (n - 1 - 2m)e_m \tag{4.12}$$

and

$$\begin{aligned} &E.(F.e_m) - F.(E.e_m) = (m + 1)(n - m - 1)e_m - m(n - m)e_m \\ &= (n - 1 - 2m)e_m = [E, F].e_m \quad . \end{aligned} \tag{4.13}$$

To check the remaining conditions is the content of the next exercise.

Irreducibility can be seen as follows. Let W be a nonzero invariant subspace of $\mathbb{C}^n$. Then $R(H)|_W$ has an eigenvector $v \in W$. But v is also an eigenvector of

$R(H)$ itself, and (because the e_m are a basis consisting of eigenvectors of H with distinct eigenvalues) has to be of the form $v = \lambda e_m$, for some $m \in \{0, \dots, n-1\}$ and $\lambda \in \mathbb{C}$. Thus W contains in particular the vector e_m Starting from e_m one can obtain all other e_k by acting with E and F. Thus W has to contain all e_k and hence $W = \mathbb{C}^n$. $\square$

Exercise 4.8:
Check that the representation of $sl(2, \mathbb{C})$ defined in the lecture indeed also obeys $[H, E].v = 2E.v$ and $[H, F].v = -2F.v$ for all $v \in \mathbb{C}^n$.

Proof of Theorem 4.7, part I:
Lemma 4.9 shows that the map dim( ) in the statement of Theorem 4.7 is surjective.

Exercise 4.9:
Let (W, R) be a finite-dimensional, irreducible representation of $sl(2, \mathbb{C})$. Show that for some $n \in \mathbb{Z}_{\geq 0}$ there is an injective intertwiner $\varphi : V_n \to W$.
Hint: (recall exercise 4.6)
1) Find a $v_0 \in W$ such that $E.v_0 = 0$ and $H.v_0 = \lambda_0 v_0$ for some $h \in \mathbb{Z}$.
2) Set $v_m = F^m.v_0$. Show that there exists an n such that $v_m = 0$ for $m \geq n$. Choose the smallest such n.
3) Show that $\varphi(e_m) = v_m$ for $m = 0, \dots, n-1$ defines an injective intertwiner.

Proof of Theorem 4.7, part II:
Suppose (W, R) is a finite-dimensional irreducible representation of $sl(2, \mathbb{C})$. By exercise 4.9 there is an injective intertwiner $\varphi : V_n \to W$. By Schur's lemma, as φ is nonzero, it has to be an isomorphism. This shows that the map dim( ) in the statement of Theorem 4.7 is injective. Since we already saw that it is also surjective, it is indeed a bijection. $\square$

4.3 Direct sums and tensor products

Definition 4.10:
Let U, V be two $\mathbb{K}$-vector spaces.
(i) The *direct sum* of U and V is the set

$$U \oplus V = \{ (u, v) \,|\, u \in U, v \in V \} \tag{4.14}$$

with addition and scalar multiplication defined to be

$$(u, v) + (u', v') = (u + u', v + v') \qquad \text{and} \qquad \lambda(u, v) = (\lambda u, \lambda v) \tag{4.15}$$

for all $u \in U$, $v \in V$, $\lambda \in \mathbb{K}$. We will write $u \oplus v \equiv (u, v)$.
(ii) The *tensor product* of U and V is the quotient vector space

$$U \otimes V = \mathrm{span}_{\mathbb{K}}((u, v) |\, u \in U, v \in V)/W \tag{4.16}$$

where W is the $\mathbb{K}$-vector space spanned by the vectors

$$(\lambda_1 u_1 + \lambda_2 u_2, v) - \lambda_1(u_1, v) - \lambda_2(u_2, v) \quad , \ \lambda_1, \lambda_2 \in \mathbb{K} \ , \ u_1, u_2 \in U \ , \ v \in V \ .$$

$$(u, \lambda_1 v_1 + \lambda_2 v_2) - \lambda_1(u, v_1) - \lambda_2(u, v_2) \quad , \ \lambda_1, \lambda_2 \in \mathbb{K} \ , \ u \in U \ , \ v_1, v_2 \in V \ .$$

The equivalence class containing (u, v) is denoted by $(u, v) + W$ or by $u \otimes v$.

What the definition of the tensor product means is explained in the following lemma, which can also be understood as a pragmatic definition of $U \otimes V$.

Lemma 4.11:
(i) Every element of $U \otimes V$ can be written in the form $u_1 \otimes v_1 + \cdots + u_n \otimes v_n$.
(ii) In $U \otimes V$ we can use the following rules

$$(\lambda_1 u_1 + \lambda_2 u_2) \otimes v = \lambda_1 \, u_1 \otimes v + \lambda_2 \, u_2 \otimes v \quad , \ \lambda_1, \lambda_2 \in \mathbb{K} \ , \ u_1, u_2 \in U \ , \ v \in V \ .$$

$$u \otimes (\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \, u \otimes v_1 + \lambda_2 \, u \otimes v_2 \quad , \ \lambda_1, \lambda_2 \in \mathbb{K} \ , \ u \in U \ , \ v_1, v_2 \in V \ .$$

Proof:
(ii) is an immediate consequence of the definition: Take the first equality as an example. The difference between the representative $(\lambda_1 u_1 + \lambda_2 u_2, v)$ of the equivalence class on the lhs and the representative $\lambda_1(u_1, v) + \lambda_2(u_2, v)$ of the equivalence class on rhs lies in W, i.e. in the equivalence class of zero.
(i) By definition, any $q \in U \otimes V$ is the equivalence class of an element of the form

$$q = \lambda_1(u_1, v_1) + \cdots + \lambda_n(u_n, v_n) + W \tag{4.17}$$

for some $n > 0$. But this is just the equivalence class denoted by

$$q = \lambda_1 \cdot u_1 \otimes v_1 + \cdots + \lambda_n \cdot u_n \otimes v_n \quad . \tag{4.18}$$

By part (ii), we in particular have $\lambda(u \otimes v) = (\lambda u) \otimes v$ so that the above vector can be written as

$$q = (\lambda_1 u_1) \otimes v_1 + \cdots + (\lambda_n u_n) \otimes v_n \quad , \tag{4.19}$$

which is of the desired form. □

Exercise* 4.10:
Let U, V be two finite-dimensional $\mathbb{K}$-vector spaces. Let $u_1, \ldots, u_m$ be a basis of U and let $v_1, \ldots, v_n$ be a basis of V.
(i) [Easy] Show that

$$\{u_k \oplus 0 | k = 1, \ldots, m\} \cup \{0 \oplus v_k | k = 1, \ldots, n\}$$

is a basis of $U \oplus V$.
(ii) [Harder] Show that

$$\{u_i \otimes v_j | i = 1, \ldots, m \text{ and } j = 1, \ldots, n\}$$

is a basis of $U \otimes V$.

This exercise shows in particular that

$$\dim(U \oplus V) = \dim(U) + \dim(V) \quad \text{and} \quad \dim(U \otimes V) = \dim(U)\dim(V) \ . \tag{4.20}$$

Definition 4.12:
Let g, h be Lie algebras over $\mathbb{K}$. The *direct sum* $g \oplus h$ is the Lie algebra given by the $\mathbb{K}$-vector space $g \oplus h$ with Lie bracket

$$[x \oplus y, x' \oplus y'] = [x, x'] \oplus [y, y'] \qquad \text{for all } \ x, x' \in g, \ y, y' \in h \ . \tag{4.21}$$

Exercise 4.11:
Show that for two Lie algebras g, h, the vector space $g \oplus h$ with Lie bracket as defined in the lecture is indeed a Lie algebra.

Definition 4.13:
Let g be a Lie algebra and let U, V be two representations of g.
(i) The *direct sum* of U and V is the representation of g on the vector space $U \oplus V$ with action

$$x.(u \oplus v) = (x.u) \oplus (x.v) \qquad \text{for all } \ x \in g, \ u \in U, \ v \in V \ . \tag{4.22}$$

(ii) The *tensor product* of U and V is the representation of g on the vector space $U \otimes V$ with action

$$x.(u \otimes v) = (x.u) \otimes v + u \otimes (x.v) \qquad \text{for all } \ x \in g, \ u \in U, \ v \in V \ . \tag{4.23}$$

Exercise 4.12:
Let g be a Lie algebra and let U, V be two representations of g.
(i) Show that the vector spaces $U \oplus V$ and $U \otimes V$ with g-action as defined in the lecture are indeed representations of g.
(ii) Show that the vector space $U \otimes V$ with g-action $x.(u \otimes v) = (x.u) \otimes (x.v)$ is *not* a representation of g.

Exercise 4.13:
Let V_n denote the irreducible representation of $sl(2, \mathbb{C})$ defined in the lecture. Consider the isomorphism of vector spaces $\varphi : V_1 \oplus V_3 \to V_2 \otimes V_2$ given by

$$\varphi(e_0 \oplus 0) = e_0 \otimes e_1 - e_1 \otimes e_0 \ ,$$

$$\varphi(0 \oplus e_0) = e_0 \otimes e_0 \ ,$$

$$\varphi(0 \oplus e_1) = e_0 \otimes e_1 + e_1 \otimes e_0 \ ,$$

$$\varphi(0 \oplus e_2) = 2e_1 \otimes e_1 \ ,$$

(so that V_1 gets mapped to anti-symmetric combinations and V_3 to symmetric combinations of basis elements of $V_2 \otimes V_2$). With the help of φ, show that

$$V_1 \oplus V_3 \cong V_2 \otimes V_2$$

as representations of $sl(2,\mathbb{C})$ (this involves a bit of writing).

4.4 Ideals

If U, V are sub-vector spaces of a Lie algebra g over $\mathbb{K}$ we define $[U, V]$ to be the sub-vector space

$$[U,V] = \mathrm{span}_{\mathbb{K}}\big([x,y] | x \in U, y \in V\big) \subset g \quad . \tag{4.24}$$

Definition 4.14:
Let g be a Lie algebra.
(i) A sub-vector space $h \subset g$ is an *ideal* iff $[g,h] \subset h$.
(ii) An ideal h of g is called *proper* iff $h \neq \{0\}$ and $h \neq g$.

Exercise 4.14:
Let g be a Lie algebra.
(i) Show that a sub-vector space $h \subset g$ is a Lie subalgebra of g iff $[h,h] \subset h$.
(ii) Show that an ideal of g is in particular a Lie subalgebra.
(iii) Show that for a Lie algebra homomorphism $\varphi : g \to g'$ from g to a Lie algebra g', $\ker(\varphi)$ is an ideal of g.
(iv) Show that $[g,g]$ is an ideal of g.
(v) Show that if h and h' are ideals of g, then their intersection $h \cap h'$ is an ideal of g.

Lemma 4.15:
If g is a Lie algebra and $h \subset g$ is an ideal, then quotient vector space g/h is a Lie algebra with Lie bracket

$$[x+h, y+h] = [x,y] + h \qquad \text{for} \;\; x, y \in g \quad . \tag{4.25}$$

Proof:
(i) The Lie bracket is well defined: Let $\pi : g \to g/h$, $\pi(x) = x + h$ be the canonical projection. For $a = \pi(x)$ and $b = \pi(y)$ we want to define

$$[a,b] = \pi([x,y]) \quad . \tag{4.26}$$

For this to be well defined, the rhs must only depend on a and b, but not on the specific choice of x and y. Let thus x', y' be two elements of g such that

$\pi(x') = a$, $\pi(y') = b$. Then there exist $h_x, h_y \in h$ such that $x' = x + h_x$ and $y' = y + h_y$. It follows that

$$\begin{aligned} \pi([x', y']) &= \pi([x + h_x, y + h_y]) \\ &= \pi([x, y]) + \pi([h_x, y]) + \pi([x, h_y]) + \pi([h_x, h_y]) \quad . \end{aligned} \tag{4.27}$$

But $[h_x, y]$, $[x, h_y]$ and $[h_x, h_y]$ are in h since h is an ideal, and hence

$$0 = \pi([h_x, y]) = \pi([x, h_y]) = \pi([h_x, h_y]) \quad . \tag{4.28}$$

It follows $\pi([x', y']) = \pi([x, y]) + 0$ and hence the Lie bracket on g/h is well-defined.
(ii) The Lie bracket is skew-symmetric, bilinear and solves the Jacobi-Identity: Immediate from definition. E.g.

$$[x + h, x + h] = [x, x] + h = 0 + h \quad . \tag{4.29}$$

□

Exercise 4.15:
Let g be a Lie algebra and $h \subset g$ an ideal. Show that $\pi : g \to g/h$ given by $\pi(x) = x + h$ is a surjective homomorphism of Lie algebras with kernel $\ker(\pi) = h$.

Definition 4.16:
A Lie algebra g is called
(i) *abelian* iff $[g, g] = \{0\}$.
(ii) *simple* iff it has no proper ideal and is not abelian.
(iii) *semi-simple* iff it is isomorphic to a direct sum of simple Lie algebras.
(iv) *reductive* iff it is isomorphic to a direct sum of simple and abelian Lie algebras.

Lemma 4.17:
If g is a semi-simple Lie algebra, then $[g, g] = g$.

Proof:
■ Suppose first that g is simple. We have seen in exercise 4.14(iv) that $[g, g]$ is an ideal of g. Since g is simple, $[g, g] = \{0\}$ or $[g, g] = g$. But $[g, g] = \{0\}$ implies that g is abelian, which is excluded for simple Lie algebras. Thus $[g, g] = g$.
■ Suppose now that $g = g_1 \oplus \cdots \oplus g_n$ with all g_k simple Lie algebras. Then

$$\begin{aligned} [g, g] &= \mathrm{span}_{\mathbb{K}}\big([g_k, g_l] | k, l = 1, \ldots, n\big) = \mathrm{span}_{\mathbb{K}}\big([g_k, g_k] | k = 1, \ldots, n\big) \\ &= \mathrm{span}_{\mathbb{K}}\big(g_k | k = 1, \ldots, n\big) = g \end{aligned} \tag{4.30}$$

where we first used that $[g_k, g_l] = \{0\}$ for $k \neq l$ and then that $[g_k, g_k] = g_k$ since g_k is simple. □

Exercise 4.16:
Let g, h be Lie algebras and $\varphi : g \to h$ a Lie algebra homomorphism. Show that if g is simple, then φ is either zero or injective.

4.5 The Killing form

Definition 4.18:
Let g be a finite-dimensional Lie algebra over $\mathbb{K}$. The *Killing form* $\kappa \equiv \kappa_g$ on g is the bilinear map $\kappa : g \times g \to \mathbb{K}$ given by

$$\kappa(x,y) = \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y) \qquad \text{for} \quad x, y \in g \quad . \tag{4.31}$$

Lemma 4.19:
The Killing form obeys, for all $x, y, z \in g$,
(i) $\kappa(x,y) = \kappa(y,x)$ (*symmetry*)
(ii) $\kappa([x,y],z) = \kappa(x,[y,z])$ (*invariance*)

Proof:
(i) By cyclicity of the trace we have

$$\kappa(x,y) = \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y) = \mathrm{tr}(\mathrm{ad}_y \circ \mathrm{ad}_x) = \kappa(y,x) \quad . \tag{4.32}$$

(ii) From the properties of the adjoint action and the cyclicity of the trace we get

$$\begin{aligned} &\kappa([x,y],z) = \mathrm{tr}(\mathrm{ad}_{[x,y]}\mathrm{ad}_z) = \mathrm{tr}(\mathrm{ad}_x\mathrm{ad}_y\mathrm{ad}_z - \mathrm{ad}_y\mathrm{ad}_x\mathrm{ad}_z) \\ &= \mathrm{tr}(\mathrm{ad}_x\mathrm{ad}_y\mathrm{ad}_z - \mathrm{ad}_x\mathrm{ad}_z\mathrm{ad}_y) = \kappa(x,[y,z]) \quad . \end{aligned} \tag{4.33}$$

□

Exercise 4.17:
(i) Show that for the basis of $sl(2,\mathbb{C})$ used in exercise 4.5, one has

$$\kappa(E,E) = 0 \ , \quad \kappa(E,H) = 0 \ , \quad \kappa(E,F) = 4 \ ,$$
$$\kappa(H,H) = 8 \ , \quad \kappa(H,F) = 0 \ , \quad \kappa(F,F) = 0 \ .$$

Denote by Tr the trace of 2×2-matrices. Show that for $sl(2,\mathbb{C})$ one has $\kappa(x,y) = 4\,\mathrm{Tr}(xy)$.
(ii) Evaluate the Killing form of $p(1,1)$ for all combinations of the basis elements M_{01}, P_0, P_1 (as used in exercises 3.14 and 3.16). Is the Killing form of $p(1,1)$ non-degenerate?

Exercise 4.18:
(i) Show that for $gl(n,\mathbb{C})$ one has $\kappa(x,y) = 2n\,\mathrm{Tr}(xy) - 2\mathrm{Tr}(x)\mathrm{Tr}(y)$, where Tr is the trace of $n{\times}n$-matrices.
Hint: Use the basis $\mathcal{E}_{kl}$ to compute the trace in the adjoint representation.
(ii) Show that for $sl(n,\mathbb{C})$ one has $\kappa(x,y) = 2n\,\mathrm{Tr}(xy)$.

Exercise 4.19:
Let g be a finite-dimensional Lie algebra and let $h \subset g$ be an ideal. Show that

$$h^{\perp} = \{x \in g | \kappa_g(x,y) = 0 \ \text{ for all } \ y \in h\}$$

is also an ideal of g.

The following theorem we will not prove.

Theorem 4.20:
If g is a finite-dimensional complex simple Lie algebra, then κ_g is non-degenerate.

Information 4.21:
The proof of this (and the necessary background) needs about 10 pages, and can be found e.g. in [Fulton, Harris "Representation Theory" Part II Ch. 9 and App. C Prop. C.10]. It works along the following lines. One defines

$$g^{\{0\}} = g \quad , \quad g^{\{1\}} = [g^{\{0\}}, g^{\{0\}}] \quad , \quad g^{\{2\}} = [g^{\{1\}}, g^{\{1\}}] \quad , \quad \dots \tag{4.34}$$

and calls a Lie algebra *solvable* if $g^{\{m\}} = \{0\}$ for some m. The hard part then is to prove *Cartan's criterion for solvability*, which implies that if a *complex, finite-dimensional* Lie algebra g has $\kappa_g = 0$, then g is solvable. Suppose now that g is simple. Then $[g,g] = g$, and hence g is not solvable (as $g^{\{m\}} = g$ for all m). Hence κ_g does not vanish. But the set

$$g^{\perp} = \{x \in g | \kappa_g(x,y) = 0 \ \text{ for all } \ y \in g\} \tag{4.35}$$

is an ideal (see exercise 4.19). Hence it is $\{0\}$ or g. But $g^{\perp} = g$ implies $\kappa_g = 0$, which cannot be for g simple. Thus $g^{\perp} = \{0\}$, which precisely means that κ_g is non-degenerate.

Lemma 4.22:
Let g be a finite-dimensional Lie algebra. If g contains an abelian ideal h (i.e. $[h,g] \subset h$ and $[h,h] = 0$), then κ_g is degenerate.

Exercise 4.20:
Show that if a finite-dimensional Lie algebra g contains an abelian ideal h, then the Killing form of g is degenerate. (Hint: Choose a basis of h, extend it to a basis of g, and evaluate $\kappa_g(x,a)$ with $x \in g$, $a \in h$.)

Exercise 4.21 :
Let $g = g_1 \oplus \cdots \oplus g_n$, for finite-dimensional Lie algebras g_i. Let $x = x_1 + \cdots + x_n$ and $y = y_1 + \cdots + y_n$ be elements of g such that $x_i, y_i \in g_i$. Show that

$$\kappa_g(x, y) = \sum_{i=1}^{n} \kappa_{g_i}(x_i, y_i) \ .$$

Theorem 4.23 :
For a finite-dimensional, complex Lie algebra g, the following are equivalent.
(i) g is semi-simple.
(ii) κ_g is non-degenerate.

Proof:
(i) $\Rightarrow$ (ii): We can write

$$g = g_1 \oplus \cdots \oplus g_n \tag{4.36}$$

for g_k simple Lie algebras. If $x, y \in g_k$, then $\kappa_g(x, y) = \kappa_{g_k}(x, y)$, while if $x \in g_k$ and $y \in g_l$ with $k \neq l$, we have $\kappa_g(x, y) = 0$. Let $x = x_1 + \cdots + x_n \neq 0$ be an element of g, with $x_k \in g_k$. There is at least one $x_l \neq 0$. Since g_l is simple, κ_{g_l} is non-degenerate, and there is a $y \in g_l$ such that $\kappa_{g_l}(x_l, y) \neq 0$. But

$$\kappa_g(x, y) = \kappa_{g_l}(x_l, y) \neq 0 \quad . \tag{4.37}$$

Hence κ_g is non-degenerate.
(ii) $\Rightarrow$ (i):
■ g is not abelian (or by lemma 4.22 κ_g would be degenerate). If g does not contain a proper ideal, then it is therefore simple and in particular semi-simple.
■ Suppose now that $h \subset g$ is a proper ideal and set $X = h \cap h^\perp$. Then X is an ideal. Further, $\kappa(a, b) = 0$ for all $a \in h$ and $b \in h^\perp$, so that in particular $\kappa(a, b) = 0$ for all $a, b \in X$. But then, for all $a, b \in X$ and for all $x \in g$, $\kappa(x, [a, b]) = \kappa([x, a], b) = 0$ (since $[x, a] \in X$ as X is an ideal). But κ is non-degenerate, so that this is only possible if $[a, b] = 0$. It follows that X is an abelian ideal. By the previous lemma, then $X = \{0\}$ (or κ would be degenerate).
■ In exercise 4.22 you will prove that, since κ_g is non-degenerate, $\dim(h) + \dim(h^\perp) = \dim(g)$. Since $[h, h^\perp] = \{0\}$ and $h \cap h^\perp = \{0\}$, we have $g = h \oplus h^\perp$ as Lie algebras. Apply the above argument to h and $h^\perp$ until all summands contain no proper ideals. Since g is finite-dimensional, this process will terminate. □

Exercise 4.22 :
Let g be a finite-dimensional Lie algebra with non-degenerate Killing form. Let $h \subset g$ be a sub-vector space. Show that $\dim(h) + \dim(h^\perp) = \dim(g)$.

Exercise 4.23 :
Show that the Poincaré algebra $p(1, n-1)$, $n \geq 2$, is not semi-simple.

Definition 4.24:
Let g be a Lie algebra over $\mathbb{K}$. A bilinear form $B : g \times g \to \mathbb{K}$ is called *invariant* iff $B([x,y],z) = B(x,[y,z])$ for all $x,y,z \in g$.

Clearly, the Killing form is an invariant bilinear form on g, which is in addition symmetric. The following theorem shows that for a simple Lie algebra, it is unique up to a constant.

Theorem 4.25:
Let g be a finite-dimensional, complex, simple Lie algebra and let B be an invariant bilinear form. Then $B = \lambda\kappa_g$ for some $\lambda \in \mathbb{C}$.

The proof will be given in the following exercise.

Exercise 4.24:
In this exercise we prove the theorem that for a finite-dimensional, complex, simple Lie algebra g, and for an invariant bilinear form B, we have $B = \lambda\kappa_g$ for some $\lambda \in \mathbb{C}$.
(i) Let $g^* = \{\varphi : g \to \mathbb{C} \text{ linear}\}$ be the dual space of g. The dual representation of the adjoint representation is $(g, \mathrm{ad})^+ = (g^*, -\mathrm{ad})$. Let $f_B : g \to g^*$ be given by $f_B(x) = B(x, \cdot)$, i.e. $[f_B(x)](z) = B(x,z)$. Show that f_B is an intertwiner from (g, ad) to $(g^*, -\mathrm{ad})$.
(ii) Using that g is simple, show that (g, ad) is irreducible.
(iii) Since (g, ad) and $(g^*, -\mathrm{ad})$ are isomorphic representations, also $(g^*, -\mathrm{ad})$ is irreducible. Let f_κ be defined in the same way as f_B, but with κ instead of B. Show that $f_B = \lambda f_\kappa$ for some $\lambda \in \mathbb{C}$.
(iv) Show that $B = \lambda\kappa$ for some $\lambda \in \mathbb{C}$.

5 Classification of finite-dimensional, semi-simple, complex Lie algebras

In this section we will almost exclusively work with finite-dimensional semi-simple complex Lie algebras. In order not to say that too often we abbreviate

$$\text{fssc} = \text{finite-dimensional semi-simple complex} \quad .$$

5.1 Working in a basis

Let g be a finite-dimensional Lie algebra over $\mathbb{K}$. Let $\{T^a | a = 1, \dots, \dim(g)\}$ be a basis of g. Then we can write

$$[T^a, T^b] = \sum_c f^{ab}{}_c T^c \qquad , \quad f^{ab}{}_c \in \mathbb{K} \quad . \tag{5.1}$$

The constants $f^{ab}{}_c$ are called *structure constants* of the Lie algebra g. (If g is infinite-dimensional, we cannot be sure to find a basis. But if we can, we also call the $f^{ab}{}_c$ structure constants.)

Exercise 5.1 :
Let $\{T^a\}$ be a basis of a finite-dimensional Lie algebra g over $\mathbb{K}$. For $x \in g$, let $M(x)_{ab}$ be the matrix of ad_x in that basis, i.e.

$$\mathrm{ad}_x(\sum_b v_b T^b) = \sum_a (\sum_b M(x)_{ab} v_b) T^a \quad .$$

Show that $M(T^a)_{cb} = f^{ab}{}_c$, i.e. the structure constants give the matrix elements of the adjoint action.

Exercise* 5.2 :
A fact from linear algebra: Show that for every non-degenerate symmetric bilinear form $b : V \times V \to \mathbb{C}$ on a finite-dimensional, complex vector space V there exists a basis $v_1, \dots, v_n$ (with $n = \dim(V)$) of V such that $b(v_i, v_j) = \delta_{ij}$.

If g is a fssc Lie algebra, we can hence find a basis $\{T^a | a = 1, \dots, \dim(g)\}$ such that

$$\kappa(T^a, T^b) = \delta_{ab} \quad . \tag{5.2}$$

In this basis the structure constants can be computed to be

$$\kappa(T^c, [T^a, T^b]) = \sum_d f^{ab}{}_d \kappa(T^c, T^d) = f^{ab}{}_c \quad . \tag{5.3}$$

Exercise 5.3 :
Let g be a fssc Lie algebra and $\{T^a\}$ a basis such that $\kappa(T^a, T^b) = \delta_{ab}$. Show that the structure constants in this basis are anti-symmetric in all three indices.

Exercise 5.4 :
Find a basis $\{T^a\}$ of $sl(2, \mathbb{C})$ s.t. $\kappa(T^a, T^b) = \delta_{ab}$.

5.2 Cartan subalgebras

Definition 5.1 :
An element x of a complex Lie algebra g is called *ad-diagonalisable* iff $\mathrm{ad}_x : g \to g$ is diagonalisable, i.e. iff there exists a basis T^a of g such that $[x, T^a] = \lambda_a T^a$, $\lambda_a \in \mathbb{C}$ for all a.

Lemma 5.2 :
Let g be a fssc Lie algebra g.
(i) Any $x \in g$ with $\kappa(x, x) \neq 0$ is ad-diagonalisable.
(ii) g contains at least one ad-diagonalisable element.

Proof:
(i) Let $n = \dim(g)$. The solution to exercise 5.2 shows that we can find a basis

$\{T^a \,|\, a = 1, \dots, n\}$ such that $\kappa(T^a, T^b) = \delta_{ab}$ and such that $x = \lambda T^1$ for some $\lambda \in \mathbb{C}^\times$. From exercise 5.1 we know that $M_{ba} \equiv M(T^1)_{ba} = f^{1a}{}_b$ are the matrix elements of ad_{T^1} in the basis $\{T^a\}$. Since f is totally antisymmetric (see exercise 5.3), we have

$$M_{ba} = f^{1a}{}_b = -f^{1b}{}_a = -M_{ab} \quad , \tag{5.4}$$

i.e. $M^t = -M$. In particular, $[M^t, M] = 0$, so that M is normal and can be diagonalised. Thus T^1 is ad-diagonalisable, and with it also x.
(ii) Exercise 5.2 also shows that (since κ is symmetric and non-degenerate) one can always find an $x \in g$ with $\kappa(x, x) \neq 0$. $\square$

Definition 5.3:
A sub-vector space h of a fssc Lie algebra g is a *Cartan subalgebra* iff it obeys the three properties
(i) all $x \in h$ are ad-diagonalisable.
(ii) h is abelian.
(iii) h is maximal in the sense that if h' obeys (i) and (ii) and $h \subset h'$, then already $h = h'$.

Exercise 5.5:
Show that the diagonal matrices in $sl(n, \mathbb{C})$ are a Cartan subalgebra.

■ The dimension $r = \dim(h)$ of a Cartan subalgebra is called the *rank of g*. By lemma 5.2, $r \geq 1$. It turns out (but we will not prove it in this course, but see [Fulton,Harris] § D.3) that r is independent of the choice of h and hence the rank is indeed a property of g.
■ Let $H^1, \dots, H^r$ be a basis of h. By assumption, ad_{H^i} can be diagonalised for each i. Further ad_{H^i} and ad_{H^j} commute for any $i, j \in \{1, \dots, r\}$,

$$[\mathrm{ad}_{H^i}, \mathrm{ad}_{H^j}] = \mathrm{ad}_{[H^i, H^j]} = 0 \quad . \tag{5.5}$$

Thus, all ad_{H^i} can be *simultaneously* diagonalised.
■ Let $y \in g$ be a simultaneous eigenvector for all $H \in h$,

$$\mathrm{ad}_H(y) = \alpha_y(H) y \qquad , \quad \text{for some} \quad \alpha_y(H) \in \mathbb{C} \quad . \tag{5.6}$$

The $\alpha_y(H)$ depend linearly on H. Thus we obtain a function

$$\alpha_y : h \to \mathbb{C} \quad , \tag{5.7}$$

i.e. $\alpha_y \in h^*$, the dual space of h. Conversely, given an element $\varphi \in h^*$ we set

$$g_\varphi = \{x \in g | [H, x] = \varphi(H) x \text{ for all } H \in h\} \quad . \tag{5.8}$$

Definition 5.4:
Let g be a fssc Lie algebra and h a Cartan subalgebra of g.
(i) $\alpha \in h^*$ is called a *root* of g (with respect to h) iff $\alpha \neq 0$ and $g_\alpha \neq \{0\}$.
(ii) The *root system* of g is the set

$$\Phi \equiv \Phi(g,h) = \{\alpha \in h^* | \alpha \text{ is a root}\} \quad . \tag{5.9}$$

Decomposing g into simultaneous eigenspaces of elements of h we can write

$$g = g_0 \oplus \bigoplus_{\alpha \in \Phi} g_\alpha \quad . \tag{5.10}$$

(This is a direct sum of *vector spaces* only, not of Lie algebras.)

Lemma 5.5:
(i) $[g_\alpha, g_\beta] \subset g_{\alpha+\beta}$ for all $\alpha, \beta \in h^*$.
(ii) If $x \in g_\alpha$, $y \in g_\beta$ for some $\alpha, \beta \in h^*$ s.t. $\alpha + \beta \neq 0$, then $\kappa(x,y) = 0$.
(iii) κ restricted to g_0 is non-degenerate.

Proof:
(i) Have, for all $H \in h$, $x \in g_\alpha$, $y \in g_\beta$,

$$\begin{aligned} &\mathrm{ad}_H([x,y]) = [H,[x,y]] \overset{(1)}{=} -[x,[y,H]] - [y,[H,x]] \\ &= \beta(H)[x,y] - \alpha(H)[y,x] = (\alpha+\beta)(H)\ [x,y] \end{aligned} \tag{5.11}$$

where (1) is the Jacobi identity. Thus $[x,y] \in g_{\alpha+\beta}$.
(ii) Let $H \in h$ be such that $\alpha(H) + \beta(H) \neq 0$ (H exists since $\alpha + \beta \neq 0$). Then

$$\begin{aligned} &(\alpha(H)+\beta(H))\kappa(x,y) = \kappa(\alpha(H)x,y) + \kappa(x,\beta(H)y) \\ &\overset{(1)}{=} \kappa([H,x],y) + \kappa(x,[H,y]) = -\kappa([x,H],y) + \kappa(x,[H,y]) \\ &\overset{(2)}{=} -\kappa(x,[H,y]) + \kappa(x,[H,y]) = 0 \end{aligned} \tag{5.12}$$

where (1) uses that $x \in g_\alpha$ and $y \in g_\beta$, and (2) that κ is invariant. Thus $\kappa(x,y) = 0$.
(iii) Let $y \in g_0$. Since κ is non-degenerate, there is an $x \in g$ s.t. $\kappa(x,y) \neq 0$. Write

$$x = x_0 + \sum_{\alpha \in \Phi} x_\alpha \qquad \text{where} \ \ x_0 \in g_0 \ , \quad x_\alpha \in g_\alpha \quad . \tag{5.13}$$

Then by part (ii), $\kappa(x,y) = \kappa(x_0,y)$. Thus for all $y \in g_0$ we can find an $x_0 \in g_0$ s.t. $\kappa(x_0,y) \neq 0$. □

Exercise 5.6:
Another fact about linear algebra: Let V be a finite-dimensional vector space and let $F \subset V^*$ be a proper subspace (i.e. $F \neq V^*$). Show that there exists a nonzero $v \in V$ such that $\varphi(v) = 0$ for all $\varphi \in F$.

Lemma 5.6:
Let g be a fssc Lie algebra and h a Cartan subalgebra. Then
(i) the Killing form restricted to h is non-degenerate.
(ii) $g_0 = h$.
(iii) $g_0^* = \mathrm{span}_{\mathbb{C}}(\Phi)$.

Proof:
(i) Since for all $a, b \in h$, $\mathrm{ad}_a(b) = [a, b] = 0$ we have $h \subset g_0$. Suppose there is an $a \in h$ such that $\kappa(a, b) = 0$ for all $b \in h$. Then in particular $\kappa(a, a) = 0$. As κ is non-degenerate on g_0, there is a $z \in g_0$, $z \notin h$, such that $\kappa(a, z) \neq 0$. If $\kappa(z, z) \neq 0$ set $u = z$. Otherwise set $u = a + z$ (then $\kappa(u, u) = \kappa(a + z, a + z) = 2\kappa(a, z) \neq 0$). In either case $u \notin h$ and $\kappa(u, u) \neq 0$. By lemma 5.2, u is ad-diagonalisable. Also $[b, u] = 0$ for all $b \in h$ (since $u \in g_0$). But then $\mathrm{span}_{\mathbb{C}}(h, u)$ obeys conditions (i),(ii) in the definition of a Cartan subalgebra and contains h as a proper subspace, which is a contradiction to h being a Cartan subalgebra. Hence κ has to be non-degenerate on h.
(ii) By part (i) we have subspaces

$$h \subset g_0 \subset g \tag{5.14}$$

and κ is non-degenerate on h, g_0, g. It is therefore possible to find a basis $\{T^a\}$ of g s.t.
- $\kappa(T^a, T^b) = \delta_{ab}$
- $T^a \in h$ for $a = 1, \dots, \dim(h)$ and $T^a \in g_0$ for $a = 1, \dots, \dim(g_0)$.

Let $X = T^a$ with $a = \dim(g_0)$. We have $[H, X] = 0$ for all $H \in h$ (since $X \in g_0$). Further, X is ad-diagonalisable (since $K(X, X) \neq 0$, see lemma 5.2 (i)). Thus the space $\mathrm{span}_{\mathbb{C}}(h, X)$ obeys (i) and (ii) in the definition of a Cartan subalgebra, and hence by maximality of h we have $h = \mathrm{span}_{\mathbb{C}}(h, X)$. Thus $X \in h$ and hence $\dim(g_0) = \dim(h)$.
(iii) Suppose that $\mathrm{span}_{\mathbb{C}}(\Phi)$ is a proper subspace of g_0^*. By exercise 5.6 there exists a nonzero element $H \in g_0$ s.t.

$$\alpha(H) = 0 \qquad \text{for all} \quad \alpha \in \Phi \quad . \tag{5.15}$$

Since $g_0 = h$ we have, for all $\alpha \in \Phi$ and all $x \in g_\alpha$, $[H, x] = \alpha(H)x = 0$. Thus $[H, x] = 0$ for all $x \in g$. But then $\mathrm{ad}_H = 0$, in contradiction to κ being non-degenerate (have $\kappa(y, H) = 0$ for all $y \in g$). $\square$

Exercise 5.7:
Let g be a fssc Lie algebra and let $h \subset g$ be sub-vector space such that
(1) $[h, h] = \{0\}$.
(2) κ restricted to h is non-degenerate.
(3) if for some $x \in g$ one has $[x, a] = 0$ for all $a \in h$, then already $x \in h$.
Show that h is a Cartan subalgebra of g if and only if it obeys (1)–(3) above.

5.3 Cartan-Weyl basis

Definition 5.7:
(i) For $\varphi \in g_0^*$ let $H^\varphi \in g_0$ be the unique element s.t.

$$\varphi(x) = \kappa(H^\varphi, x) \qquad \text{for all} \quad x \in g_0 \quad . \tag{5.16}$$

(ii) Define the non-degenerate pairing $(\cdot,\cdot) : g_0^* \times g_0^* \to \mathbb{C}$ via

$$(\gamma, \varphi) = \kappa(H^\gamma, H^\varphi) \quad . \tag{5.17}$$

Information 5.8:
We will see shortly that $(\alpha, \alpha) > 0$ for all $\alpha \in \Phi$. Since Φ is a finite set, there is a $\theta \in \Phi$ such that (θ, θ) is maximal. Some texts (such as [Fuchs,Schweigert] Sect. 6.3) use a a rescaled version of the Killing form κ to define $(\,\cdot\,,\,\cdot\,)$. This is done to impose the convention that the longest root lengths is $\sqrt{2}$, i.e. $(\theta, \theta) = 2$, which leads to simpler expressions in explicit calculations. But it also makes the exposition less clear, so we will stick to κ (as also done e.g. in [Fulton,Harris] § 14.2.)

Exercise 5.8:
Let $\{H^1, \dots, H^r\} \subset g_0$ be a basis of g_0 such that $\kappa(H^i, H^j) = \delta_{ij}$ (recall that $r = \dim(g_0)$ is the rank of g). Show that for $\gamma, \varphi \in g_0^*$ one has $H^\gamma = \sum_{i=1}^r \gamma(H^i) H^i$, as well as $(\gamma, \varphi) = \sum_{i=1}^r \gamma(H^i)\varphi(H^i)$ and $(\gamma, \varphi) = \gamma(H^\varphi)$.

Lemma 5.9:
Let $\alpha \in \Phi$. Then
(i) $-\alpha \in \Phi$.
(ii) If $x \in g_\alpha$ and $y \in g_{-\alpha}$ then $[x, y] = \kappa(x, y) H^\alpha$.
(iii) $(\alpha, \alpha) \neq 0$.

Proof:
(i) For $x \in g_\alpha$, the Killing form $\kappa(x, y)$ can be nonzero only for $y \in g_{-\alpha}$ (lemma 5.5(ii)). Since κ is non-degenerate, $g_{-\alpha}$ cannot be empty, and hence $-\alpha \in \Phi$.
(ii) Since $[g_\alpha, g_{-\alpha}] \subset g_0$ we have $[x, y] \in g_0$. Note that for all $H \in g_0$,

$$\begin{aligned} &\kappa(H, [x, y]) = \kappa([H, x], y) = \alpha(H)\kappa(x, y) = \kappa(H^\alpha, H)\kappa(x, y) \\ &= \kappa(H, \kappa(x, y) H^\alpha) \ . \end{aligned} \tag{5.18}$$

Since κ is non-degenerate, this implies $[x, y] = \kappa(x, y) H^\alpha$.
(iii) By exercise 5.8 we have $(\alpha, \alpha) = \alpha(H^\alpha)$. We will show that $\alpha(H^\alpha) \neq 0$.

■ Since $\alpha \neq 0$ have $H^\alpha \neq 0$. Since $g_0^* = \mathrm{span}_\mathbb{C}(\Phi)$, there exists a $\beta \in \Phi$ s.t. $\beta(H^\alpha) \neq 0$. Consider the subspace

$$U = \bigoplus_{m\in\mathbb{Z}} g_{\beta+m\alpha} \quad . \tag{5.19}$$

For $x \in g_{\beta+m\alpha}$ have $[H^\alpha, x] = (\beta(H^\alpha) + m\alpha(H^\alpha))x$ so that the trace of ad_{H^α} over U is

$$\mathrm{tr}_U(\mathrm{ad}_{H^\alpha}) = \sum_{m\in\mathbb{Z}} (\beta(H^\alpha) + m\alpha(H^\alpha)) \dim(g_{\beta+m\alpha}) \quad . \tag{5.20}$$

■ Choose a nonzero $x \in g_\alpha$. There is a $y \in g_{-\alpha}$ s.t. $\kappa(x, y) \neq 0$; we can choose y such that $\kappa(x, y) = 1$. Then $[x, y] = H^\alpha$. Since $\mathrm{ad}_x : g_\gamma \to g_{\gamma+\alpha}$ and $\mathrm{ad}_y : g_\gamma \to g_{\gamma-\alpha}$, both, ad_x and ad_y map U to U. Then we can also compute

$$\mathrm{tr}_U(\mathrm{ad}_{H^\alpha}) = \mathrm{tr}_U(\mathrm{ad}_{[x,y]}) = \mathrm{tr}_U(\mathrm{ad}_x\mathrm{ad}_y - \mathrm{ad}_y\mathrm{ad}_x) = 0 \quad , \tag{5.21}$$

by cyclicity of the trace.
■ Together with the previous expression for $\mathrm{tr}_U(\mathrm{ad}_{H^\alpha})$ this implies

$$\alpha(H^\alpha) \sum_{m\in\mathbb{Z}} m \dim(g_{\beta+m\alpha}) = -\beta(H^\alpha) \sum_{m\in\mathbb{Z}} \dim(g_{\beta+m\alpha}) \tag{5.22}$$

The rhs is nonzero (as $\beta(H^\alpha) \neq 0$ by construction, and $\dim(g_\beta) \neq 0$ since β is a root), and hence the lhs as to be nonzero. In particular, $\alpha(H^\alpha) \neq 0$. $\square$

Recall the standard basis E, F, H of $sl(2, \mathbb{C})$ we introduced in section 4.2.

Theorem and Exercise 5.9:
Let g be a fssc Lie algebra and g_0 a Cartan subalgebra. Let $\alpha \in \Phi(g, g_0)$. Choose $e \in g_\alpha$ and $f \in g_{-\alpha}$ such that $\kappa(e, f) = \frac{2}{(\alpha,\alpha)}$. Show that $\varphi : sl(2, \mathbb{C}) \to g$, given by

$$\varphi(E) = e \quad , \quad \varphi(F) = f \quad , \quad \varphi(H) = \frac{2}{(\alpha, \alpha)} H^\alpha \quad ,$$

is an injective homomorphism of Lie algebras.

This implies in particular that g can be turned into a finite-dimensional representation (g, R_φ) of $sl(2, \mathbb{C})$ via

$$R_\varphi(x)z = \mathrm{ad}_{\varphi(x)}z \qquad \text{for all} \quad x \in sl(2, \mathbb{C}) \ , \quad z \in g \quad , \tag{5.23}$$

i.e. by restricting the adjoint representation of g to $sl(2, \mathbb{C})$. For $z \in g_\beta$ we find

$$R_\varphi(H)z = \frac{2}{(\alpha, \alpha)}[H^\alpha, z] = \frac{2}{(\alpha, \alpha)}\beta(H^\alpha)z = \frac{2(\alpha, \beta)}{(\alpha, \alpha)}z \quad . \tag{5.24}$$

From corollary 4.6 we know that in a finite-dimensional representation of $sl(2, \mathbb{C})$, all eigenvalues of $R_\varphi(H)$ have to be integers. Thus

$$\frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z} \qquad \text{for all} \quad \alpha, \beta \in \Phi \quad . \tag{5.25}$$

Theorem 5.10:
Let g be a fssc Lie algebra and g_0 a Cartan subalgebra. Then
(i) if $\alpha \in \Phi$ and $\lambda\alpha \in \Phi$ for some $\lambda \in \mathbb{C}$, then $\lambda \in \{\pm 1\}$.
(ii) $\dim(g_\alpha) = 1$ for all $\alpha \in \Phi$.

The proof will be given in the following exercise.

Exercise* 5.10:
In this exercise we will show that $\dim(g_\alpha) = 1$ for all $\alpha \in \Phi$. On the way we will also see that if $\alpha \in \Phi$ and $\lambda\alpha \in \Phi$ for some $\lambda \in \mathbb{C}$, then $\lambda \in \{\pm 1\}$.
(i) Choose $\alpha \in \Phi$. Let $L = \{m \in \mathbb{Z} | m\alpha \in \Phi\}$. Since Φ is a finite set, so is L. Let n_+ be the largest integer in L, n_- the smallest integer in L. Show that $n_+ \geq 1$ and $n_- \leq -1$.
(ii) We can assume that $n_+ \geq |n_-|$. Otherwise we exchange α for $-\alpha$. Pick $e \in g_\alpha$, $f \in g_{-\alpha}$ s.t. $\kappa(e, f) = \frac{2}{(\alpha,\alpha)}$ and define $\varphi : sl(2, \mathbb{C}) \to g$ as in exercise 5.9. Show that

$$U = \mathbb{C}H^\alpha \oplus \bigoplus_{m \in L} g_{m\alpha}$$

is an invariant subspace of the representation (g, R_φ) of $sl(2, \mathbb{C})$.
(iii) Show that for $z \in g_{m\alpha}$ one has $R_\varphi(H)z = 2mz$.
(iv) By (ii), (U, R_φ) is also a representation of $sl(2, \mathbb{C})$. Show that $V = \mathbb{C}e \oplus \mathbb{C}H^\alpha \oplus \mathbb{C}f$ is an invariant subspace of (U, R_φ). Show that the representation (V, R_φ) is isomorphic to the irreducible representation V_3 of $sl(2, \mathbb{C})$.
(v) Choose an element $v_0 \in g_{n_+\alpha}$. Set $v_{k+1} = R_\varphi(F)v_k$ and show that

$$W = \mathrm{span}_{\mathbb{C}}(v_0, v_1, \ldots, v_{2n_+})$$

is an invariant subspace of U.
(vi) (W, R_φ) is isomorphic to the irreducible representation V_{2n_++1} of $sl(2, \mathbb{C})$. Show that the intersection $X = V \cap W$ is an invariant subspace of V and W. Show that X contains the element H^α and hence $X \neq \{0\}$. Show that $X = V$ and $X = W$.
We have learned that for any choice of v_0 in $g_{n_+\alpha}$ we have $V = W$. This can only be if $n_+ = 1$ and $\dim(g_\alpha) = 1$. Since $1 \leq |n_-| \leq n_+$, also $n_- = 1$. Since $\kappa : g_\alpha \times g_{-\alpha}$ is non-degenerate, also $\dim(g_{-\alpha}) = 1$.

Definition 5.11:
Let g be a fssc Lie algebra. A subset

$$\{H^i, i = 1, \ldots, r\} \cup \{E^\alpha | \alpha \in \Phi\}$$

of g, for Φ a finite subset of $\big(\mathrm{span}_{\mathbb{C}}(H^1, \ldots, H^r)\big)^* - \{0\}$, is called a *Cartan-Weyl basis* of g iff it is a basis of g, and $[H^i, H^j] = 0$, $[H^i, E^\alpha] = \alpha(H^i)\, E^\alpha$,

$$[E^\alpha, E^\beta] = \begin{cases} 0 & ; \alpha + \beta \notin \Phi \\ N_{\alpha,\beta} E^{\alpha+\beta} & ; \alpha + \beta \in \Phi \\ \frac{2}{(\alpha,\alpha)} H^\alpha & ; \alpha = -\beta \end{cases}$$

where $N_{\alpha,\beta} \in \mathbb{C}$ are some constants.

The analysis up to now implies the following theorem.

Theorem 5.12 :
For any fssc Lie algebra g, there exists a Cartan-Weyl basis.

Exercise 5.11 :
Let $\{H^i\} \cup \{E^\alpha\}$ be a Cartan-Weyl basis of a fssc Lie algebra g. Show that
(i) $\mathrm{span}_{\mathbb{C}}(H^1, \dots, H^r)$ is a Cartan subalgebra of g.
(ii) $\kappa(E^\alpha, E^{-\alpha}) = \frac{2}{(\alpha,\alpha)}$.

Lemma 5.13 :
(i) $\kappa(G,H) = \sum_{\alpha\in\Phi} \alpha(G)\alpha(H)$ for all $G, H \in g_0$.
(ii) $(\lambda,\mu) = \sum_{\alpha\in\Phi}(\lambda,\alpha)(\alpha,\mu)$ for all $\lambda,\mu \in g_0^*$.
(iii) $(\alpha,\beta) \in \mathbb{R}$ and $(\alpha,\alpha) > 0$ for all $\alpha,\beta \in \Phi$.

Proof:
(i) Let $\{H^i\} \cup \{E^\alpha\}$ be a Cartan-Weyl basis of g. One computes

$$\kappa(G,H) = \sum_{i=1}^{r} H^{i*}\big([G,[H,H^i]]\big) + \sum_{\alpha\in\Phi} E^{\alpha *}\big([G,[H,E^\alpha]]\big) = \sum_{\alpha\in\Phi} \alpha(H)\alpha(G) \quad . \tag{5.26}$$

(ii) Using part (i) we get

$$(\lambda,\mu) = \kappa(H^\lambda, H^\mu) = \sum_{\alpha\in\Phi} \alpha(H^\lambda)\alpha(H^\mu) = \sum_{\alpha\in\Phi}(\alpha,\mu)(\alpha,\lambda) \tag{5.27}$$

(iii) Using part (ii) we compute

$$(\alpha,\alpha) = \sum_{\beta\in\Phi}(\alpha,\beta)(\alpha,\beta) \quad . \tag{5.28}$$

Multiplying both sides by $4/(\alpha,\alpha)^2$ yields

$$\frac{4}{(\alpha,\alpha)} = \sum_{\beta\in\Phi}\Big(\frac{2(\alpha,\beta)}{(\alpha,\alpha)}\Big)^2 \quad . \tag{5.29}$$

We have already seen that $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$. Thus the rhs is real and non-negative. Since $(\alpha,\alpha) \neq 0$ (see lemma 5.9 (iii)) it follows that $(\alpha,\alpha) > 0$. Together with $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ this in turn implies that $(\alpha,\beta) \in \mathbb{R}$. $\square$

Exercise 5.12:
Let $g = g_1 \oplus g_2$ with g_1, g_2 fssc Lie algebras. For $k = 1, 2$, let h_k be a Cartan subalgebra of g_k.
(i) Show that $h = h_1 \oplus h_2$ is a Cartan subalgebra of g.
(ii) Show that the root system of g is $\Phi(g, h) = \Phi_1 \cup \Phi_2 \subset h_1^* \oplus h_2^*$ where $\Phi_1 = \{\alpha \oplus 0 | \alpha \in \Phi(g_1, h_1)\}$ and $\Phi_2 = \{0 \oplus \beta | \beta \in \Phi(g_2, h_2)\}$.
(iii) Show that $(\alpha, \beta) = 0$ for all $\alpha \in \Phi_1$ and $\beta \in \Phi_2$.

Lemma 5.14:
Let g be a fssc Lie algebra and g_0 be a Cartan subalgebra. The following are equivalent.
(i) g is simple.
(ii) One cannot write $\Phi(g, g_0) = \Phi_1 \cup \Phi_2$ where Φ_1, Φ_2 are non-empty and $(\alpha, \beta) = 0$ for all $\alpha \in \Phi_1$, $\beta \in \Phi_2$.

Proof:
$\neg$(i)$\Rightarrow$ $\neg$(ii): This amounts to exercise 5.12.
$\neg$(ii)$\Rightarrow$ $\neg$(i): Let $\Phi \equiv \Phi(g, g_0) = \Phi_1 \cup \Phi_2$ with the properties stated in (ii).
■ If $\alpha \in \Phi_1$, $\beta \in \Phi_2$ then $\alpha + \beta \notin \Phi_2$, since

$$(\alpha, \alpha + \beta) = (\alpha, \alpha) \neq 0 \quad . \tag{5.30}$$

Similarly, since $(\beta, \alpha + \beta) \neq 0$ we have $\alpha + \beta \notin \Phi_1$. Thus $\alpha + \beta \notin \Phi$.
■ If $\alpha, \beta \in \Phi_1$, $\alpha \neq -\beta$, then

$$0 \neq (\alpha + \beta, \alpha + \beta) = (\alpha + \beta, \alpha) + (\alpha + \beta, \beta) \tag{5.31}$$

so that $\alpha + \beta \notin \Phi_2$.
■ Let $\{H^i\} \cup \{E^\alpha\}$ be a Cartan-Weyl basis of g. Let $h_1 = \mathrm{span}_{\mathbb{C}}(H^\alpha | \alpha \in \Phi_1\}$ and

$$g_1 = h_1 \oplus \bigoplus_{\alpha \in \Phi_1} \mathbb{C} E^\alpha \quad . \tag{5.32}$$

Claim: g_1 is a proper ideal of g. The proof of this claim is the subject of the next exercise. Thus g has a proper ideal and hence is not simple. $\square$

Exercise 5.13:
Let g be a fssc Lie algebra and g_0 be a Cartan subalgebra. Suppose $\Phi(g, g_0) = \Phi_1 \cup \Phi_2$ where Φ_1, Φ_2 are non-empty and $(\alpha, \beta) = 0$ for all $\alpha \in \Phi_1$, $\beta \in \Phi_2$. Let $\{H^i\} \cup \{E^\alpha\}$ be a Cartan-Weyl basis of g. Show that

$$g_1 = \mathrm{span}_{\mathbb{C}}(H^\alpha | \alpha \in \Phi_1\} \oplus \bigoplus_{\alpha \in \Phi_1} \mathbb{C} E^\alpha$$

is a proper ideal in g.

The following theorem we will not prove. (For a proof see [Fulton,Harris] § 22.1.)

Theorem 5.15:
Two fssc Lie algebras g, g' are isomorphic iff there is an isomorphism ${g_0}^* \to {g'_0}^*$ of vector spaces that preserves $(\cdot, \cdot)$ and maps $\Phi(g, g_0)$ to $\Phi(g', g'_0)$.

Definition 5.16:
Let g be a fssc Lie algebra g with Cartan subalgebra g_0. The *root space* is the real span

$$R \equiv R(g, g_0) = \mathrm{span}_{\mathbb{R}}(\Phi(g, g_0)) \quad . \tag{5.33}$$

[The term *root spaces* is also used for the spaces g_α, so one has to be a bit careful.]

In particular, R is a *real* vector space.

Exercise 5.14:
Let R the root space of a fssc Lie algebra with Cartan subalgebra g_0.
(i) Show that the bilinear form $(\cdot, \cdot)$ on g_0^* restricts to a real valued *positive definite* inner product on R.
(ii) Use the Gram-Schmidt procedure to find an orthonormal basis $\{\varepsilon_1, \dots, \varepsilon_m\}$ of R (over $\mathbb{R}$). Show that $m = r$ (where $r = \dim(g_0)$ is the rank of g) and that $\{\varepsilon_1, \dots, \varepsilon_r\}$ is a basis of g_0^* (over $\mathbb{C}$).
(iii) Show that there exists a basis $\{H^i | i = 1, \dots, r\}$ of g_0 such that $\alpha(H^i) \in \mathbb{R}$ for all $i = 1, \dots, r$ and $\alpha \in \Phi$.

The basis $\{\varepsilon_1, \dots, \varepsilon_r\}$ provides an identification of R and $\mathbb{R}^r$, whereby the inner product $(\,\cdot\,,\,\cdot\,)$ on R becomes the usual inner product $g(x, y) = \sum_{i=1}^r x_i y_i$ on $\mathbb{R}^r$. [In other words, R and $\mathbb{R}^r$ are isomorphic as inner product spaces.]

5.4 Examples: $sl(2, \mathbb{C})$ and $sl(n, \mathbb{C})$

Recall the basis

$$H = \mathcal{E}_{11} - \mathcal{E}_{22} \quad , \quad E = \mathcal{E}_{12} \quad , \quad F = \mathcal{E}_{21} \tag{5.34}$$

of $sl(2, \mathbb{C})$, with Lie brackets

$$[E, F] = H \quad , \quad [H, E] = 2E \quad , \quad [H, F] = -2F \quad . \tag{5.35}$$

Define the linear forms ω_1 and ω_2 on diagonal 2×2-matrices as

$$\omega_i(\mathcal{E}_{jj}) = \delta_{ij} \ . \tag{5.36}$$

Fix the Cartan subalgebra $h = \mathbb{C}H$. Define $\alpha \equiv \alpha_{12} = \omega_1 - \omega_2 \in h^*$. Then α is a root since

$$\alpha(H) = (\omega_1 - \omega_2)(\mathcal{E}_{11} - \mathcal{E}_{22}) = 2 \quad \text{and} \quad [H, E] = 2E = \alpha(H)E \quad . \tag{5.37}$$

Let us now work out H^α. After a bit of staring one makes the ansatz

$$H^\alpha = \frac{1}{4}\big(\mathcal{E}_{11} - \mathcal{E}_{22}\big) \ . \tag{5.38}$$

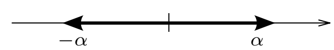
As h is one-dimensional, to verify this it is enough to check that $\kappa(H^\alpha, H) = \alpha(H)$. Recall from exercise 4.17 that $\kappa(x,y) = 4\mathrm{Tr}(xy)$. Thus

$$\kappa(H^\alpha, H) = 4\,\mathrm{Tr}(H^\alpha H) = \mathrm{Tr}\big((\mathcal{E}_{11} - \mathcal{E}_{22})(\mathcal{E}_{11} - \mathcal{E}_{22})\big) = 2 = \alpha(H) \ . \tag{5.39}$$

In the same way one gets $(\alpha,\alpha) = \kappa(H^\alpha, H^\alpha) = \frac{1}{2}$. Finally, note that

$$[E,F] = H = \frac{2}{(\alpha,\alpha)} H^\alpha \ . \tag{5.40}$$

Altogether this shows that $\{H, E^\alpha \equiv E, E^{-\alpha} \equiv F\}$ already is a Cartan-Weyl basis of $sl(2,\mathbb{C})$. We can draw the following picture,

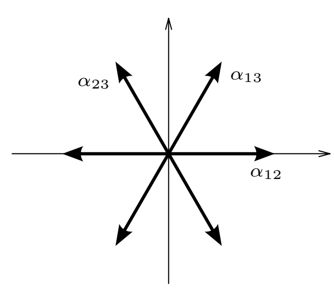


Such a picture is called a *root diagram*. The real axis is identified with the root space R, and the root α has length $1/\sqrt{2}$.

Exercise 5.15:
In this exercise we construct a Cartan-Weyl basis for $sl(n,\mathbb{C})$. As Cartan subalgebra h we take the trace-less diagonal matrices.
(i) Define the linear forms ω_i, $i = 1,\dots,n$ on diagonal $n{\times}n$-matrices as $\omega_i(\mathcal{E}_{jj}) = \delta_{ij}$. Define $\alpha_{kl} = \omega_k - \omega_l$. Show that for $k \neq l$, α_{kl} is a root.
Hint: Write a general element $H \in h$ as $H = \sum_{k=1}^n a_k \mathcal{E}_{kk}$ with $\sum_{k=1}^n a_k = 0$. Show that $[H, \mathcal{E}_{kl}] = \alpha_{kl}(H)\mathcal{E}_{kl}$.
(ii) Show that $H^{\alpha_{kl}} = \frac{1}{2n}(\mathcal{E}_{kk} - \mathcal{E}_{ll})$.
Hint: Use exercise 4.18 to verify $\kappa(H^{\alpha_{kl}}, H) = \alpha_{kl}(H)$ for all $H \in h$.
(iii) Show that $(\alpha_{kl},\alpha_{kl}) = 1/n$ and that $[\mathcal{E}_{kl}, \mathcal{E}_{lk}] = 2/(\alpha_{kl},\alpha_{kl}) \cdot H^{\alpha_{kl}}$
(iv) Show that, with $\Phi = \{\alpha_{kl}\,|\,k,l = 1,\dots,n\,,\ k \neq l\}$ and $E^{\alpha_{kl}} = \mathcal{E}_{kl}$,

$$\big\{H^{\alpha_{k,k+1}} \,\big|\, k = 1,\dots,n{-}1\big\} \cup \big\{E^\alpha \,\big|\, \alpha \in \Phi\big\}$$

is a Cartan-Weyl basis of $sl(n,\mathbb{C})$.
(v) Show that the root diagram of $sl(3,\mathbb{C})$ is



where each arrow has length $1/\sqrt{3}$ and the angle between the arrows is $60°$.

5.5 The Weyl group

Let g be a fssc Lie algebra with Cartan subalgebra g_0. For each $\alpha \in \Phi(g, g_0)$, define a linear map

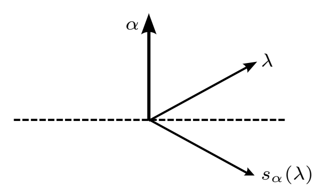
$$s_\alpha : g_0^* \longrightarrow g_0^* \quad , \quad s_\alpha(\lambda) = \lambda - 2\frac{(\alpha,\lambda)}{(\alpha,\alpha)}\alpha \quad . \tag{5.41}$$

The s_α, $\alpha \in \Phi$ are called *Weyl reflections*.

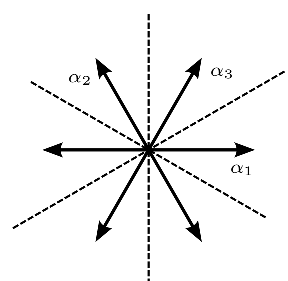
Exercise 5.16:
Let s_α be a Weyl reflection. Show that

$$s_\alpha(\alpha) = -\alpha \quad , \quad (\alpha, \lambda) = 0 \Rightarrow s_\alpha(\lambda) = \lambda \quad , \quad s_\alpha \circ s_\alpha = \mathrm{id} \quad , \quad s_{-\alpha} = s_\alpha \quad .$$

Thus s_α is indeed a reflection, we have the following picture,



And in $sl(3, \mathbb{C})$ we find



So in this example, it seems that roots get mapped to roots under Weyl reflections. This is true in general.

Theorem 5.17:
Let g be a fssc Lie algebra with Cartan subalgebra g_0. If $\alpha, \beta \in \Phi$, then also $s_\alpha(\beta) \in \Phi$.

Proof:
■ Let $\alpha \in \Phi$. Recall the injective homomorphism of Lie algebras $\varphi \equiv \varphi_\alpha : sl(2, \mathbb{C}) \to g$,

$$\varphi(H) = \tfrac{2}{(\alpha,\alpha)} H^\alpha \quad , \quad \varphi(E) = E^\alpha \quad , \quad \varphi(F) = E^{-\alpha} \quad . \tag{5.42}$$

This turns g into a representation (g, R_φ) of $sl(2,\mathbb{C})$. For a root $\beta \in \Phi$ have

$$R_\varphi(H)E^\beta = \tfrac{2}{(\alpha,\alpha)}[H^\alpha, E^\beta] = \tfrac{2(\alpha,\beta)}{(\alpha,\alpha)}E^\beta = mE^\beta \tag{5.43}$$

for some integer m. We may assume $m \geq 0$ (otherwise replace $\alpha \to -\alpha$ and start again). The representation theory of $sl(2,\mathbb{C})$ tells us that then also $-m$ has to be an eigenvalue of $R_\varphi(H)$ with eigenvector

$$v = (R_\varphi(F))^m E^\beta \neq 0 \quad . \tag{5.44}$$

But

$$v = (R_\varphi(F))^m E^\beta = [E^{-\alpha}, [\dots, [E^{-\alpha}, E^\beta]\dots] \in g_{\beta - m\alpha} \quad . \tag{5.45}$$

Since $v \neq 0$ have $g_{\beta-m\alpha} \neq \{0\}$ so that $\beta - m\alpha \in \Phi$.
■ Now evaluate the Weyl reflection

$$s_\alpha(\beta) = \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha = \beta - m\alpha \quad , \tag{5.46}$$

which we have shown to be a root. Thus $\alpha, \beta \in \Phi$ implies that $s_\alpha(\beta) \in \Phi$. □

Definition 5.18:
Let g be a fssc Lie algebra with Cartan subalgebra g_0. The *Weyl group* W of g is the subgroup of $GL(g_0^*)$ generated by the Weyl reflections,

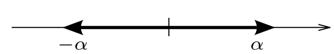
$$W = \{s_{\beta_1} \cdots s_{\beta_m} \,|\, \beta_1, \dots, \beta_m \in \Phi, m = 0, 1, 2, \dots\} \quad . \tag{5.47}$$

Exercise 5.17:
Show that the Weyl group of a fssc Lie algebra is a finite group (i.e. contains only a finite number of elements).

5.6 Simple Lie algebras of rank 1 and 2

Let g be a fssc Lie algebra and let $R = \text{span}_\mathbb{R}(\Phi)$ be the root space. Recall that on R, $(\cdot,\cdot)$ is a positive definite inner product.

Suppose g has rank 1, i.e. $R = \text{span}_\mathbb{R}(\Phi)$ is one-dimensional. By theorem 5.10, if $\alpha \in \Phi$ and $\lambda\alpha \in \Phi$, then $\lambda \in \{\pm 1\}$. Hence the root system has to be



This is the root system of $sl(2,\mathbb{C})$. Thus by theorem 5.15 any fssc Lie algebra of rank 1 is isomorphic to $sl(2,\mathbb{C})$. This Lie algebra is also called A_1.

To proceed we need to have a closer look at the inner product of roots. By the *Cauchy-Schwartz inequality*,

$$\text{for all} \;\; u, v \in R \;, \quad (u,v)^2 \leq (u,u)(v,v) \quad . \tag{5.48}$$

Also, $(u,v)^2 = (u,u)(v,v)$ iff u and v are colinear.

For two roots $\alpha, \beta \in \Phi$, $\beta \neq \pm\alpha$, this means $(\alpha,\beta)^2 < (\alpha,\alpha)(\beta,\beta)$, i.e.

$$p \cdot q < 4 \quad \text{where} \quad p = 2\tfrac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z} \;\; \text{and} \;\; q = 2\tfrac{(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z} \quad . \tag{5.49}$$

The angle between two roots α and β is

$$\cos(\theta)^2 = \frac{(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)} = \frac{pq}{4} \quad . \tag{5.50}$$

If $(\alpha,\beta) \neq 0$ we can also compute the ratio of length between the two roots,

$$\frac{(\beta,\beta)}{(\alpha,\alpha)} = \frac{p}{q} \quad . \tag{5.51}$$

If $(\alpha,\beta) = 0$ then $p = q = 0$ and we obtain no condition for the length ratio.
Now suppose

- $(\alpha,\beta) \neq 0$ (i.e. α and β are not orthogonal)
- $\beta \neq \pm\alpha$ (i.e. α and β are not colinear)

Then we can in addition assume

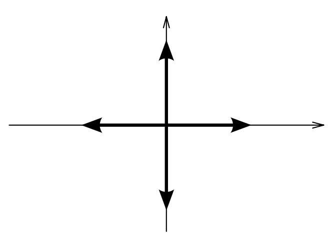
- $(\alpha,\beta) > 0$ (otherwise replace $\alpha \to -\alpha$)
- $(\beta,\beta) \geq (\alpha,\alpha)$ (otherwise exchange $\alpha \leftrightarrow \beta$)

Then $p \geq q > 0$. Altogether, the only allowed pairs (p,q) are

p	q	$(\cos\theta)^2 = pq/4$	$\frac{(\beta,\beta)}{(\alpha,\alpha)} = p/q$
3	1	$\frac{3}{4}$ ($\theta = \pm 30°$)	3
2	1	$\frac{1}{2}$ ($\theta = \pm 45°$)	2
1	1	$\frac{1}{4}$ ($\theta = \pm 60°$)	1
0	0	0 ($\theta = \pm 90°$)	no cond.

Consider now a Lie algebra of rank 2. Let θ_m be the *smallest* angle between two distinct roots that occurs. The following are all possible root systems:

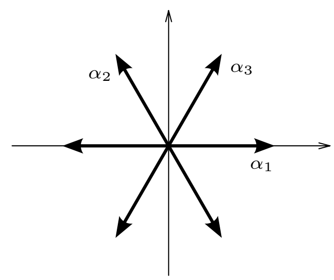
- $\theta_m = 90°$. The root system is



By lemma 5.14 this Lie algebra is not simple. It has to be the direct sum of two rank 1 Lie algebras. Up to isomorphism, there is only one such algebra, hence

$$g = sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C}) \quad . \tag{5.52}$$

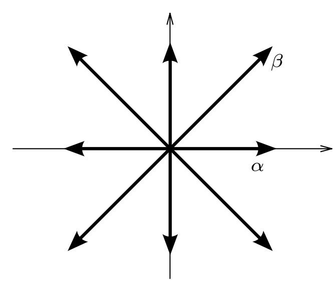
■ $\theta_m = 60°$. Let α_1, α_3 be two roots with this angle. Then by the above table, α_1 and α_3 have the same length,



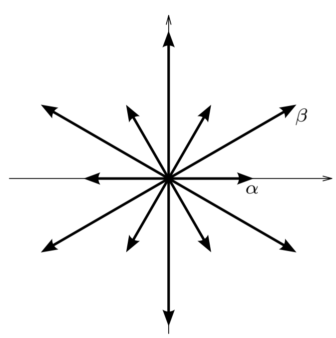
The root α_2 has been obtained by completing this picture with respect to Weyl reflections. Also for each root α, a root $-\alpha$ has been added. There can be no further roots, or one would have a minimum angle less than $60°$. Thus

$$g = sl(3, \mathbb{C}) \quad . \tag{5.53}$$

This Lie algebra is also called A_2.
■ $\theta_m = 45°$. Let α, β be two roots with this angle. Then by the above table, $(\beta, \beta) = 2(\alpha, \alpha)$,



Again, the root system has been completed with respect to Weyl reflections and $\alpha \to -\alpha$. This Lie algebra is called B_2. [It is a complexification of $so(5)$, see section 5.8.]
■ $\theta_m = 30°$. Let α, β be two roots with this angle. Then by the above table, $(\beta, \beta) = 3(\alpha, \alpha)$, and the roots system (completed with respect to Weyl reflections and $\alpha \to -\alpha$) is



This Lie algebra is called G_2. [See [Fulton-Harris, chapter 22] for more on G_2.]

Exercise 5.18 :
Give the dimension (over $\mathbb{C}$) of all rank two fssc Lie algebras as found in section 5.6.

5.7 Dynkin diagrams

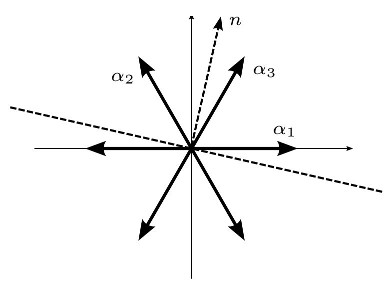
Let g be a fssc Lie algebra and $R = \mathrm{span}_{\mathbb{R}}(\Phi)$. Pick a vector $n \in R$ such that the hyperplane

$$H = \{v \in R|(v,n) = 0\} \tag{5.54}$$

does not contain an element of Φ (i.e. $H \cap \Phi = \emptyset$). Define
- *positive roots* $\Phi_+ = \{\alpha \in \Phi|(\alpha,n) > 0\}$
- *negative roots* $\Phi_- = \{\alpha \in \Phi|(\alpha,n) < 0\}$

Exercise 5.19 :
Let g be a fssc Lie algebra and let Φ_+ and Φ_- be the positive and negative roots with respect to some hyperplane. Show that
(i) $\Phi = \Phi_+ \cup \Phi_-$.
(ii) $\alpha \in \Phi_+ \Leftrightarrow -\alpha \in \Phi_-$ and $|\Phi_+| = |\Phi_-|$ (the number of elements in a set S is denoted by $|S|$).
(iii) $\mathrm{span}_{\mathbb{R}}(\Phi_+) = \mathrm{span}_{\mathbb{R}}(\Phi)$.

For example, for $sl(3,\mathbb{C})$ we can take



(5.55)

Let Φ_s be all elements of Φ_+ that *cannot* be written as a linear combination of elements of Φ_+ with *positive coefficients* and at least two terms [this is to exclude the trivial linear combination $\alpha = 1 \cdot \alpha$]. The roots $\Phi_s \subset \Phi_+$ are called *simple roots*. For example, for $sl(3,\mathbb{C})$ (with the choice of n as in (5.55)) we get $\Phi_s = \{\alpha_1, \alpha_2\}$.

Properties of simple roots (which we will not prove):
- Φ_s is a basis of R. [It is easy to see that $\mathrm{span}_{\mathbb{R}}(\Phi_s) = R$, linear independence not so obvious.]
- Let $\Phi_s = \{\alpha^{(1)}, \ldots, \alpha^{(r)}\}$. If $i \neq j$ then $(\alpha^{(i)}, \alpha^{(j)}) \leq 0$. [Also not so obvious.]

Definition 5.19:
Let g be a fssc Lie algebra and let $\Phi_s \subset \Phi$ be a choice of simple roots. The *Cartan matrix* is the $r \times r$ matrix with entries

$$A^{ij} = \frac{2(\alpha^{(i)}, \alpha^{(j)})}{(\alpha^{(j)}, \alpha^{(j)})} \quad . \tag{5.56}$$

Exercise 5.20:
Using the properties of simple roots stated in the lecture, prove the following properties of the Cartan matrix.
(i) $A^{ij} \in \mathbb{Z}$.
(ii) $A^{ii} = 2$ and $A^{ij} \leq 0$ if $i \neq j$.
(iii) $A^{ij} A^{ji} \in \{0, 1, 2, 3\}$ if $i \neq j$.

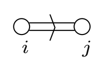
For $sl(3, \mathbb{C})$ we can choose $\Phi_s = \{\alpha^1, \alpha^2\}$. The off-diagonal entries of the Cartan matrix are

$$A^{12} = \frac{2(\alpha^1, \alpha^2)}{(\alpha^2, \alpha^2)} = \frac{2 \cdot (-1)}{2} = -1 = A^{21} \quad . \tag{5.57}$$

Thus the Cartan matrix of $sl(3, \mathbb{C})$ is

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad . \tag{5.58}$$

A *Dynkin diagram* is a pictorial representation of a Cartan matrix obtained as follows:
■ Draw dots (called vertices) labelled $1, \dots, r$.
■ For $i \neq j$ draw $A^{ij} A^{ji}$ lines between the vertices i and j.
■ If $|A^{ij}| > |A^{ji}|$ draw an arrowhead '>' on the lines between i and j pointing from i to j.
■ Remove the labels $1, \dots, r$.

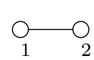
Notes:
(1) If there is an arrow from node i to node j 

then

$$\frac{(\alpha^{(i)}, \alpha^{(i)})}{(\alpha^{(j)}, \alpha^{(j)})} = \frac{|A^{ij}|}{|A^{ji}|} > 1 \quad , \tag{5.59}$$

i.e. the root $\alpha^{(i)}$ is longer than the root $\alpha^{(j)}$.
(2) When giving a Dynkin diagram, we will often also include a labelling of the vertices. However, this is not part of the definition of a Dynkin diagram. Instead, it constitutes an additional choice, namely a choice of numbering of the simple roots.

For $sl(3, \mathbb{C})$ we get the Dynkin diagram (together with a choice of labelling for the vertices, which is not part of the Dynkin diagram)



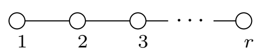
(5.60)

Exercise 5.21:
A Dynkin diagram is *connected* if for any two vertices $i \neq j$ there is a sequence of nodes $k_1, \dots, k_m$ with $k_1 = i$, $k_m = j$ such that $A^{k_s,k_{s+1}} \neq 0$ for $s = 1, \dots, m-1$. [In words: one can go from any vertex to any other by walking only along lines.] Show that if the Dynkin diagram of a fssc Lie algebra is connected, then the Lie algebra is simple. [The converse follows from the classification theorem of Killing and Cartan, which shows explicitly that simple Lie algebras have connected Dynkin diagrams.]

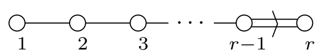
The following two theorems we will not prove [but at least we can understand their contents]. (For a proof see [Fulton,Harris] § 21.2 and § 21.3.)

Theorem 5.20:
Two fssc Lie algebras g, g' are isomorphic iff they have the same Dynkin diagram.

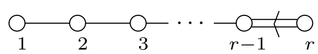
Theorem 5.21:
(Killing, Cartan) Let g be a simple finite-dimensional complex Lie algebra. The Dynkin diagram of g is one of the following:

$A_r =$ 

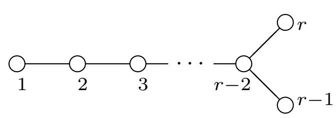
for $r \geq 1$

$B_r =$ 

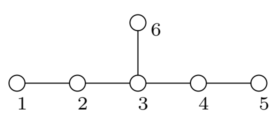
for $r \geq 2$

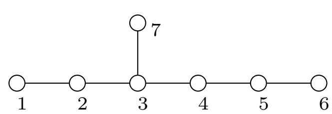
$C_r =$ 

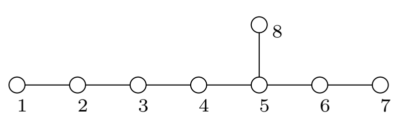
for $r \geq 3$

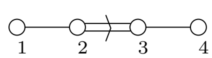
$D_r =$ 

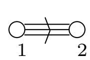
for $r \geq 4$

$E_6 =$ 

$E_7 =$ 

$E_8 =$ 

$F_4 =$ 

$G_2 =$ 

(The names of the Dynkin diagrams above are also used to denote the corresponding Lie algebras. The choice for the labelling of vertices made in the list above is the same as e.g. in [Fuchs, Schweigert, Table IV].)

Exercise 5.22 :
Compute the Dynkin diagrams of all rank two fssc Lie algebras using the root diagrams obtained in section 5.6.

Exercise 5.23 :
The Dynkin diagram (together with a choice for the numbering of the vertices) determines the Cartan matrix uniquely. Write out the Cartan matrix for the Lie algebras A_4, B_4, C_4, D_4 and F_4.

5.8 Complexification of real Lie algebras

In sections 3.3–3.6 we studied the Lie algebras of matrix Lie groups. Those were defined to be *real* Lie algebras. In this section we will make the connection to the complex Lie algebras studied chapter 5.

Definition 5.22 :
Let V be a real vector space. The *complexification* $V_{\mathbb{C}}$ of V is defined as the quotient

$$\mathrm{span}_{\mathbb{C}}((\lambda, v)|\lambda \in \mathbb{C}, v \in V)/W \tag{5.61}$$

where W is the vector space spanned (over $\mathbb{C}$) by the vectors

$$(\lambda, r_1 v_1 + r_2 v_2) - \lambda r_1(1, v_1) - \lambda r_2(1, v_2) \tag{5.62}$$

for all $\lambda \in \mathbb{C}$, $r_1, r_2 \in \mathbb{R}$, $v_1, v_2 \in V$. [This is nothing but to say that $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$.]

Elements of $V_{\mathbb{C}}$ are equivalence classes. The equivalence class containing the pair (λ, v) will be denoted $(\lambda, v) + W$, as usual. $V_{\mathbb{C}}$ is a complex vector space. All elements of $V_{\mathbb{C}}$ are complex linear combinations of elements of the form $(\lambda, v) + W$. But

$$(\lambda, v) + W = \lambda(1, v) + W \quad , \tag{5.63}$$

so that all elements of $V_{\mathbb{C}}$ are linear combinations of elements of the form $(1, v) + W$. We will use the shorthand notation $v \equiv (1, v) + W$.

Exercise* 5.24 :
Let V be a real vector space. Show that every $v \in V_{\mathbb{C}}$ can be *uniquely* written as $v = (1, a) + i(1, b) + W$, with $a, b \in V$, i.e., using the shorthand notation, $v = a + ib$.

Exercise 5.25:
Let V be a finite-dimensional, real vector space and let $\{v_1, \dots, v_n\}$ be a basis of V. Show that $\{(1, v_1) + W, \dots, (1, v_n) + W\}$ is a basis of $V_{\mathbb{C}}$. (You may want to use the result of exercise 5.24.)

Remark 5.23:
The abbreviation $v \equiv (1, v) + W$ has to be used with some care. Consider the complex numbers $\mathbb{C}$ as two-dimensional real vectors space. Every element of $\mathbb{C}$ can be written uniquely as $a+ib$ with $a, b \in \mathbb{R}$. Thus a basis of $\mathbb{C}$ (over $\mathbb{R}$) is given by $e_1 = 1$ and $e_2 = i$. The complexification $\mathbb{C}_{\mathbb{C}}$ therefore has the basis $\{e_1, e_2\}$ (over $\mathbb{C}$). In particular, in $\mathbb{C}_{\mathbb{C}}$ we have $ie_1 \neq e_2$ (or e_1 and e_2 are not linearly independent). The shorthand notation might suggest that $ie_1 = i(1) = (i) = e_2$, but this is *not true*, as in full notation

$$ie_1 = i\big((1, 1) + W\big) = (i, 1) + W \neq (1, i) + W = e_2 \quad . \tag{5.64}$$

Definition 5.24:
Let h be a real Lie algebra.
(i) The *complexification* $h_{\mathbb{C}}$ of h is the complex vector space $h_{\mathbb{C}}$ together with the Lie bracket

$$[\lambda x, \mu y] = \lambda\mu \cdot [x, y] \quad \text{for all} \quad \lambda, \mu \in \mathbb{C} \ , \quad x, y \in h \quad . \tag{5.65}$$

(ii) Let g be a complex Lie algebra. h is called a *real form* of g iff $h_{\mathbb{C}} \cong g$ as complex Lie algebras.

Exercise 5.26:
Let h be a finite-dimensional real Lie algebra and let g be a finite-dimensional complex Lie algebra. Show that the following are equivalent.
(1) h is a real form of g.
(2) There exist bases $\{T^a | a = 1, \dots, n\}$ of h (over $\mathbb{R}$) and $\{\tilde{T}^a | a = 1, \dots, n\}$ of g (over $\mathbb{C}$) such that

$$[T^a, T^b] = \sum_{c=1}^{n} f^{ab}_{\ \ c}\, T^c \quad \text{and} \quad [\tilde{T}^a, \tilde{T}^b] = \sum_{c=1}^{n} f^{ab}_{\ \ c}\, \tilde{T}^c$$

with the same structure constants $f^{ab}_{\ \ c}$.

The following is an instructive example.

Lemma 5.25:
(i) $su(2)$ is a real form of $sl(2, \mathbb{C})$.
(ii) $sl(2, \mathbb{R})$ is a real form of $sl(2, \mathbb{C})$.
(iii) $su(2)$ and $sl(2, \mathbb{R})$ are not isomorphic as real Lie algebras.

Proof:
(i) Recall that

$$\begin{aligned} su(2) &= \{M \in \mathrm{Mat}(2,\mathbb{C}) | M + M^\dagger = 0, \mathrm{tr}(M) = 0\} \quad , \\ sl(2,\mathbb{C}) &= \{M \in \mathrm{Mat}(2,\mathbb{C}) | \mathrm{tr}(M) = 0\} \quad . \end{aligned} \tag{5.66}$$

In both cases the Lie bracket is given by the matrix commutator. A basis of $su(2)$ (over $\mathbb{R}$) is

$$T^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad , \quad T^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad T^3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad . \tag{5.67}$$

By exercise 5.25, the vectors T^a also provide a basis (over $\mathbb{C}$) of $su(2)_\mathbb{C}$. A basis of $sl(2,\mathbb{C})$ (over $\mathbb{C}$) is

$$\tilde{T}^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad , \quad \tilde{T}^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad \tilde{T}^3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad . \tag{5.68}$$

Since these are the the same matrices, their matrix commutator agrees, and hence $\varphi : su(2)_\mathbb{C} \to sl(2,\mathbb{C})$, $\varphi(T^a) = \tilde{T}^a$ is an isomorphism of complex Lie algebras.
(ii) The proof goes in the same way as part (i). Recall that

$$sl(2,\mathbb{R}) = \{M \in \mathrm{Mat}(2,\mathbb{R}) | \mathrm{tr}(M) = 0\} \quad . \tag{5.69}$$

A basis of $sl(2,\mathbb{R})$ (over $\mathbb{R}$) is

$$T^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad T^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad T^3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad . \tag{5.70}$$

A basis of $sl(2,\mathbb{C})$ (over $\mathbb{C}$) is

$$\tilde{T}^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \tilde{T}^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad \tilde{T}^3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad . \tag{5.71}$$

As before, $\varphi : sl(2,\mathbb{R})_\mathbb{C} \to sl(2,\mathbb{C})$, $\varphi(T^a) = \tilde{T}^a$ an isomorphism of complex Lie algebras.
(iii) This is shown in the next exercise. $\square$

Exercise 5.27:
Show that the Killing form of $su(2)$ is negative definite (i.e. $\kappa(x,x) < 0$ for all $x \in su(2)$) and that the one of $sl(2,\mathbb{R})$ is not. Conclude that there exists no isomorphism $\varphi : sl(2,\mathbb{R}) \to su(2)$ of real Lie algebras.

Here is a list of real Lie algebras whose complexifications give the simple complex Lie algebras A_r, B_r, C_r and D_r. As we have seen, several real Lie algebras can have the same complexification, so the list below is only one possible choice.

real Lie algebra h	$su(r{+}1)$	$so(2r{+}1)$	$sp(2r)$	$so(2r)$
complex Lie algebra $g \cong h_{\mathbb{C}}$	A_r	B_r	C_r	D_r

Lemma 5.26 :
Let g be a finite-dimensional complex Lie algebra and let h be a real form of g. Then κ_g is non-degenerate iff κ_h is non-degenerate.

Proof:
Let T^a be a basis of h and $\tilde{T}^a$ be a basis of g such that

$$[T^a, T^b] = \sum_{c=1}^{n} f^{ab}_{\ \ c} T^c \quad \text{and} \quad [\tilde{T}^a, \tilde{T}^b] = \sum_{c=1}^{n} f^{ab}_{\ \ c} \tilde{T}^c \quad . \tag{5.72}$$

Then

$$\begin{aligned} \kappa_h(T^a, T^b) &= \sum_c (T^c)^*([T^a,[T^b,T^c]]) = \sum_{c,d} f^{bc}_{\ \ d} f^{ad}_{\ \ c} \\ &= \sum_c (\tilde{T}^c)^*([\tilde{T}^a,[\tilde{T}^b,\tilde{T}^c]]) = \kappa_g(\tilde{T}^a, \tilde{T}^b) \quad . \end{aligned} \tag{5.73}$$

This shows that in the bases we have chosen, the matrix elements of κ_h and κ_g agree. In particular, the statement that κ_h is non-degenerate is equivalent to the statement that κ_g is non-degenerate. $\square$

Exercise 5.28 :
Show that $o(1,n{-}1)_{\mathbb{C}} \cong so(n)_{\mathbb{C}}$ as complex Lie algebras.

Recall that the Lie algebra of the four-dimensional Lorentz group is $o(1,3)$. The exercise shows in particular, that $o(1,3)$ has the same complexification as $so(4)$. Since the Killing form of $so(4)$ is non-degenerate, we know that $so(4)_{\mathbb{C}}$ is semi-simple.

Lemma 5.27 :
$so(4) \cong su(2) \oplus su(2)$ as real Lie algebras.

Proof:
Consider the following basis for $so(4)$,

$$\begin{aligned} &X_1 = \mathcal{E}_{23} - \mathcal{E}_{32} \quad , \qquad X_2 = \mathcal{E}_{31} - \mathcal{E}_{13} \quad , \qquad X_3 = \mathcal{E}_{12} - \mathcal{E}_{21} \quad , \\ &Y_1 = \mathcal{E}_{14} - \mathcal{E}_{41} \quad , \qquad Y_2 = \mathcal{E}_{24} - \mathcal{E}_{42} \quad , \qquad Y_3 = \mathcal{E}_{34} - \mathcal{E}_{43} \quad , \end{aligned} \tag{5.74}$$

i.e. in short

$$X_a = \sum_{b,c=1}^{3} \varepsilon_{abc} \mathcal{E}_{bc} \quad , \qquad Y_a = \mathcal{E}_{a4} - \mathcal{E}_{4a} \quad , \qquad \text{for} \;\; a = 1,2,3 \quad . \tag{5.75}$$

The X_1, X_2, X_3 are just the basis of the $so(3)$ subalgebra of $so(4)$ obtained by considering only the upper left 3×3-block. Their commutator has been computed in exercise 3.9,

$$[X_a, X_b] = -\sum_{c=1}^{3} \varepsilon_{abc} X_c \quad . \tag{5.76}$$

To obtain the remaining commutators we first compute

$$\begin{aligned} X_a Y_b &= \sum_{c,d=1}^{3} \varepsilon_{acd} \mathcal{E}_{cd} (\mathcal{E}_{b4} - \mathcal{E}_{4b}) = \sum_{c=1}^{3} \varepsilon_{acb} \mathcal{E}_{c4} \quad , \\ Y_b X_a &= \sum_{c,d=1}^{3} \varepsilon_{acd} (\mathcal{E}_{b4} - \mathcal{E}_{4b}) \mathcal{E}_{cd} = \sum_{d=1}^{3} \varepsilon_{abd} \mathcal{E}_{4d} \quad , \\ Y_a Y_b &= (\mathcal{E}_{a4} - \mathcal{E}_{4a})(\mathcal{E}_{b4} - \mathcal{E}_{4b}) = -\mathcal{E}_{ab} - \delta_{ab} \mathcal{E}_{44} \quad , \end{aligned} \tag{5.77}$$

and from this, and exercise 3.8(ii), we get

$$\begin{aligned} [X_a, Y_b] &= \sum_{c=1}^{3} \varepsilon_{abc} (-\mathcal{E}_{c4} + \mathcal{E}_{4c}) = -\sum_{c=1}^{3} \varepsilon_{abc} Y_c \quad , \\ [Y_a, Y_b] &= -\mathcal{E}_{ab} + \mathcal{E}_{ba} = -\sum_{c,d=1}^{3} (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) \mathcal{E}_{cd} \\ &= -\sum_{c,d,x=1}^{3} \varepsilon_{abx} \varepsilon_{xcd} \mathcal{E}_{cd} = -\sum_{x=1}^{3} \varepsilon_{abx} X_x \quad . \end{aligned} \tag{5.78}$$

Set now

$$J_a^+ = \tfrac{1}{2}(X_a + Y_a) \qquad , \qquad J_a^- = \tfrac{1}{2}(X_a - Y_a) \quad . \tag{5.79}$$

Then

$$\begin{aligned} [J_a^+, J_b^+] &= \tfrac{1}{4}\big([X_a, X_b] + [X_a, Y_b] + [Y_a, X_b] + [Y_a, Y_b]\big) \\ &= -\tfrac{1}{4} \sum_{c=1}^{3} \varepsilon_{abc} (X_c + Y_c + Y_c + X_c) = -\sum_{c=1}^{3} \varepsilon_{abc} J_c^+ \quad , \\ [J_a^+, J_b^-] &= \cdots = 0 \quad , \\ [J_a^-, J_b^-] &= \cdots = -\sum_{c=1}^{3} \varepsilon_{abc} J_c^- \quad . \end{aligned} \tag{5.80}$$

We see that we get two commuting copies of $so(3)$, i.e. we have shown (note that we have only used real coefficients)

$$so(4) \cong so(3) \oplus so(3) \tag{5.81}$$

as real Lie algebras. Together with $so(3) \cong su(2)$ (as real Lie algebras) this implies the claim. $\square$

Since $su(2)_{\mathbb{C}} \cong sl(2,\mathbb{C})$ (as complex Lie algebras), complexification immediately yields the identity

$$so(4)_{\mathbb{C}} \cong sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C}) \quad . \tag{5.82}$$

It follows that for the complexification of the Lorentz algebra $o(1,3)$ we equally get $o(1,3)_{\mathbb{C}} \cong sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C})$.

Information 5.28:
This isomorphism is used in theoretical physics (in particular in the contexts of relativistic quantum field theory and of supersymmetry) to describe representations of the Lorentz algebra $o(1,3)$ in terms of representations of $sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C})$. Irreducible representations V_d of $sl(2,\mathbb{C})$ are labelled by their dimension d. It is also customary to denote representations of $sl(2,\mathbb{C})$ by their 'spin' (this comes from $sl(2,\mathbb{C}) \cong su(2)_{\mathbb{C}}$); V_d then has spin $s = (d-1)/2$, i.e. $V_1, V_2, V_3, \dots$ have spin $0, \frac{1}{2}, 1, \dots$. Representations of $o(1,3)_{\mathbb{C}}$ are then labelled by a pair of spins (s_1, s_2).

6 Epilogue

A natural question for a mathematician would be "*What are all Lie groups? What are their representations?*". A physicist would ask the same question, but would use different words: "*What continuous symmetries can occur in nature? How do they act on the space of quantum states?*"

In this course we have answered neither of these questions, but we certainly went into the good direction.

- In the beginning we have studied matrix Lie groups. They are typically defined by non-linear equations, and it is easier to work with a 'linearised version', which is provided by a Lie algebra. To this end we have seen that

  a (matrix) Lie group G gives rise to a real Lie algebra g.

  We have not shown, but it is nonetheless true, that a representation of a Lie group also gives rise to a representation of the corresponding real Lie algebra.

- So before addressing the question "What are all Lie groups?" we can try to answer the simpler question "What are all real Lie algebras?" However, it turns out that it is much simpler to work with complex Lie algebras than with real Lie algebras. To obtain a complex Lie algebra we used that

  every real Lie algebra g gives rise to a complex Lie algebra $g_{\mathbb{C}}$.

  For representations one finds (but we did not) that a (real) representation of a real Lie algebra gives rise to a (complex) representation of its complexification.

- To classify all complex Lie algebras is still to hard a problem. But if one demands two additional properties, namely that the complex Lie algebra is finite-dimensional and that its Killing form is non-degenerate (these were precisely the fssc Lie algebras), then a complete classification can be achieved.

  All finite-dimensional simple complex Lie algebras are classified in the Theorem of Killing and Cartan.

  It turns out (but we did not treat this in the course) that one can equally classify all finite-dimensional representations of fssc Lie algebras.

One can now wonder what the classification result, i.e. the answer to "*What are all finite-dimensional simple complex Lie algebras?*", has to do with the original question "*What are all Lie groups?*". It turns out that one can retrace one's steps and arrive instead at an answer for the question "*What are all compact connected simple Lie groups?*" (A group is called *simple* if has no normal subgroups other than $\{e\}$ and itself, and if it is not itself the trivial group. A connected Lie group is called simple if it does not contain connected normal subgroups other than $\{e\}$ and itself.) A similar route can be taken to obtain the finite-dimensional representations of a compact simple Lie group.

A Appendix: Collected exercises

Exercise 0.1:
I certainly did not manage to remove all errors from this script. So the first exercise is to find all errors and tell them to me.

Exercise 1.1:
Show that for a real or complex vector space V, a bilinear map $b(\cdot,\cdot) : V\times V \to V$ obeys $b(u,v) = -b(v,u)$ (for all u,v) if and only if $b(u,u) = 0$ (for all u). [If you want to know, the formulation $[X,X] = 0$ in the definition of a Lie algebra is preferable because it also works for the field $\mathbb{F}_2$. There, the above equivalence is not true because in $\mathbb{F}_2$ we have $1+1=0$.]

Exercise 2.1:
Prove the following consequences of the group axioms: The unit is unique. The inverse is unique. The map $x \mapsto x^{-1}$ is invertible as a map from G to G. $e^{-1} = e$. If $gg = g$ for some $g \in G$, then $g = e$. The set of integers together with addition $(\mathbb{Z}, +)$ forms a group. The set of integers together with multiplication $(\mathbb{Z}, \cdot)$ does not form a group.

Exercise 2.2:
Verify the group axioms for $GL(n,\mathbb{R})$. Show that $\mathrm{Mat}(n,\mathbb{R})$ (with matrix multiplication) is not a group.

Exercise 2.3:
Let $\varphi : G \to H$ be a group homomorphism. Show that $\varphi(e) = e$ (the units in G and H, respectively), and that $\varphi(g^{-1}) = \varphi(g)^{-1}$.

Exercise 2.4:
Show that $\mathrm{Aut}(G)$ is a group.

Exercise 2.5:
(i) Show that a subgroup $H \leq G$ is in particular a group, and show that it has the same unit element as G.
(ii) Show that $SO(n)$ is a subgroup of $GL(n,\mathbb{R})$.

Exercise 2.6:
Prove that
(i*) for every $f \in E(n)$ there is a unique $T \in O(n)$ and $u \in \mathbb{R}^n$, s.t. $f(v) = Tv+u$ for all $v \in \mathbb{R}^n$.
(ii) for $T \in O(n)$ and $u \in \mathbb{R}^n$ the map $v \mapsto Tv + u$ is in $E(n)$.

Exercise 2.7:
(i) Starting from the definition of the semidirect product, show that $H \ltimes_\varphi N$ is indeed a group. [To see why the notation H and N is used for the two groups, look up “semidirect product” on wikipedia.org or eom.springer.de.]

(ii) Show that the direct product is a special case of the semidirect product.
(iii) Show that the multiplication rule $(T,x)\cdot(R,y) = (TR, Ty+x)$ found in the study of $E(n)$ is that of the semidirect product $O(n) \ltimes_\varphi \mathbb{R}^n$, with $\varphi : O(n) \to \mathrm{Aut}(\mathbb{R}^n)$ given by $\varphi_T(u) = Tu$.

Exercise 2.8:
Show that $O(1,n-1)$ can equivalently be written as

$$O(1,n-1) = \{M \in GL(n,\mathbb{R}) | M^t J M = J\}$$

where J is the diaognal matrix with entries $J = \mathrm{diag}(1,-1,\dots,-1)$.

Exercise 2.9:
(i*) Prove that for every $f \in P(1,n-1)$ there is a unique $\Lambda \in O(1,n-1)$ and $u \in \mathbb{R}^n$, s.t. $f(v) = \Lambda v + u$ for all $v \in \mathbb{R}^n$.
(ii) Show that the Poincaré group is isomorphic to the semidirect product $O(1,n-1) \ltimes \mathbb{R}^n$ with multiplication

$$(\Lambda, u)\cdot(\Lambda', u') = (\Lambda\Lambda', \Lambda u' + u) \quad .$$

Exercise 2.10:
Verify that the commutator $[A,B] = AB - BA$ obeys the Jacobi identity.

Exercise 2.11:
(i) Consider a rotation around the 3-axis,

$$(U_{\mathrm{rot}}(\theta)\psi)(q_1,q_2,q_3) = \psi(q_1\cos\theta - q_2\sin\theta, q_2\cos\theta + q_1\sin\theta, q_3)$$

and check that infinitesimally

$$U_{\mathrm{rot}}(\theta) = \mathbf{1} + i\theta L_3 + O(\theta^2) \quad .$$

(ii) Using $[q_r, p_s] = i\delta_{rs}$ (check!) verify the commutator

$$[L_r, L_s] = i\sum_{t=1}^{3} \varepsilon_{rst} L_t \quad .$$

(You might need the relation $\sum_{k=1}^{3} \varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ (check!).)

Exercise 3.1:
(i) Show that $U(n)$ and $SU(n)$ are indeed groups.
(ii) Let $(A^\dagger)_{ij} = (A_{ji})^*$ be the hermitian conjugate. Show that the condition $(Au, Av) = (u,v)$ for all $u,v \in \mathbb{C}^n$ is equivalent to $A^\dagger A = \mathbf{1}$, i.e.

$$U(n) = \{A \in \mathrm{Mat}(n,\mathbb{C}) \,|\, A^\dagger A = \mathbf{1}\} \ .$$

(iii) Show that $U(n)$ and $SU(n)$ are matrix Lie groups.

Exercise 3.2:
(i) Using the definition of the matrix exponential in terms of the infinite sum, show that for $\lambda \in \mathbb{C}$,

$$\exp\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = e^{\lambda} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad .$$

(ii) Let $A \in \mathrm{Mat}(n, \mathbb{C})$. Show that for any $U \in GL(n, \mathbb{C})$

$$U^{-1} \exp(A) U = \exp(U^{-1} A U) \quad .$$

(iii) Recall that a complex $n \times n$ matrix A can always be brought to Jordan normal form, i.e. there exists an $U \in GL(n, \mathbb{C})$ s.t.

$$U^{-1} A U = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_r \end{pmatrix} \quad ,$$

where each Jordan block is of the form

$$J_k = \begin{pmatrix} \lambda_k & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_k \end{pmatrix} \quad , \quad \lambda_k \in \mathbb{C} \quad .$$

In particular, if all Jordan blocks have size 1, the matrix A is diagonalisable. Compute

$$\exp\begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} \quad \text{and} \quad \exp\begin{pmatrix} 5 & 9 \\ -1 & -1 \end{pmatrix} \quad .$$

Exercise 3.3:
Let $A \in \mathrm{Mat}(\mathrm{n}, \mathbb{C})$.
(i) Let $f(t) = \det(\exp(tA))$ and $g(t) = \exp(t\,\mathrm{tr}(A))$. Show that $f(t)$ and $g(t)$ both solve the first order DEQ $u' = \mathrm{tr}(A)\, u$.
(ii) Using (i), show that

$$\det(\exp(A)) = \exp(\mathrm{tr}(A)) \quad .$$

Exercise 3.4:
Show that if A and B commute (i.e. if $AB = BA$), then $\exp(A)\exp(B) = \exp(A + B)$.

Exercise* 3.5:
Let G be a matrix Lie group and let g be the Lie algebra of G.
(i) Show that if $A \in g$, then also $sA \in g$ for all $s \in \mathbb{R}$.

(ii) The following formulae hold for $A, B \in \mathrm{Mat}(n, \mathbb{K})$: the *Trotter Product Formula*,

$$\exp(A+B) = \lim_{n\to\infty} \Big(\exp(A/n)\exp(B/n) \Big)^n \quad ,$$

and the *Commutator Formula*,

$$\exp([A,B]) = \lim_{n\to\infty} \Big(\exp(A/n)\exp(B/n)\exp(-A/n)\exp(-B/n) \Big)^{n^2} \quad .$$

(For a proof see [Baker, Theorem 7.26]). Use these to show that if $A, B \in g$, then also $A+B \in g$ and $[A,B] \in g$. (You will need that a matrix Lie group is closed.) Note that part (i) and (ii) combined prove Theorem 3.9.

Exercise 3.6:
Prove that $SP(2n)$ is a matrix Lie group.

Exercise 3.7:
In the table of matrix Lie algebras, verify the entries for $SL(n,\mathbb{C})$, $SP(2n)$, $U(n)$ and confirm the dimension of $SU(n)$.

Exercise 3.8:
(i) Show that $\mathcal{E}_{ab}\mathcal{E}_{cd} = \delta_{bc}\mathcal{E}_{ad}$.
(ii) Show that $\sum_{x=1}^{3} \varepsilon_{abx}\varepsilon_{cdx} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}$.

Exercise 3.9:
(i) Show that the generators J_1, J_2, J_3 can also be written as $J_a = \sum_{b,c=1}^{3} \varepsilon_{abc}\mathcal{E}_{bc}$, $a \in \{1,2,3\}$.
(ii) Show that $[J_a, J_b] = -\sum_{c=1}^{3} \varepsilon_{abc} J_c$
(iii) Check that $R_3(\theta) = \exp(-\theta J_3)$ is given by

$$R_3(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad .$$

This is a rotation by an angle θ around the 3-axis. Check explicitly that $R_3(\theta) \in SO(3)$.

Exercise 3.10:
Show that for $a, b \in \{1,2,3\}$, $[\sigma_a, \sigma_b] = 2i \sum_c \varepsilon_{abc}\sigma_c$.

Exercise 3.11:
(i) Show that the set $\{i\sigma_1, i\sigma_2, i\sigma_3\}$ is a basis of $su(2)$ as a real vector space. Convince yourself that the set $\{\sigma_1, \sigma_2, \sigma_3\}$ does *not* form a basis of $su(2)$ as a real vector space.
(ii) Show that $[i\sigma_a, i\sigma_b] = -2\sum_{c=1}^{3} \varepsilon_{abc} i\sigma_c$.

Exercise 3.12:
Show that $so(3)$ and $su(2)$ are isomorphic as real Lie algebras.

Exercise 3.13:
Show that the Lie algebra of $O(1, n{-}1)$ is

$$o(1, n{-}1) = \{A \in \mathrm{Mat}(n, \mathbb{R}) | A^t J + JA = 0\} \quad .$$

Exercise 3.14:
Check that the commutator of the M_{ab}'s is

$$[M_{ab}, M_{cd}] = \eta_{ad} M_{bc} + \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac} \quad .$$

Exercise 3.15:
(i) Show that, for $A \in \mathrm{Mat}(n, \mathbb{R})$ and $u \in \mathbb{R}^n$,

$$\exp\left(\begin{array}{c|c} A & u \\ \hline 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} e^A & Bu \\ \hline 0 & 1 \end{array}\right) \qquad , \quad B = \sum_{n=1}^{\infty} \frac{1}{n!} A^{n-1} \quad .$$

[If A is invertible, then $B = A^{-1}(e^A - \mathbf{1})$.]
(ii) Show that the Lie algebra of $\tilde{P}(1, n{-}1)$ (the Poincaré group embedded in $\mathrm{Mat}(n{+}1, \mathbb{R})$) is

$$p(1, n{-}1) = \left\{ \left(\begin{array}{c|c} A & x \\ \hline 0 & 0 \end{array}\right) \middle| A \in o(1, n{-}1) \ , \ x \in \mathbb{R}^n \right\} \quad .$$

Exercise 3.16:
Show that, for $a, b, c \in \{0, 1, \dots, n{-}1\}$,

$$[M_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b \quad , \quad [P_a, P_b] = 0 \quad .$$

Exercise 3.17:
There are some variants of the BCH identity which are also known as *Baker-Campbell-Hausdorff formulae*. Here we will prove some.
Let $\mathrm{ad}(A) : \mathrm{Mat}(n, \mathbb{C}) \to \mathrm{Mat}(n, \mathbb{C})$ be given by $\mathrm{ad}(A)B = [A, B]$. [This is called the *adjoint action*.]
(i) Show that for $A, B \in \mathrm{Mat}(n, \mathbb{C})$,

$$f(t) = e^{tA} B e^{-tA} \quad \text{and} \quad g(t) = e^{t\,\mathrm{ad}(A)} B$$

both solve the first order DEQ

$$\frac{d}{dt} u(t) = [A, u(t)] \quad .$$

(ii) Show that

$$e^A B e^{-A} = e^{\mathrm{ad}(A)} B = B + [A, B] + \tfrac{1}{2}[A, [A, B]] + \dots$$

(iii) Show that

$$e^A e^B e^{-A} = \exp(e^{\mathrm{ad}(A)} B)$$

(iv) Show that if $[A, B]$ commutes with A and B,

$$e^A e^B = e^{[A,B]} e^B e^A \quad .$$

(v) Suppose $[A, B]$ commutes with A and B. Show that $f(t) = e^{tA}e^{tB}$ and $g(t) = e^{tA+tB+\frac{1}{2}t^2[A,B]}$ both solve $\frac{d}{dt}u(t) = (A + B + t[A, B])u(t)$. Show further that

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]} \quad .$$

Exercise 4.1:
It is also common to use 'modules' instead of representations. The two concepts are equivalent, as will be clear by the end of this exercise.
Let g be a Lie algebra over $\mathbb{K}$. A *g-module* V is a $\mathbb{K}$-vector space V together with a bilinear map $. : g \times V \to V$ such that

$$[x, y].w = x.(y.w) - y.(x.w) \qquad \text{for all} \quad x, y \in g, w \in V \quad . \tag{A.1}$$

(i) Show that given a g-module V, one gets a representation of g by setting $R(x)w = x.w$.
(ii) Given a representation (V, R) of g, show that setting $x.w = R(x)w$ defines a g-module on V.

Exercise 4.2:
Show that for the Lie algebra $u(1)$, the trivial and the adjoint representation are isomorphic.

Exercise 4.3:
Show that if $(\mathbb{K}^n, R)$ is a representation of g, then so is $(\mathbb{K}^n, R+)$ with $R^+(x) = -R(x)^t$.

Exercise 4.4:
Let $f : V \to W$ be an intertwiner of two representations V, W of g. Show that $\ker(f) = \{v \in V | f(v) = 0\}$ and $\mathrm{im}(f) = \{w \in W | w = f(v) \text{ for some } v \in V\}$ are invariant subspaces of V and W, respectively.

Exercise 4.5:
Check that for the basis elements of $sl(2, \mathbb{C})$ one has $[H, E] = 2E$, $[H, F] = -2F$ and $[E, F] = H$.

Exercise 4.6:
Let (V, R) be a representation of $sl(2, \mathbb{C})$. Show that if $R(H)$ has an eigenvector with non-integer eigenvalue, then V is infinite-dimensional.
Hint: Let $H.v = \lambda v$ with $\lambda \notin \mathbb{Z}$. Proceed as follows.
1) Set $w = E.v$. Show that either $w = 0$ or w is an eigenvector of $R(H)$ with

eigenvalue $\lambda + 2$.
2) Show that either V is infinite-dimensional or there is an eigenvector v_0 of $R(H)$ of eigenvalue $\lambda_0 \notin \mathbb{Z}$ such that $E.v_0 = 0$.
3) Let $v_m = F^m.v_0$ and define $v_{-1} = 0$. Show by induction on m that

$$H.v_m = (\lambda_0 - 2m)v_m \quad \text{and} \quad E.v_m = m(\lambda_0 - m + 1)v_{m-1} \quad .$$

4) Conclude that if $\lambda_0 \notin \mathbb{Z}_{\geq 0}$ all v_m are nonzero.

Exercise 4.7:
The Lie algebra $h = \mathbb{C}H$ is a subalgebra of $sl(2, \mathbb{C})$. Show that h has finite-dimensional representations where $R(H)$ has non-integer eigenvalues.

Exercise 4.8:
Check that the representation of $sl(2, \mathbb{C})$ defined in the lecture indeed also obeys $[H, E].v = 2E.v$ and $[H, F].v = -2F.v$ for all $v \in \mathbb{C}^n$.

Exercise 4.9:
Let (W, R) be a finite-dimensional, irreducible representation of $sl(2, \mathbb{C})$. Show that for some $n \in \mathbb{Z}_{\geq 0}$ there is an injective intertwiner $\varphi : V_n \to W$.
Hint: (recall exercise 4.6)
1) Find a $v_0 \in W$ such that $E.v_0 = 0$ and $H.v_0 = \lambda_0 v_0$ for some $h \in \mathbb{Z}$.
2) Set $v_m = F^m.v_0$. Show that there exists an n such that $v_m = 0$ for $m \geq n$. Choose the smallest such n.
3) Show that $\varphi(e_m) = v_m$ for $m = 0, \dots, n-1$ defines an injective intertwiner.

Exercise* 4.10:
Let U, V be two finite-dimensional $\mathbb{K}$-vector spaces. Let $u_1, \dots, u_m$ be a basis of U and let $v_1, \dots, v_n$ be a basis of V.
(i) [Easy] Show that

$$\{u_k \oplus 0 | k = 1, \dots, m\} \cup \{0 \oplus v_k | k = 1, \dots, n\}$$

is a basis of $U \oplus V$.
(ii) [Harder] Show that

$$\{u_i \otimes v_j | i = 1, \dots, m \text{ and } j = 1, \dots, n\}$$

is a basis of $U \otimes V$.

Exercise 4.11:
Show that for two Lie algebras g, h, the vector space $g \oplus h$ with Lie bracket as defined in the lecture is indeed a Lie algebra.

Exercise 4.12:
Let g be a Lie algebra and let U, V be two representations of g.
(i) Show that the vector spaces $U \oplus V$ and $U \otimes V$ with g-action as defined in

the lecture are indeed representations of g.
(ii) Show that the vector space $U \otimes V$ with g-action $x.(u \otimes v) = (x.u) \otimes (x.v)$ is *not* a representation of g.

Exercise 4.13:
Let V_n denote the irreducible representation of $sl(2,\mathbb{C})$ defined in the lecture. Consider the isomorphism of vector spaces $\varphi : V_1 \oplus V_3 \to V_2 \otimes V_2$ given by

$$\varphi(e_0 \oplus 0) = e_0 \otimes e_1 - e_1 \otimes e_0 \quad ,$$

$$\varphi(0 \oplus e_0) = e_0 \otimes e_0 \quad ,$$

$$\varphi(0 \oplus e_1) = e_0 \otimes e_1 + e_1 \otimes e_0 \quad ,$$

$$\varphi(0 \oplus e_2) = 2e_1 \otimes e_1 \quad ,$$

(so that V_1 gets mapped to anti-symmetric combinations and V_3 to symmetric combinations of basis elements of $V_2 \otimes V_2$). With the help of φ, show that

$$V_1 \oplus V_3 \cong V_2 \otimes V_2$$

as representations of $sl(2,\mathbb{C})$ (this involves a bit of writing).

Exercise 4.14:
Let g be a Lie algebra.
(i) Show that a sub-vector space $h \subset g$ is a Lie subalgebra of g if and only if $[h,h] \subset h$.
(ii) Show that an ideal of g is in particular a Lie subalgebra.
(iii) Show that for a Lie algebra homomorphism $\varphi : g \to g'$ from g to a Lie algebra g', $\ker(\varphi)$ is an ideal of g.
(iv) Show that $[g,g]$ is an ideal of g.
(v) Show that if h and h' are ideals of g, then their intersection $h \cap h'$ is an ideal of g.

Exercise 4.15:
Let g be a Lie algebra and $h \subset g$ an ideal. Show that $\pi : g \to g/h$ given by $\pi(x) = x + h$ is a surjective homomorphism of Lie algebras with kernel $\ker(\pi) = h$.

Exercise 4.16:
Let g, h be Lie algebras and $\varphi : g \to h$ a Lie algebra homomorphism. Show that if g is simple, then φ is either zero or injective.

Exercise 4.17:
(i) Show that for the basis of $sl(2,\mathbb{C})$ used in exercise 4.5, one has

$$\kappa(E,E) = 0 \ , \quad \kappa(E,H) = 0 \ , \quad \kappa(E,F) = 4 \ ,$$

$$\kappa(H,H) = 8 \ , \quad \kappa(H,F) = 0 \ , \quad \kappa(F,F) = 0 \ .$$

Denote by Tr the trace of 2×2-matrices. Show that for $sl(2,\mathbb{C})$ one has $\kappa(x,y) = 4\,\mathrm{Tr}(xy)$.
(ii) Evaluate the Killing form of $p(1,1)$ for all combinations of the basis elements M_{01}, P_0, P_1 (as used in exercises 3.14 and 3.16). Is the Killing form of $p(1,1)$ non-degenerate?

Exercise 4.18:
(i) Show that for $gl(n,\mathbb{C})$ one has $\kappa(x,y) = 2n\,\mathrm{Tr}(xy) - 2\mathrm{Tr}(x)\mathrm{Tr}(y)$, where Tr is the trace of $n\times n$-matrices.
Hint: Use the basis $\mathcal{E}_{kl}$ to compute the trace in the adjoint representation.
(ii) Show that for $sl(n,\mathbb{C})$ one has $\kappa(x,y) = 2n\,\mathrm{Tr}(xy)$.

Exercise 4.19:
Let g be a finite-dimensional Lie algebra and let $h \subset g$ be an ideal. Show that

$$h^\perp = \{x \in g | \kappa_g(x,y) = 0 \text{ for all } y \in h\}$$

is also an ideal of g.

Exercise 4.20:
Show that if a finite-dimensional Lie algebra g contains an abelian ideal h, then the Killing form of g is degenerate. (Hint: Choose a basis of h, extend it to a basis of g, and evaluate $\kappa_g(x,a)$ with $x \in g$, $a \in h$.)

Exercise 4.21:
Let $g = g_1 \oplus \cdots \oplus g_n$, for finite-dimensional Lie algebras g_i. Let $x = x_1 + \cdots + x_n$ and $y = y_1 + \cdots + y_n$ be elements of g such that $x_i, y_i \in g_i$. Show that

$$\kappa_g(x,y) = \sum_{i=1}^{n} \kappa_{g_i}(x_i, y_i) \ .$$

Exercise 4.22:
Let g be a finite-dimensional Lie algebra with non-degenerate Killing form. Let $h \subset g$ be a sub-vector space. Show that $\dim(h) + \dim(h^\perp) = \dim(g)$.

Exercise 4.23:
Show that the Poincaré algebra $p(1, n-1)$, $n \geq 2$, is not semi-simple.

Exercise 4.24:
In this exercise we prove the theorem that for a finite-dimensional, complex, simple Lie algebra g, and for an invariant bilinear form B, we have $B = \lambda\kappa_g$ for some $\lambda \in \mathbb{C}$.
(i) Let $g^* = \{\varphi : g \to \mathbb{C} \text{ linear}\}$ be the dual space of g. The dual representation of the adjoint representation is $(g, \mathrm{ad})^+ = (g^*, -\mathrm{ad})$. Let $f_B : g \to g^*$ be given by $f_B(x) = B(x, \cdot)$, i.e. $[f_B(x)](z) = B(x,z)$. Show that f_B is an intertwiner from (g, ad) to $(g^*, -\mathrm{ad})$.

(ii) Using that g is simple, show that (g, ad) is irreducible.
(iii) Since (g, ad) and $(g^*, -\mathrm{ad})$ are isomorphic representations, also $(g^*, -\mathrm{ad})$ is irreducible. Let f_κ be defined in the same way as f_B, but with κ instead of B. Show that $f_B = \lambda f_\kappa$ for some $\lambda \in \mathbb{C}$.
(iv) Show that $B = \lambda\kappa$ for some $\lambda \in \mathbb{C}$.

Exercise 5.1:
Let $\{T^a\}$ be a basis of a finite-dimensional Lie algebra g over $\mathbb{K}$. For $x \in g$, let $M(x)_{ab}$ be the matrix of ad_x in that basis, i.e.

$$\mathrm{ad}_x(\sum_b v_b T^b) = \sum_a (\sum_b M(x)_{ab} v_b) T^a \quad .$$

Show that $M(T^a)_{cb} = f^{ab}{}_c$, i.e. the structure constants give the matrix elements of the adjoint action.

Exercise* 5.2:
A fact from linear algebra: Show that for every non-degenerate symmetric bilinear form $b : V \times V \to \mathbb{C}$ on a finite-dimensional, complex vector space V there exists a basis $v_1, \dots, v_n$ (with $n = \dim(V)$) of V such that $b(v_i, v_j) = \delta_{ij}$.

Exercise 5.3:
Let g be a fssc Lie algebra and $\{T^a\}$ a basis such that $\kappa(T^a, T^b) = \delta_{ab}$. Show that the structure constants in this basis are anti-symmetric in all three indices.

Exercise 5.4:
Find a basis $\{T^a\}$ of $sl(2, \mathbb{C})$ s.t. $\kappa(T^a, T^b) = \delta_{ab}$.

Exercise 5.5:
Show that the diagonal matrices in $sl(n, \mathbb{C})$ are a Cartan subalgebra.

Exercise 5.6:
Another fact about linear algebra: Let V be a finite-dimensional vector space and let $F \subset V^*$ be a proper subspace (i.e. $F \neq V^*$). Show that there exists a nonzero $v \in V$ such that $\varphi(v) = 0$ for all $\varphi \in F$.

Exercise 5.7:
Let g be a fssc Lie algebra and let $h \subset g$ be sub-vector space such that
(1) $[h, h] = \{0\}$.
(2) κ restricted to h is non-degenerate.
(3) if for some $x \in g$ one has $[x, a] = 0$ for all $a \in h$, then already $x \in h$.
Show that h is a Cartan subalgebra of g if and only if it obeys (1)–(3) above.

Exercise 5.8:
Let $\{H^1, \dots, H^r\} \subset g_0$ be a basis of g_0 such that $\kappa(H^i, H^j) = \delta_{ij}$ (recall that $r = \dim(g_0)$ is the rank of g). Show that for $\gamma, \varphi \in g_0^*$ one has $H^\gamma = \sum_{i=1}^r \gamma(H^i) H^i$, as well as $(\gamma, \varphi) = \sum_{i=1}^r \gamma(H^i)\varphi(H^i)$ and $(\gamma, \varphi) = \gamma(H^\varphi)$.

Exercise 5.9:
Let g be a fssc Lie algebra and g_0 a Cartan subalgebra. Let $\alpha \in \Phi(g, g_0)$. Choose $e \in g_\alpha$ and $f \in g_{-\alpha}$ such that $\kappa(e, f) = \frac{2}{(\alpha,\alpha)}$. Show that $\varphi : sl(2, \mathbb{C}) \to g$ given by

$$\varphi(E) = e \quad , \quad \varphi(F) = f \quad , \quad \varphi(H) = \frac{2}{(\alpha, \alpha)} H^\alpha$$

is an injective homomorphism of Lie algebras.

Exercise* 5.10:
In this exercise we will show that $\dim(g_\alpha) = 1$ for all $\alpha \in \Phi$. On the way we will also see that if $\alpha \in \Phi$ and $\lambda\alpha \in \Phi$ for some $\lambda \in \mathbb{C}$, then $\lambda \in \{\pm 1\}$.
(i) Choose $\alpha \in \Phi$. Let $L = \{m \in \mathbb{Z} | m\alpha \in \Phi\}$. Since Φ is a finite set, so is L. Let n_+ be the largest integer in L, n_- the smallest integer in L. Show that $n_+ \geq 1$ and $n_- \leq -1$.
(ii) We can assume that $n_+ \geq |n_-|$. Otherwise we exchange α for $-\alpha$. Pick $e \in g_\alpha$, $f \in g_{-\alpha}$ s.t. $\kappa(e, f) = \frac{2}{(\alpha,\alpha)}$ and define $\varphi : sl(2, \mathbb{C}) \to g$ as in exercise 5.9. Show that

$$U = \mathbb{C}H^\alpha \oplus \bigoplus_{m \in L} g_{m\alpha}$$

is an invariant subspace of the representation (g, R_φ) of $sl(2, \mathbb{C})$.
(iii) Show that for $z \in g_{m\alpha}$ one has $R_\varphi(H)z = 2mz$.
(iv) By (ii), (U, R_φ) is also a representation of $sl(2, \mathbb{C})$. Show that $V = \mathbb{C}e \oplus \mathbb{C}H^\alpha \oplus \mathbb{C}f$ is an invariant subspace of (U, R_φ). Show that the representation (V, R_φ) is isomorphic to the irreducible representation V_3 of $sl(2, \mathbb{C})$.
(v) Choose an element $v_0 \in g_{n_+\alpha}$. Set $v_{k+1} = R_\varphi(F)v_k$ and show that

$$W = \mathrm{span}_\mathbb{C}(v_0, v_1, \ldots, v_{2n_+})$$

is an invariant subspace of U.
(vi) (W, R_φ) is isomorphic to the irreducible representation V_{2n_++1} of $sl(2, \mathbb{C})$. Show that the intersection $X = V \cap W$ is an invariant subspace of V and W. Show that X contains the element H^α and hence $X \neq \{0\}$. Show that $X = V$ and $X = W$.
We have learned that for any choice of v_0 in $g_{n_+\alpha}$ we have $V = W$. This can only be if $n_+ = 1$ and $\dim(g_\alpha) = 1$. Since $1 \leq |n_-| \leq n_+$, also $n_- = 1$. Since $\kappa : g_\alpha \times g_{-\alpha}$ is non-degenerate, also $\dim(g_{-\alpha}) = 1$.

Exercise 5.11:
Let $\{H^i\} \cup \{E^\alpha\}$ be a Cartan-Weyl basis of a fssc Lie algebra g. Show that
(i) $\mathrm{span}_\mathbb{C}(H^1, \ldots, H^r)$ is a Cartan subalgebra of g.
(ii) $\kappa(E^\alpha, E^{-\alpha}) = \frac{2}{(\alpha,\alpha)}$.

Exercise 5.12:
Let $g = g_1 \oplus g_2$ with g_1, g_2 fssc Lie algebras. For $k = 1, 2$, let h_k be a Cartan subalgebra of g_k.

(i) Show that $h = h_1 \oplus h_2$ is a Cartan subalgebra of g.
(ii) Show that the root system of g is $\Phi(g,h) = \Phi_1 \cup \Phi_2 \subset h_1^* \oplus h_2^*$ where $\Phi_1 = \{\alpha \oplus 0 | \alpha \in \Phi(g_1,h_1)\}$ and $\Phi_2 = \{0 \oplus \beta | \beta \in \Phi(g_2,h_2)\}$.
(iii) Show that $(\alpha,\beta) = 0$ for all $\alpha \in \Phi_1$ and $\beta \in \Phi_2$.

Exercise 5.13:
Let g be a fssc Lie algebra and g_0 be a Cartan subalgebra. Suppose $\Phi(g,g_0) = \Phi_1 \cup \Phi_2$ where Φ_1, Φ_2 are non-empty and $(\alpha,\beta) = 0$ for all $\alpha \in \Phi_1$, $\beta \in \Phi_2$. Let $\{H^i\} \cup \{E^\alpha\}$ be a Cartan-Weyl basis of g. Show that

$$g_1 = \mathrm{span}_{\mathbb{C}}(H^\alpha | \alpha \in \Phi_1\} \oplus \bigoplus_{\alpha \in \Phi_1} \mathbb{C} E^\alpha$$

is a proper ideal in g.

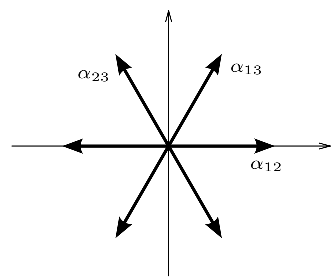
Exercise 5.14:
Let R the root space of a fssc Lie algebra with Cartan subalgebra g_0.
(i) Show that the bilinear form $(\cdot,\cdot)$ on g_0^* restricts to a real valued *positive definite* inner product on R.
(ii) Use the Gram-Schmidt procedure to find an orthonormal basis $\{\varepsilon_1,\dots,\varepsilon_m\}$ of R (over $\mathbb{R}$). Show that $m = r$ (where $r = \dim(g_0)$ is the rank of g) and that $\{\varepsilon_1,\dots,\varepsilon_r\}$ is a basis of g_0^* (over $\mathbb{C}$).
(iii) Show that there exists a basis $\{H^i | i = 1,\dots,r\}$ of g_0 such that $\alpha(H^i) \in \mathbb{R}$ for all $i = 1,\dots,r$ and $\alpha \in \Phi$.

Exercise 5.15:
In this exercise we construct a Cartan-Weyl basis for $sl(n,\mathbb{C})$. As Cartan subalgebra h we take the trace-less diagonal matrices.
(i) Define the linear forms ω_i, $i = 1,\dots,n$ on diagonal $n \times n$-matrices as $\omega_i(\mathcal{E}_{jj}) = \delta_{ij}$. Define $\alpha_{kl} = \omega_k - \omega_l$. Show that for $k \neq l$, α_{kl} is a root.
Hint: Write a general element $H \in h$ as $H = \sum_{k=1}^n a_k \mathcal{E}_{kk}$ with $\sum_{k=1}^n a_k = 0$. Show that $[H, \mathcal{E}_{kl}] = \alpha_{kl}(H)\mathcal{E}_{kl}$.
(ii) Show that $H^{\alpha_{kl}} = \frac{1}{2n}(\mathcal{E}_{kk} - \mathcal{E}_{ll})$.
Hint: Use exercise 4.18 to verify $\kappa(H^{\alpha_{kl}}, H) = \alpha_{kl}(H)$ for all $H \in h$.
(iii) Show that $(\alpha_{kl},\alpha_{kl}) = 1/n$ and that $[\mathcal{E}_{kl},\mathcal{E}_{lk}] = 2/(\alpha_{kl},\alpha_{kl}) \cdot H^{\alpha_{kl}}$
(iv) Show that, with $\Phi = \{\alpha_{kl} \,|\, k,l = 1,\dots,n\,,\, k \neq l\}$ and $E^{\alpha_{kl}} = \mathcal{E}_{kl}$,

$$\big\{ H^{\alpha_{k,k+1}} \,\big|\, k = 1,\dots,n{-}1 \big\} \cup \big\{ E^\alpha \,\big|\, \alpha \in \Phi \big\}$$

is a Cartan-Weyl basis of $sl(n,\mathbb{C})$.

(v) Show that the root diagram of $sl(3,\mathbb{C})$ is



where each arrow has length $1/\sqrt{3}$ and the angle between the arrows is $60°$.

Exercise 5.16:
Let s_α be a Weyl reflection. Show that

$$s_\alpha(\alpha) = -\alpha \quad , \quad (\alpha, \lambda) = 0 \Rightarrow s_\alpha(\lambda) = \lambda \quad , \quad s_\alpha \circ s_\alpha = \mathrm{id} \quad , \quad s_{-\alpha} = s_\alpha \quad .$$

Exercise 5.17:
Show that the Weyl group of a fssc Lie algebra is a finite group (i.e. contains only a finite number of elements).

Exercise 5.18:
Give the dimension (over $\mathbb{C}$) of all rank two fssc Lie algebras as found in section 5.6.

Exercise 5.19:
Let g be a fssc Lie algebra and let Φ_+ and Φ_- be the positive and negative roots with respect to some hyperplane. Show that
(i) $\Phi = \Phi_+ \cup \Phi_-$.
(ii) $\alpha \in \Phi_+ \Leftrightarrow -\alpha \in \Phi_-$ and $|\Phi_+| = |\Phi_-|$ (the number of elements in a set S is denoted by $|S|$).
(iii) $\mathrm{span}_{\mathbb{R}}(\Phi_+) = \mathrm{span}_{\mathbb{R}}(\Phi)$.

Exercise 5.20:
Using the properties of simple roots stated in the lecture, prove the following properties of the Cartan matrix.
(i) $A^{ij} \in \mathbb{Z}$.
(ii) $A^{ii} = 2$ and $A^{ij} \leq 0$ if $i \neq j$.
(iii) $A^{ij}A^{ji} \in \{0, 1, 2, 3\}$ if $i \neq j$.

Exercise 5.21:
A Dynkin diagram is *connected* if for any two vertices $i \neq j$ there is a sequence of nodes $k_1, \dots, k_m$ with $k_1 = i$, $k_m = j$ such that $A^{k_s,k_{s+1}} \neq 0$ for $s = 1, \dots, m-1$. [In words: one can go from any vertex to any other by walking only along lines.]

Show that if the Dynkin diagram of a fssc Lie algebra is connected, then the Lie algebra is simple. [The converse follows from the classification theorem of Killing and Cartan, which shows explicitly that simple Lie algebras have connected Dynkin diagrams.]

Exercise 5.22:
Compute the Dynkin diagrams of all rank two fssc Lie algebras using the root diagrams obtained in section 5.6.

Exercise 5.23:
The Dynkin diagram (together with a choice for the numbering of the vertices) determines the Cartan matrix uniquely. Write out the Cartan matrix for the Lie algebras A_4, B_4, C_4, D_4 and F_4.

Exercise* 5.24:
Let V be a real vector space. Show that every $v \in V_{\mathbb{C}}$ can be *uniquely* written as $v = (1, a) + i(1, b) + W$, with $a, b \in V$, i.e., using the shorthand notation, $v = a + ib$.

Exercise 5.25:
Let V be a finite-dimensional, real vector space and let $\{v_1, \ldots, v_n\}$ be a basis of V. Show that $\{(1, v_1) + W, \ldots, (1, v_n) + W\}$ is a basis of $V_{\mathbb{C}}$. (You may want to use the result of exercise 5.24.)

Exercise 5.26:
Let h be a finite-dimensional real Lie algebra and let g be a finite-dimensional complex Lie algebra. Show that the following are equivalent.
(1) h is a real form of g.
(2) There exist bases $\{T^a | a = 1, \ldots, n\}$ of h (over $\mathbb{R}$) and $\{\tilde{T}^a | a = 1, \ldots, n\}$ of g (over $\mathbb{C}$) such that

$$[T^a, T^b] = \sum_{c=1}^{n} f^{ab}_{\ c} T^c \quad \text{and} \quad [\tilde{T}^a, \tilde{T}^b] = \sum_{c=1}^{n} f^{ab}_{\ c} \tilde{T}^c$$

with the same structure constants $f^{ab}_{\ c}$.

Exercise 5.27:
Show that the Killing form of $su(2)$ is negative definite (i.e. $\kappa(x, x) < 0$ for all $x \in su(2)$) and that the one of $sl(2, \mathbb{R})$ is not. Conclude that there exists no isomorphism $\varphi : sl(2, \mathbb{R}) \to su(2)$ of real Lie algebras.

Exercise 5.28:
Show that $o(1, n-1)_{\mathbb{C}} \cong so(n)_{\mathbb{C}}$ as complex Lie algebras.

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