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Form factors of Ising spin and disorder fields on the Poincaré disk

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Abstract

Using recent results concerning form factors of certain scaling fields in the massive Dirac theory on the Poincaré disk, we find expressions for the form factors of Ising spin and disorder fields in the massive Majorana theory on the Poincaré disk. In particular, we verify that these recent results agree with the factorization properties of the fields in the Dirac theory representing tensor products of spin and of disorder fields in the Majorana theory.

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1 Introduction

Quantum Field Theory (QFT) in curved space-time is a subject of great importance which has been studied from many viewpoints (see for instance [1]). One of the main applications of QFT in Euclidean space is the study of classical statistical systems near their critical point, where the correlation length is of the order of the size of the system. It is important to extend such application to the use of QFT on Euclidean-signature curved space-time for the study of statistical systems on curved space, as the effects of curvature on the properties of critical points are not well understood. A simple but non-trivial curved space is the Poincaré disk. It is maximally symmetric, which allows for the extension of some techniques on two-dimensional flat space to this space, and has a negative Gaussian curvature. As was argued in [2], a negative curvature can be used as an infrared regulator for Euclidean QFT; it is interesting to analyze further the effects of such a curvature on the critical point of a statistical system.

Recently, correlation functions of certain scaling fields in the Dirac theory on the Poincaré disk were studied [3, 4]. The scaling fields in question, $\mathcal{O}_\alpha = \mathcal{O}^\dagger_{-\alpha}$, $-1 < \alpha < 1$, are spinless, $U(1)$-neutral and have scaling dimension α^2. They are not mutually local with respect to the Dirac field. Their mutual locality index with the Dirac field is α, that is, the Dirac field Ψ takes a factor, $\Psi \to e^{2\pi i\alpha}\Psi$, when continued counterclockwise around the field $\mathcal{O}_\alpha$. In [3], using a generalization of the method of isomonodromic deformations to the study of determinants of Dirac operators on the Poincaré disk, the authors obtained differential equations of Painlevé type for correlation functions of such fields. In [4], we solved the associated connection problem for the two-point function, obtained its long distance expansion by developing a form factor decomposition, and evaluated the one-point function.

A physical application of these fields, with which the present paper is concerned, comes from the fact that correlators of some of them are simply related to the scaling limit of correlators of local variables in the lattice Ising model [5, 6, 7]. Let us explain in more details what this relation is on flat space, where the results are well known. The lattice Ising model at zero magnetic field and at a temperature very near to its critical temperature (more precisely, in the scaling limit) is described by the quantum field theory of a free massive Majorana fermion [8, 9] (cf. [10]). Recall that in the Ising model, one can consider the spin variable and its dual, the disorder variable [11], related by the duality transformation that takes the system from its low temperature regime to its high temperature regime and vice versa. Correspondingly in the Majorana theory, one can define the spin field σ and the disorder field μ. These fields are local, but not mutually local with respect to the fermion fields or with respect to each other. Correlation functions involving these fields give the scaling limit of correlation functions involving spin and disorder variables in the Ising model. Now, the tensor product of two independent copies of the Majorana theory can be equivalently described by a single copy of the Dirac theory. One can then represent the tensor product of two spin fields acting non trivially on independent copies of the Majorana theory as a single field in the Dirac theory, and similarly for the tensor product of two disorder fields. Taking the Majorana theory (with positive mass) to represent the scaling limit of the Ising model in its low-temperature regime (that is, for temperatures near to but smaller than the critical temperature), one has the following equivalences [5, 6]:

$$\sigma \otimes \sigma = \mathcal{O}^{(+)}, \quad \mu \otimes \mu = \mathcal{O}^{(-)} \tag{1.1}$$

where the fields $\mathcal{O}^{(+)}$ and $\mathcal{O}^{(-)}$ belong to the Dirac theory. They can be expressed in terms of the fields $\mathcal{O}_\alpha$ described above:

$$\mathcal{O}^{(+)} = \frac{1}{\sqrt{2}}\left(\mathcal{O}_{\frac{1}{2}} + \mathcal{O}_{-\frac{1}{2}}\right), \quad \mathcal{O}^{(-)} = \frac{1}{\sqrt{2}}\left(\mathcal{O}_{\frac{1}{2}} - \mathcal{O}_{-\frac{1}{2}}\right). \tag{1.2}$$

In fact, once the field $\mathcal{O}^{(+)}$ is given to represent $\sigma \otimes \sigma$, the field $\mathcal{O}^{(-)}$ for representing $\mu \otimes \mu$ can be deduced using the OPE's in the Dirac theory and in the Majorana theory[1].

[1]Such considerations also lead to the relations:

$$\begin{aligned} \mathcal{O}^{(+)}(x_1)\cdots\mathcal{O}^{(+)}(x_n) &= (\sigma(x_1)\cdots\sigma(x_n))\otimes(\sigma(x_1)\cdots\sigma(x_n)) \\ \mathcal{O}^{(-)}(x_1)\cdots\mathcal{O}^{(-)}(x_n) &= (-1)^n(\mu(x_1)\cdots\mu(x_n))\otimes(\mu(x_1)\cdots\mu(x_n)) \end{aligned} \tag{1.3}$$

Having recalled some results on flat space, let us now turn to the Dirac theory on the Poincaré disk. Again it is equivalent to a tensor product of two copies of the Majorana theory, but on the Poincaré disk. Then we expect that the fields $\mathcal{O}^{(+)}$ and $\mathcal{O}^{(-)}$ given by (1.2) factorize as a tensor product of fields belonging to the Majorana theory, the way they do in (1.1). Although such a factorization is not *a priori* obvious from the definitions of $\mathcal{O}^{(+)}$ and $\mathcal{O}^{(-)}$, it is a local property and should not be affected by the curvature. The two fields σ and μ thus defined in the Majorana theory on the Poincaré disk will still be called spin and disorder fields. We expect that these fields be further related to spin and disorder variables in a lattice Ising model on the Poincaré disk, although the precise relation is not yet clarified. In any case, the study of the fields $\mathcal{O}_\alpha$ in [3, 4] should give information about spin and disorder fields in the Majorana theory on the Poincaré disk. It is not straightforward to obtain most of this information, and a full analysis, based on more efficient methods, will be exposed in another publication [12]. However, it is a simple matter to specialize some of the results of [4] to obtain expressions for form factors, and in particular vacuum expectation values, of spin and disorder fields. This is what we do in this paper. This involves, in particular, a verification that results of [4] indeed respect the factorization properties of the fields $\mathcal{O}^{(+)}$ and $\mathcal{O}^{(-)}$. More precisely, since the Hilbert space of the Dirac theory on the Poincaré disk is a tensor product of two copies of the Hilbert space of the Majorana theory on the Poincaré disk, we verify that in the tensor product basis of the Dirac Hilbert space, matrix elements between vacuum and excited states, or form factors, of the fields $\mathcal{O}^{(+)}$ and $\mathcal{O}^{(-)}$ factorize.

Form factors are useful quantities to study, in particular because of their relation with long distance expansions of correlations functions. We will not discuss such expansions in this paper; in a future publication [12], form factors and correlation functions of spin and disorder fields on the Poincaré disk will be studied by a different, much simpler method. Some of the results of [12] concerning form factors are already available to us, and will be compared with the expressions obtained in the present paper; this will give a further verification of the more general form factor results of [4]. However, the method of [12] does not give an explicit expression for the vacuum expectation value of the spin field, which in fact is simple to read off from a result of [4]. In the Majorana theory with fermion mass m on the Poincaré disk with Gaussian curvature $-1/R^2$, it is given by

$$\langle\sigma\rangle^2 = (R/2)^{-\frac{1}{4}} \prod_{n=1}^{\infty} \left(\frac{1-\frac{1}{4(mR+n)^2}}{1-\frac{1}{4n^2}} \right)^n . \tag{1.4}$$

Here we use the following normalization of the fields $\mathcal{O}^{(+)}$ and $\mathcal{O}^{(-)}$ in the Dirac theory on the Poincaré disk:

$$\langle\mathcal{O}^{(+)}(x)\mathcal{O}^{(+)}(y)\rangle \sim \mathrm{d}(x,y)^{-\frac{1}{2}}, \quad \langle\mathcal{O}^{(-)}(x)\,\mathcal{O}^{(-)}(y)\rangle \sim -\mathrm{d}(x,y)^{-\frac{1}{2}} \quad \text{as } x\to y\ , \tag{1.5}$$

where $\mathrm{d}(x,y)$ is the geodesic distance between the points x and y. This corresponds to the following normalization for the spin and disorder fields:

$$\langle\sigma(x)\sigma(y)\rangle \sim \mathrm{d}(x,y)^{-\frac{1}{4}}, \quad \langle\mu(x)\mu(y)\rangle \sim \mathrm{d}(x,y)^{-\frac{1}{4}} \quad \text{as } x\to y\ .$$

Note that in the theory on flat space, form factors were first calculated in [13], and the vacuum expectation value of the spin field was first calculated in [7].

The plan of the paper is as follows. In section 2, we recall the structure of the Hilbert space of the Dirac theory on the Poincaré disk, and explicitly factorize it in a tensor product of two copies of the Hilbert space of the Majorana theory. In section 3, we briefly recall and analyze some results of [4] concerning form factors in the Dirac theory of the fields $\mathcal{O}_{\pm\frac{1}{2}}$ used in the definition of $\mathcal{O}^{(+)}$ and $\mathcal{O}^{(-)}$ (1.2). In section 4, we verify the factorization properties for the field $\mathcal{O}^{(+)}$ and calculate multi-particle form factors of the spin field. Finally, in section 5, we verify the factorization properties for the field $\mathcal{O}^{(-)}$ and calculate multi-particle form factors of the disorder field.

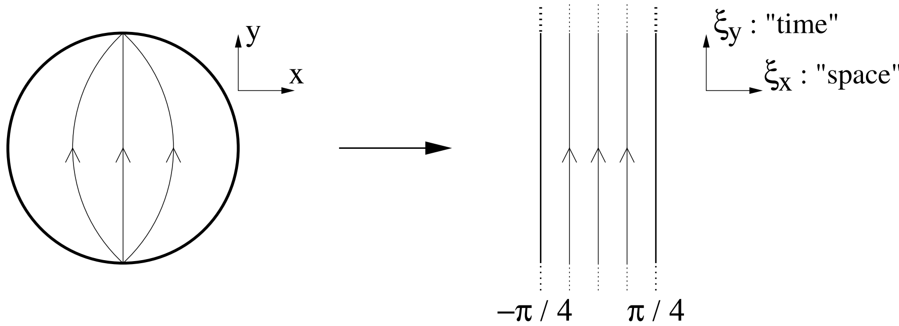

Figure 1: The mapping from the Poincaré disk to the strip is described by $\mathrm{x}+i\mathrm{y} = \tan(\xi_x + i\xi_y)$, $\mathrm{x}-i\mathrm{y} = \tan(\xi_x - i\xi_y)$. Lines with an arrow represent orbits of a non-compact subgroup of the isometry group of the Poincaré disk.

2 Hilbert space

In [4], the Hilbert space of the free Dirac theory on the Poincaré disk was constructed in various quantization schemes, and form factors of the scaling fields $\mathcal{O}_\alpha$ were calculated in two (related) schemes. It was pointed out that in a quantization scheme where the Hamiltonian is taken as the generator of a non-compact subgroup of the $SU(1,1)$ isometry group of the Poincaré disk, the spectrum is discrete. This quantization scheme is the most convenient one for obtaining the long distance expansion of correlation functions by using a resolution the identity on the Hilbert space. Also, in this scheme, form factors of the fields $\mathcal{O}_\alpha$ seem to have the simplest expressions. Here we will recall the structure of the Hilbert space of the Dirac theory on the Poincaré disk, and its relation to the Hilbert space of the Majorana theory on the Poincaré disk, within this quantization scheme only.

In order to describe the quantization scheme, it is convenient to consider a system of coordinates $-\pi/4 < \xi_\mathrm{x} < \pi/4$, $\xi_y \in \mathbb{R}$ covering a "Poincaré strip" instead of the Poincaré disk, with the mapping shown in Figure 1. For the "Poincaré strip" of Gaussian curvature $-1/R^2$, the metric is given by

$$ds^2 = \frac{(2R)^2}{\cos^2(2\xi_\mathrm{x})}\,(d\xi_\mathrm{x}^2 + d\xi_\mathrm{y}^2).$$

In this system of coordinates, the action for a free Dirac fermion field $\Psi = \begin{pmatrix} \Psi_R \\ \Psi_L \end{pmatrix}$ of mass m is

$$\mathcal{A} = \int_{-\infty}^{\infty} d\xi_\mathrm{y} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\xi_\mathrm{x}\, \bar\Psi \left(\gamma^\mathrm{x} \frac{\partial}{\partial\xi_\mathrm{x}} + \gamma^\mathrm{y} \frac{\partial}{\partial\xi_\mathrm{y}} + \frac{2\nu}{\cos(2\xi_\mathrm{x})} \right) \Psi \tag{2.1}$$

where $\nu = mR$ and $\bar\Psi = \Psi^\dagger\gamma^\mathrm{y}$. Here the Dirac matrices are taken to be

$$\gamma^\mathrm{x} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \gamma^\mathrm{y} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For simplicity, we will assume that $\nu > 1/2$, as was assumed in [4] [2] (we expect the Hilbert space to have more structure than what is described below in the case $\nu < 1/2$; this case will be studied in more details in [12]).

The translation $\xi_\mathrm{y} \to \xi_\mathrm{y} + q$ is a translation along an orbit of a non-compact subgroup of the isometry group. The quantization scheme in which we are interested is then obtained by taking ξ_y as the "time". The corresponding Hilbert space $\mathcal{H}$ is a module for the canonical equal-"time" anti-commutation relations

$$\{\Psi(\xi_\mathrm{x},\xi_\mathrm{y}), \Psi^\dagger(\xi'_\mathrm{x},\xi_\mathrm{y})\} = \mathbf{1}\,\delta(\xi_\mathrm{x} - \xi'_\mathrm{x}).$$

[2] Note that in [4], the combination mR was denoted μ instead of ν.

A basis for $\mathcal{H}$ can be taken as the discrete set of states diagonalizing the Hamiltonian corresponding to the action (2.1):

$$|k_1,\ldots,k_n\rangle_{\epsilon_1,\ldots,\epsilon_n},\ k_j\in\mathbb{N},\ \epsilon_j=\pm_j,\ n=0,1,2,\ldots,\ k_1<\cdots<k_n. \tag{2.2}$$

They correspond to eigenvalues $\lambda_1+\ldots+\lambda_n$ of the Hamiltonian, where

$$\lambda_j = 1+2\nu+2k_j.$$

The vacuum state, with $n=0$, will be denoted $|vac\rangle$. In particular, correlation functions, denoted $\langle\cdots\rangle$, are vacuum expectation values of "time"-ordered operators, where the "time"-ordering brings operators at lower values of $\xi_{\rm y}$ to the right of those at higher values of $\xi_{\rm y}$. The number of arguments, n, in the state (2.2) should be interpreted as the number of free fermionic particles forming the state, the integer k_j as the discrete energy level of the $j^{\rm th}$ particle and the sign ϵ_j as its $U(1)$ charge. These states are normalized by

$$_{\epsilon_n,\ldots,\epsilon_1}\langle k_n,\ldots,k_1|k_1',\ldots,k_n'\rangle_{\epsilon_1',\ldots,\epsilon_n'} = \delta_{k_1,k_1'}\cdots\delta_{k_n,k_n'}\ \delta_{\epsilon_1+\epsilon_1'}\cdots\delta_{\epsilon_n+\epsilon_n'}$$

and states with different number of particles are orthogonal. States with different orderings of the energy levels k_j are defined by the fact that exchanging simultaneously the positions of two values of energy levels k_i, k_j and of $U(1)$ charges ϵ_i, ϵ_j in $|k_1,\ldots,k_n\rangle_{\epsilon_1,\ldots,\epsilon_n}$ brings a factor of (-1).

In parallel to the case of the theory on flat space, the free massive Dirac theory on the Poincaré disk is equivalent to two independent copies of a free massive Majorana theory on the Poincaré disk. Consider four fermion fields $\psi_a,\psi_b,\bar\psi_a,\bar\psi_b$, defined via

$$\Psi_R=\frac{1}{\sqrt2}(\psi_a+i\psi_b),\quad \Psi_L=\frac{1}{\sqrt2}(\bar\psi_b-i\bar\psi_a).$$

It is easy to verify that correlators of these fields factorize; for instance:

$$\langle\psi_a(x_1)\cdots\psi_a(x_n)\,\psi_b(x_1')\cdots\psi_b(x_m')\rangle = \langle\psi_a(x_1)\cdots\psi_a(x_n)\rangle\,\langle\psi_b(x_1')\cdots\psi_b(x_m')\rangle.$$

This factorization can be expressed by writing the fields $\psi_a,\psi_b,\bar\psi_a,\bar\psi_b$ as tensor products of fields in two independent copies of the Majorana theory:

$$\psi_a=\psi\otimes\mathbf{1},\quad \psi_b=\mathbf{1}\otimes\psi,\quad \bar\psi_a=\bar\psi\otimes\mathbf{1},\quad \bar\psi_b=\mathbf{1}\otimes\bar\psi.$$

Here $\psi,\bar\psi$ are (real) Majorana fields that satisfy the equations of motion

$$\frac{\partial}{\partial\xi}\bar\psi=-\frac{\nu}{\cos(2\xi_{\rm x})}\psi,\quad \frac{\partial}{\partial\bar\xi}\psi=-\frac{\nu}{\cos(2\xi_{\rm x})}\bar\psi$$

with $\xi=\xi_{\rm x}+i\xi_{\rm y}$ and $\bar\xi=\xi_{\rm x}-i\xi_{\rm y}$ and have short distance normalization given by

$$\langle\psi(\xi_1,\bar\xi_1)\psi(\xi_2,\bar\xi_2)\rangle\sim-\frac{1}{2\pi i}\frac{1}{\xi_1-\xi_2},\quad \langle\bar\psi(\xi_1,\bar\xi_1)\bar\psi(\xi_2,\bar\xi_2)\rangle\sim\frac{1}{2\pi i}\frac{1}{\bar\xi_1-\bar\xi_2}.$$

A precise correspondence between a product of fermion fields in the Dirac theory and a tensor product of products of fermion fields in the Majorana theory must take into account the signs coming from the fact that two Dirac fields anti-commute. We will define, for instance,

$$\psi_a(x_1)\cdots\psi_a(x_n)\,\psi_b(x_1')\cdots\psi_b(x_m') = (\psi(x_1)\cdots\psi(x_n))\,\otimes\,(\psi(x_1')\cdots\psi(x_m')),$$

and appropriately include extra minus signs for other orderings of ψ_a's with respect to ψ_b's.

Consequent to this decomposition of the Dirac fermion field, the Hilbert space of the Dirac theory can be written as a tensor product of two copies of the Hilbert space of the Majorana theory: $\mathcal{H}=\mathcal{H}_M\otimes\mathcal{H}_M$. It can be verified that the Hilbert space $\mathcal{H}_M$, within the quantization scheme we are considering, has a structure similar to that of $\mathcal{H}$. An explicit construction of $\mathcal{H}_M$ will be done in [12]. For now, we simply need to know that a basis for $\mathcal{H}_M$ can be taken as the discrete set of states diagonalizing the Hamiltonian of the Majorana theory in this quantization scheme:

$$|k_1,\ldots,k_n\rangle_M,\ k_j\in\mathbb{N},\ n=0,1,2,\ldots,\ k_1<\cdots<k_n,$$

with vacuum denoted by $|vac\rangle_M$. These states correspond to eigenvalues $\lambda_1+\ldots+\lambda_n$. States with different orderings of energy levels are defined by the fact that exchanging the positions of two arguments $k_i,\ k_j$ brings a factor of (-1). In order to obtain a precise correspondence between the Dirac Hilbert space and a tensor product of two copies of the Majorana Hilbert space, define one-particle states $|k\rangle_a$ and $|k\rangle_b$ in the Dirac theory by

$$|k\rangle_a=\frac{1}{\sqrt{2}}(\,|k\rangle_+ + |k\rangle_-\,),\quad |k\rangle_b=\frac{i}{\sqrt{2}}(\,|k\rangle_+ - |k\rangle_-\,),\tag{2.3}$$

and multi-particle states involving states of type a and b by forming exterior products of these one-particle states. Then,

$$|vac\rangle=|vac\rangle_M\otimes|vac\rangle_M$$

and

$$|k_1,\ldots,k_n,k'_1,\ldots,k'_m\rangle_{\underbrace{a,a,\ldots}_{n},\underbrace{b,b,\ldots}_{m}}=|k_1,\ldots,k_n\rangle_M\,\otimes\,|k'_1,\ldots,k'_m\rangle_M.$$

Here we have fixed some of the phases by requiring the charge conjugation symmetry in the Dirac theory to be implemented by

$$|k\rangle_+\leftrightarrow|k\rangle_-,\quad \Psi_R^\dagger\leftrightarrow\Psi_R,\quad \Psi_L^\dagger\leftrightarrow-\Psi_L.$$

In what follows, we will omit the subscript M on Majorana states unless required for clarity.

The expected correspondence (1.1) between fields $\mathcal{O}^{(+)}$ and $\mathcal{O}^{(-)}$ in the Dirac theory and fields σ and μ in the Majorana theory, and the correspondence described above between the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_M$ of both theories, allow us to write matrix elements in $\mathcal{H}$ of the fields $\mathcal{O}^{(+)}$ and $\mathcal{O}^{(-)}$ in terms of matrix elements in $\mathcal{H}_M$ of the fields σ and μ. Having expressions for form factors of the fields $\mathcal{O}^{(+)}$ and $\mathcal{O}^{(-)}$ [4], this in turn gives us expressions for form factors of the fields σ and μ. We will verify in the next sections that one can indeed define form factors of a spin field σ acting on the Majorana Hilbert space $\mathcal{H}_M$ by the identification

$$\langle vac|\mathcal{O}^{(+)}|k_1,\ldots,k_n,k'_1,\ldots,k'_m\rangle_{\underbrace{a,a,\ldots}_{n},\underbrace{b,b,\ldots}_{m}}=\langle vac|\sigma|k_1,\ldots,k_n\rangle\langle vac|\sigma|k'_1,\ldots,k'_m\rangle\tag{2.4}$$

where $\mathcal{O}^{(+)}$ is defined in (1.2). In particular, from the vacuum expectation value of $\mathcal{O}^{(+)}$, the vacuum expectation value $\langle\sigma\rangle\equiv\langle vac|\sigma|vac\rangle$ of the field σ is given by (1.4). Here and below, the fields of which we take matrix elements are assumed to be at the center of the Poincaré disk. Using their transformation properties under the $SU(1,1)$ isometry group, they can always be translated to any other points in the Poincaré disk. In a similar way, we will verify that one can define form factors of a disorder field μ acting on $\mathcal{H}_M$ by[3]

$$\langle vac|\mathcal{O}^{(-)}|k_1,\ldots,k_n,k'_1,\ldots,k'_m\rangle_{\underbrace{a,a,\ldots}_{n},\underbrace{b,b,\ldots}_{m}}=(-1)^n\langle vac|\mu|k_1,\ldots,k_n\rangle\langle vac|\mu|k'_1,\ldots,k'_m\rangle\tag{2.5}$$

where the field $\mathcal{O}^{(-)}$ is defined in (1.2). In particular, the vacuum expectation value of the disorder field is zero: $\langle\mu\rangle=0$.

[3]The sign $(-1)^n$ comes from the identification

$$\mathcal{O}^{(-)}(x)\psi_a(x_1)\cdots\psi_a(x_n)\,\psi_b(x'_1)\cdots\psi_b(x'_m)=(-1)^n\,(\mu(x)\psi(x_1)\cdots\psi(x_n))\,\otimes\,(\mu(x)\psi(x'_1)\cdots\psi(x'_m)),$$

which can be obtained, for instance, by analyzing the OPE's in the Dirac and in the Majorana theories.

3 Two-particle form factors of the scaling fields $\mathcal{O}_{\pm\frac{1}{2}}$ in the Dirac theory

In this section we will specialize some of the results of [4] to the two-particle form factors of the scaling fields $\mathcal{O}_{\pm\frac{1}{2}}$ evaluated in the discrete basis discussed in the previous section. These form factors will then be used to construct form factors of spin and disorder fields in the subsequent sections.

Define the functions $f_\pm(k_1, k_2)$ by

$$f_\pm(k_1,k_2) \equiv \frac{\langle vac|\mathcal{O}_{\pm\frac{1}{2}}|k_1,k_2\rangle_{+,-}}{\langle\mathcal{O}_{\pm\frac{1}{2}}\rangle}$$

where the field $\mathcal{O}_{\pm\frac{1}{2}}$ are at the center of the Poincaré disk. Formulas in Appendix B.2 of [4] give expressions for these form factors (they were obtained using methods of angular quantization). Multiplying these expressions by a phase factor $i^{k_1+k_2+2}$ for convenience (this corresponds to redefining the eigenstates by multiplying them by a phase), we have

$$f_\pm(k_1,k_2) = 2^{2\nu+1} i^{k_2-k_1+1} \sqrt{\frac{\Gamma(1+2\nu+k_1)\Gamma(1+2\nu+k_2)}{k_1!k_2!}}\, \frac{G_{\pm;k_1,k_2}}{\Gamma(1+2\nu)^2\cos(\pi\nu)} \tag{3.1}$$

where

$$G_{\pm;\,k_1,k_2} = \sum_{m_1=0}^{k_1}\sum_{m_2=0}^{k_2} \frac{(-k_1)_{m_1}(-k_2)_{m_2}2^{m_1+m_2}}{(2\nu+1)_{m_1}(2\nu+1)_{m_2}m_1!m_2!} H_{\pm;\,m_1,m_2}$$

with

$$\begin{aligned} H_{\pm;\,m_1,m_2} &= \frac{\Gamma(1+\nu\pm1/2+m_1)\Gamma(1+\nu\pm1/2+m_2)}{\Gamma(1-\nu\pm1/2)\Gamma(1+\nu\pm1/2)}\times \\ &\quad \times\, {}_3F_2(1,1+\nu\pm1/2+m_1,1+\nu\pm1/2+m_2;1+\nu\pm1/2,1-\nu\pm1/2;1). \end{aligned}$$

The ${}_3F_2$ hypergeometric function above can be evaluated in closed form, for any given integer m_1 and m_2, in terms of Gamma functions and rational functions of ν. It can be checked that

$$f_+(k_1,k_2) = -f_-(k_2,k_1).$$

As this identity relates $f_-(k_2,k_1)$ to $f_+(k_1,k_2)$, we need only use $f_+(k_1,k_2)$ in the following sections, which we will denote by $f(k_1,k_2)$. It can be verified that this function satisfies the following identities:

$$\begin{aligned} f(k_1,k_2) &= (-1)^{\frac{k_1+k_2}{2}}\sqrt{f(k_1,k_1)}\sqrt{f(k_2,k_2)} \quad (k_1+k_2 \text{ even}) \\ f(k_1,k_2) &= -f(k_2,k_1) \quad (k_1 \text{ or } k_2 \text{ odd}) \\ f(k_1,k_2) &= 0 \quad (k_1 \text{ and } k_2 \text{ odd}). \end{aligned} \tag{3.2}$$

Of course, the last identity is just a consequence of the first and the second. In the first identity, as well as in equation (3.1) and in other equations below, square roots $\sqrt{z}$ assume their branch delimited by the region $-\pi \le \arg(z) < \pi$ with $\sqrt{z} \ge 0$ for $\arg(z) = 0$.

4 Form factors of spin field

We now verify the factorization properties of the field $\mathcal{O}^{(+)}$ as defined by (1.2) and calculate the multi-particle form factors of the spin field σ in the Majorana theory. Using formulas of Appendix B.3 of [4], which essentially state that multi-particle form factors of the fields $\mathcal{O}_\alpha$ can be evaluated in terms of their two-particle form factors through Wick's theorem, the multi-particle form factors of $\mathcal{O}^{(+)}$ in the Dirac theory can be written in the form

$$\frac{1}{\langle \mathcal{O}^{(+)} \rangle} \langle vac | \mathcal{O}^{(+)} | k_1, \ldots, k_n, k'_1, \ldots, k'_n \rangle_{\underbrace{+,+,\ldots}_{n}, \underbrace{-,-,\ldots}_{n}} = (-1)^{n(n-1)/2} \frac{\det(A_+) + \det(A_-)}{2} \tag{4.1}$$

where the $n \times n$ matrices A_+ and A_- have matrix elements

$$[A_+]_{ij} = f(k_i, k'_j), \quad [A_-]_{ij} = -f(k'_j, k_i).$$

In Appendix A, it is shown that

$$\frac{\det(A_+) + \det(A_-)}{2} = \det\left(\frac{A_+ + A_-}{2} \right). \tag{4.2}$$

This equation simply means that we can calculate the form factor in (4.1) by using Wick's theorem to pair energy levels in the asymptotic state, the contractions being given by

$$\overbracket{|k_1\rangle_+ \quad |k_2\rangle_-} = \frac{1}{2}(f(k_1, k_2) - f(k_2, k_1)) = \left\{ \begin{array}{l} 0 \quad (k_1 + k_2 \text{ even}) \\ f(k_1, k_2) \quad (k_1 + k_2 \text{ odd}) \end{array} \right. \tag{4.3}$$

where we used properties (3.2) (other contractions being zero). Changing basis to $|k\rangle_a$ and $|k\rangle_b$ through (2.3), we can calculate the multi-particle form factors by using Wick's theorem with the contractions

$$\overbracket{|k_1\rangle_a \quad |k_2\rangle_a} = \overbracket{|k_1\rangle_b \quad |k_2\rangle_b} = \overbracket{|k_1\rangle_+ \quad |k_2\rangle_-}$$

and

$$\overbracket{|k_1\rangle_a \quad |k_2\rangle_b} = 0.$$

We then obtain

$$\frac{1}{\langle \mathcal{O}^{(+)} \rangle} \langle vac | \mathcal{O}^{(+)} | k_1, \ldots, k_n, k'_1, \ldots, k'_m \rangle_{\underbrace{a,a,\ldots}_{n}, \underbrace{b,b,\ldots}_{m}} = \mathrm{Pf}(\Sigma)\mathrm{Pf}(\Sigma') \tag{4.4}$$

for n and m even, the form factor being zero otherwise. Here Pf means the Pfaffian of a matrix. The $n \times n$ matrix Σ and the $m \times m$ matrix Σ' have matrix elements

$$[\Sigma]_{j,l} = \overbracket{|k_j\rangle_+ \quad |k_l\rangle_-} , \quad [\Sigma']_{j,l} = \overbracket{|k'_j\rangle_+ \quad |k'_l\rangle_-} .$$

The factorized form of the right-hand side of (4.4) strongly suggests that we can identify the field $\mathcal{O}^{(+)}$ in the Dirac theory on the Poincaré disk with a tensor product of spin fields σ in two independent copies of the Majorana theory on the Poincaré disk, as in (1.1). Comparing with (2.4), equation (4.4) then leads to the form factors of the spin field:

$$\frac{\langle vac | \sigma | k_1, \ldots, k_n \rangle}{\langle \sigma \rangle} = \mathrm{Pf}(\Sigma) \tag{4.5}$$

for n even, zero otherwise. In particular, this gives the two-particle form factor as

$$\frac{\langle vac | \sigma | k_1, k_2 \rangle}{\langle \sigma \rangle} = \left\{ \begin{array}{l} 0 \quad (k_1 + k_2 \text{ even}) \\ f(k_1, k_2) \quad (k_1 + k_2 \text{ odd}) \end{array} \right. \tag{4.6}$$

(where we recall that $f(k_1, k_2) = f_+(k_1, k_2)$ is given by (3.1)), and says that we can calculate multi-particle form factors of the spin field by using Wick's theorem to pair energy levels in the asymptotic states, the contractions being given by the two-particle form factors.

Although it is not straightforward, it is possible to verify that the two-particle form factor (4.6) is in agreement with results of [12], which directly give the expression:

$$\frac{\langle vac|\sigma|k_1,k_2\rangle}{\langle\sigma\rangle} = (-1)^{\frac{k_1+k_2+1}{2}} \frac{\sqrt{k_2(2\nu+k_2)}}{\pi(1+2\nu+k_1+k_2)} \sqrt{\frac{\Gamma\left(\nu+\frac{1}{2}+\frac{k_1}{2}\right)\Gamma\left(\nu+\frac{k_2}{2}\right)\Gamma\left(\frac{1}{2}+\frac{k_1}{2}\right)\Gamma\left(\frac{k_2}{2}\right)}{\Gamma\left(\nu+1+\frac{k_1}{2}\right)\Gamma\left(\nu+\frac{1}{2}+\frac{k_2}{2}\right)\Gamma\left(1+\frac{k_1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{k_2}{2}\right)}}$$

for k_1 even and k_2 odd. For k_1 odd and k_2 even, one can use $\langle vac|\sigma|k_1,k_2\rangle = -\langle vac|\sigma|k_2,k_1\rangle$, and in other cases the two-particle form factor is zero. Properties and significance of this expression will be discussed in [12].

5 Form factors of disorder field

Now we proceed to verify the factorization properties of the field $\mathcal{O}^{(-)}$ and to calculate the multi-particle form factors of the disorder field μ in the Majorana theory. As in the previous section, using definitions (1.2) and formulas of Appendix B.3 of [4], the multi-particle form factor of the field $\mathcal{O}^{(-)}$ in the Dirac theory can be written in the form

$$\frac{1}{\langle\mathcal{O}^{(+)}\rangle}\langle vac|\mathcal{O}^{(-)}|k_1,\ldots,k_n,k_1',\ldots,k_n'\rangle_{\underbrace{+,+\ldots}_{n},\underbrace{-,-\ldots}_{n}} = (-1)^{n(n-1)/2}\frac{\det(A_+)-\det(A_-)}{2} \tag{5.1}$$

where, again as in the previous section, the $n\times n$ matrices A_+ and A_- have matrix elements

$$[A_+]_{ij} = f(k_i,k_j'),\quad [A_-]_{ij} = -f(k_j',k_i).$$

In Appendix A, it is shown that

$$\frac{\det(A_+)-\det(A_-)}{2} = \mathrm{Res}_w \det(A_+(w)) \tag{5.2}$$

where $A_+(w)$ is a matrix with matrix elements depending on an auxiliary (formal) variable w:

$$[A_+(w)]_{ij} = f(k_i,k_j')\cdot\left\{\begin{array}{l} w^{-1}\quad (k_i \text{ and } k_j' \text{ even}) \\ 1\quad (k_i \text{ or } k_j' \text{ odd}).\end{array}\right.$$

In equation (5.2), the symbol Res_w is just a convenient way of saying that one must keep only the coefficient of the monomial w^{-1} in the determinant $\det(A_+(w))$, that is, one must take the formal residue in the variable w. Equation (5.2) means that we can calculate the form factor (5.1) by using Wick's theorem with contractions given by

$$\overbrace{|k_1\rangle_+\ |k_2\rangle_-} = f(k_1,k_2)\cdot\left\{\begin{array}{l} w^{-1}\quad (k_1 \text{ and } k_2 \text{ even}) \\ 1\quad (k_1 \text{ or } k_2 \text{ odd})\end{array}\right.$$

(other contractions being zero) and by taking the formal residue in w of the resulting sum of products of contractions. Changing basis to $|k\rangle_a$ and $|k\rangle_b$, we can calculate the multi-particle form factors by using Wick's theorem with the contractions

$$\overbrace{|k_1\rangle_a\ |k_2\rangle_a} = \overbrace{|k_1\rangle_b\ |k_2\rangle_b} = \frac{1}{2}(f(k_1,k_2)-f(k_2,k_1)) = \frac{\langle vac|\sigma|k_1,k_2\rangle}{\langle\sigma\rangle}$$

and

$$\overbrace{|k_1\rangle_a\ |k_2\rangle_b} = -\frac{i}{2}w^{-1}(f(k_1,k_2)+f(k_2,k_1)) = -i\,w^{-1}\,(-1)^{\frac{k_1}{2}}\sqrt{f(k_1,k_1)}\,(-1)^{\frac{k_2}{2}}\sqrt{f(k_2,k_2)}$$

and by taking the residue in w (here we used the first and second equations of (3.2)). Then we find

$$\frac{1}{\langle\mathcal{O}^{(+)}\rangle}\langle vac|\mathcal{O}^{(-)}|k_1,\ldots,k_n,k'_1,\ldots,k'_m\rangle_{\underbrace{a,a,\ldots}_{n},\underbrace{b,b,\ldots}_{m}} =$$

$$-\left(\sum_{j=1}^{n}(-1)^{j-1}\,(-1)^{\frac{k_j}{2}}\sqrt{if(k_j,k_j)}\;\frac{\langle vac|\sigma|k_1,\ldots,\widehat{k_j},\ldots,k_n\rangle}{\langle\sigma\rangle}\right)\times$$
$$\times\left(\sum_{j=1}^{m}(-1)^{j-1}\,(-1)^{\frac{k'_j}{2}}\sqrt{if(k'_j,k'_j)}\;\frac{\langle vac|\sigma|k'_1,\ldots,\widehat{k'_j},\ldots,k'_m\rangle}{\langle\sigma\rangle}\right) \tag{5.3}$$

for n and m odd, the form factor being zero otherwise (where the hat means omission of the argument). Again, as in the case of the field $\mathcal{O}^{(+)}$, the factorized form of the right-hand side of (5.3) strongly suggests that we can identify the field $\mathcal{O}^{(-)}$ in the Dirac theory on the Poincaré disk with a tensor product of disorder fields μ in two independent copies of the Majorana theory on the Poincaré disk, as in (1.1). Comparing with (2.5), equation (5.3) then leads to the form factors of the disorder field:

$$\langle vac|\mu|k_1,\ldots,k_n\rangle = \sum_{j=1}^{n}(-1)^{j-1}\langle vac|\mu|k_j\rangle\;\frac{\langle vac|\sigma|k_1,\ldots,\widehat{k_j},\ldots,k_n\rangle}{\langle\sigma\rangle} \tag{5.4}$$

for n odd, zero otherwise. Here the one-particle form factor $\langle vac|\mu|k\rangle$, up to a sign factor independent of k, is given by

$$\langle vac|\mu|k\rangle = \langle\sigma\rangle\,(-1)^{\frac{k}{2}}\sqrt{if(k,k)} \tag{5.5}$$

(where we recall that $f(k,k)=f_+(k,k)$ is given by (3.1)). From the last property of (3.2), the one-particle form factor $\langle vac|\mu|k\rangle$ is non-zero only for k even, and it is real since $if(k,k)$ is real and positive. The ambiguous sign factor was chosen to make $\langle vac|\mu|0\rangle$ positive. This ambiguity is related to the ambiguity in the choice of branch on which to evaluate the correlation function $\langle\psi(x)\mu(y)\rangle$ in the Majorana theory. The precise relation between these two ambiguities will be clarify in [12].

It is possible to show that the one-particle form factor (5.5) is in agreement with results of [12], which directly give the expression:

$$\frac{\langle vac|\mu|k\rangle}{\langle\sigma\rangle} = (-1)^{\frac{k}{2}}\sqrt{\frac{\Gamma\left(\nu+\frac{1}{2}+\frac{k}{2}\right)\Gamma\left(\frac{1}{2}+\frac{k}{2}\right)}{\pi\Gamma\left(\nu+1+\frac{k}{2}\right)\Gamma\left(1+\frac{k}{2}\right)}}\quad\text{for } k \text{ even, 0 otherwise.}$$

Properties and significance of this expression will be discussed in [12].

6 Conclusion

In [4], we had found expressions for form factors of particular scaling fields in the Dirac theory on the Poincaré disk. One can relate the Hilbert space of the Dirac theory on the Poincaré disk to two independent copies of the Hilbert space of the Majorana theory on the Poincaré disk in a way similar to what can be done on flat space. If one can then factorize the fields $\mathcal{O}^{(+)}$ and $\mathcal{O}^{(-)}$ (1.2) in the Dirac theory in terms of spin and disorder fields in the Majorana theory in the way it is done on flat space, as in (1.1), then one can obtain, from the expressions of [4], expressions for form factors of spin and disorder fields in the Majorana theory on the Poincaré disk. In the present paper, we verified that the expressions for form factors of $\mathcal{O}^{(+)}$ and $\mathcal{O}^{(-)}$ in the Dirac theory on the Poincaré disk that we found previously agree with this factorization property. As a result, we obtained expressions for form factors and vacuum expectation values of spin and disorder fields in the Majorana theory on the Poincaré disk.

Spin and disorder fields in the Majorana theory on the Poincaré disk should be related to spin and disorder variables in the lattice Ising model on the Poincaré disk. The analysis of their correlation functions and of their form factors should then give information concerning the statistical properties of such an Ising model near "criticality". This clearly is a very interesting prospect, given that the effect of a curvature on the critical point of a statistical model is not well understood. Such an analysis is currently being developed in some depth [12]; the present paper in particular provides a link between the work [4] and this future publication.

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A Proof of formulas (4.2) and (5.2)

We see that $[A_+]_{ij} = [A_-]_{ij}$ when k_i or k'_j is odd, and that $[A_+]_{ij} = -[A_-]_{ij}$ when k_i and k'_j are even. Also, when k_i and k'_j are even, the matrix elements factorize. Arrange the order of the k_i's and k'_j's so that all even ones are at the beginning: k_i is even if and only if $i < I$ and k'_j is even if and only if $j < J$. Then the matrices A_+ and A_- have the following form:

$$A_+ = M + S, \quad A_- = M - S \tag{A.1}$$

where

$$[S]_{ij} = s_i s'_j, \quad s_i = 0 \text{ if } i \geq I, \quad s'_j = 0 \text{ if } j \geq J, \quad [M]_{ij} = 0 \text{ if } i < I \text{ and } j < J.$$

Using the technique of minors to calculate determinants, the determinant of A_+, for instance, can always be written as a sum $\sum_i a_i$. In this sum, each term a_i can be factorized as $a_i = b_i \det(B_i)$, where $\det(B_i)$ is the determinant of a sub-matrix B_i of A_+ that has the same horizontal dimension as that of S, and that contains a certain number (if any) of full lines of S. When written in such a way, in each term $b_i \det(B_i)$, the only factor where matrix elements $[S]_{jk}$ of S enter is in the determinant $\det(B_i)$. A similar expression can be written for $\det(A_-)$, with the sub-matrix S replaced by the sub-matrix $-S$. If more than one line of S is contained in B_i, then $\det(B_i) = 0$ because the elements of S factorize. If only one line of S is contained in B_i, then the same term will appear in both the expressions for $\det(A_+)$ and for $\det(A_-)$ but with opposite signs. If no line of S is contained in B_i, then the same term will appear in both the expressions for $\det(A_+)$ and for $\det(A_-)$ (with the same sign).

From these properties, in the sum of the expressions for the determinants of A_+ and A_-, the only terms remaining are those containing no elements of S as factors. Hence, $\det(A_+) + \det(A_-) = 2\det(M)$. This proves Equation (4.2). On the other hand, in the difference of the expressions for the determinants of A_+ and A_-, the only terms remaining are the terms which are linear in elements of S. This prescription can be implemented by multiplying the elements of the sub-matrix S of A_+ by the inverse of a formal variable, w^{-1}, thus forming a matrix which we denote $A_+(w)$, and by taking the formal residue of the determinant of $A_+(w)$. This proves Equation (5.2).

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