Homework 2 – due 19 March 2009

A model of quantum field theory possesses one particle of mass M in its spectrum, without any additional quantum number. Hence, a basis of states, for instance the *in* basis, can be taken as $|\theta_1, \theta_2, \ldots, \theta_n\rangle$ for $n = 0, 1, 2, 3, \ldots$ (denoted $|\text{vac}\rangle$ in the case $n = 0$), where $\theta_1 > \theta_2 > \ldots > \theta_n$. In terms of these, we have the decomposition of the identity:

$$\mathbf{1} = \sum_{n=0}^{\infty} \frac{1}{(2\pi)^n} \int_{\theta_1 > \ldots > \theta_n} d\theta_1 \cdots d\theta_n |\theta_1, \ldots, \theta_n\rangle\langle\theta_1, \ldots, \theta_n|. \tag{1}$$

For a local field $\hat{\mathcal{O}}(x,t)$, the matrix elements

$$F^{\hat{\mathcal{O}}}(\theta_1, \ldots, \theta_n) = \langle\text{vac}|\hat{\mathcal{O}}(0,0)|\theta_1, \ldots, \theta_n\rangle$$

are called *form factors*. In fact, they are defined here for ordered rapidities, but as functions of the rapidities, we can analytically continue them to any order, and to complex rapidities. Form factors are these analytically continued matrix elements. A straightforward property of form factors is:

- Relativistic covariance:

$$F^{\hat{\mathcal{O}}}(\theta_1 + \beta, \ldots, \theta_n + \beta) = e^{s(\hat{\mathcal{O}})\beta} F^{\hat{\mathcal{O}}}(\theta_1, \ldots, \theta_n)$$

  where $s(\hat{\mathcal{O}})$ is the spin of $\hat{\mathcal{O}}$.

If the model under consideration is integrable, form factors have additional properties, which are called, by a slight abuse of language, *form-factor equations*. They in fact constitute a (generalisation of a) *Riemann-Hilbert problem*, and it is believed that the set of all solutions to these properties, given a scattering matrix, is the set of all local fields of the model with that scattering matrix. Given the two-particle scattering matrix $S(\theta_1 - \theta_2)$ (which just depends on the difference of rapidities by relativistic invariance), the form-factor equations are as follows:

- Meromorphicity: as functions of the variable $\theta_i - \theta_j$, for any $i, j \in \{1, \ldots, n\}$, form factors are analytic inside the strip $0 < \text{Im}(\theta_i - \theta_j) < 2\pi$ except for simple poles;

- Generalized Watson's theorem:

$$F^{\hat{\mathcal{O}}}(\theta_1, \ldots, \theta_j, \theta_{j+1}, \ldots, \theta_n) = S(\theta_j - \theta_{j+1}) F^{\hat{\mathcal{O}}}(\theta_1, \ldots, \theta_{j+1}, \theta_j, \ldots, \theta_n)$$

- Semi-locality:

$$F^{\hat{\mathcal{O}}}(\theta_1, \ldots, \theta_{n-1}, \theta_n + 2\pi\mathrm{i}) = (-1)^{f_{\hat{\mathcal{O}}} f_{\hat{\Psi}}} e^{2\pi\mathrm{i}\omega(\hat{\mathcal{O}}, \Psi)} F^{\hat{\mathcal{O}}}(\theta_n, \theta_1, \ldots, \theta_{n-1})$$

  where $f_{\hat{\mathcal{O}}}$ is 1 if $\hat{\mathcal{O}}$ is fermionic, 0 if it is bosonic, $\hat{\Psi}$ is the fundamental field associated to the particle, and $\omega(\hat{\mathcal{O}}, \hat{\Psi})$ is the *semi-locality index* (or mutual locality index) of $\hat{\mathcal{O}}$ with respect to $\hat{\Psi}$;

- Kinematic pole: as a function of the variable θ_n, there are poles at $\theta_j + i\pi$ for $j \in \{1,\ldots,n-1\}$. For $j = n-1$, the residue is given by

$$-\mathrm{i}F^{\hat{\mathcal{O}}}(\theta_1,\ldots,\theta_n) \sim \frac{F(\theta_1,\ldots,\theta_{n-2})}{\theta_n - \theta_{n-1} - i\pi} \times \\ \left(1 - (-1)^{f_{\hat{\mathcal{O}}} f_{\hat{\Psi}}} e^{2\pi i \omega(\hat{\mathcal{O}},\hat{\Psi})} S(\theta_{n-1}-\theta_{n-2}) S(\theta_{n-1}-\theta_{n-3}) \cdots S(\theta_{n-1}-\theta_1)\right).$$

(In fact, there are other form factor equations, having to do with the possible presence of bound states; also, the semi-locality equation can be modified to account for local fields with more general semi-locality property with respect to the fundamental field $\hat{\Psi}$.)

In our model, two Hermitian homoneous local fields, $\hat{\epsilon}(x,t)$ and $\hat{\sigma}(x,t)$, have the following form factors:

$$\begin{aligned} \langle \mathrm{vac}|\hat{\epsilon}(0,0)|\mathrm{vac}\rangle &= aM \\ \langle \mathrm{vac}|\hat{\epsilon}(0,0)|\theta_1,\theta_2\rangle &= -\frac{\mathrm{i}M}{2\pi}\sinh\left(\frac{\theta_1-\theta_2}{2}\right) \end{aligned}$$

and

$$\langle \mathrm{vac}|\hat{\sigma}(0,0)|\theta_1,\theta_2,\ldots,\theta_{2l}\rangle = bM^{\frac{1}{8}}\mathrm{i}^l \prod_{1\leq j<k\leq 2l} \tanh\left(\frac{\theta_k-\theta_j}{2}\right) \qquad (l = 0,1,2,\ldots)$$

for some constants a and b, other form factors being exacly zero.

1. Using the decomposition of the identity, write down the two-point function $\langle \mathrm{vac}|\hat{\epsilon}(x,t)\hat{\epsilon}(0,0)|\mathrm{vac}\rangle$ as a sum of convergent integrals involving the relativistic space-like distance $s = \sqrt{x^2-t^2}$ (there is no need to evaluate the integrals). Do the same for $\langle \mathrm{vac}|\hat{\sigma}(x,t)\hat{\sigma}(0,0)|\mathrm{vac}\rangle$, but using the decomposition of the identity up to, and including, the two-particle sector only.

2. Deduce, from the form factors, the spins of $\hat{\epsilon}$ and of $\hat{\sigma}$. Knowing that the model is integrable, deduce the two-particle scattering matrix. Knowing further that both fields have bosonic statistics, deduce the semi-locality index of the fields $\hat{\epsilon}$ and $\hat{\sigma}$. Verify that the form factors satisfy the correct remaining form-factor equations.

The model we are dealing with, characterised by the Hilbert space and the scattering matrix you found, is the so-called free Majorana fermion model. It turns out that it is the scaling limit of the two-dimensional lattice Ising model. The field $\hat{\sigma}$ is the so-called spin field (the scaling limit of the spin variable σ_j in the classical Ising model), and the field $\hat{\epsilon}$ is the energy field (the scaling limit of the density $\sum_{j \text{ neighbour of } i} \sigma_i\sigma_j$ of energy of the classical Ising model). The name "spin field" for $\hat{\sigma}$ is not to be confused with its actual spin, a property it has as a quantum fields. The field $\hat{\epsilon}$ is not to be confused with the Hamiltonian density $\hat{h}$ of the quantum field theory model, which has different form factors.

Answers.

1.

In general, for $\hat{\mathcal{O}}(x)$ a homogeneous local fields ($\hat{\mathcal{O}}(x) = e^{-\mathrm{i}\hat{P}x}\hat{\mathcal{O}}(0)e^{\mathrm{i}\hat{P}x}$) that is Hermitian ($\hat{\mathcal{O}}(x)^\dagger = \hat{\mathcal{O}}(x)$), we have

$$\begin{aligned}
\langle \mathrm{vac}|\hat{\mathcal{O}}(x,t)\hat{\mathcal{O}}(0,0)|\mathrm{vac}\rangle &= \sum_{n=0}^{\infty}\frac{1}{(2\pi)^n}\int_{\theta_1>\ldots>\theta_n} d\theta_1\cdots d\theta_n \langle \mathrm{vac}|\hat{\mathcal{O}}(x,t)|\theta_1,\ldots,\theta_n\rangle\langle\theta_1,\ldots,\theta_n|\hat{\mathcal{O}}(0,0)|\mathrm{vac}\rangle \\
&= \sum_{n=0}^{\infty}\frac{1}{(2\pi)^n}\int_{\theta_1>\ldots>\theta_n} d\theta_1\cdots d\theta_n \\
&\qquad \langle \mathrm{vac}|e^{\mathrm{i}\hat{H}t-\mathrm{i}\hat{P}x}\hat{\mathcal{O}}(0,0)e^{-\mathrm{i}\hat{H}t+\mathrm{i}\hat{P}x}|\theta_1,\ldots,\theta_n\rangle\langle\theta_1,\ldots,\theta_n|\hat{\mathcal{O}}(0,0)|\mathrm{vac}\rangle \\
&= \sum_{n=0}^{\infty}\frac{1}{(2\pi)^n}\int_{\theta_1>\ldots>\theta_n} d\theta_1\cdots d\theta_n \\
&\qquad e^{-\mathrm{i}t\sum_{j=1}^n E_{\theta_j}+\mathrm{i}x\sum_{j=1}^n p_{\theta_j}}\langle \mathrm{vac}|\hat{\mathcal{O}}(0,0)|\theta_1,\ldots,\theta_n\rangle\langle\theta_1,\ldots,\theta_n|\hat{\mathcal{O}}(0,0)|\mathrm{vac}\rangle \\
&= \sum_{n=0}^{\infty}\frac{1}{(2\pi)^n}\int_{\theta_1>\ldots>\theta_n} d\theta_1\cdots d\theta_n \\
&\qquad e^{-\mathrm{i}t\sum_{j=1}^n E_{\theta_j}+\mathrm{i}x\sum_{j=1}^n p_{\theta_j}}\left|\langle \mathrm{vac}|\hat{\mathcal{O}}(0,0)|\theta_1,\ldots,\theta_n\rangle\right|^2.
\end{aligned}$$

In the case of the field $\hat{\epsilon}$, the only non-zero form factors are those with zero particle (the vacuum expectation value) and those with two particles. So, the full series simplifies to

$$\begin{aligned}
\langle \mathrm{vac}|\hat{\epsilon}(x,t)\hat{\epsilon}(0,0)|\mathrm{vac}\rangle &= |\langle \mathrm{vac}|\hat{\epsilon}(0,0)|\mathrm{vac}\rangle|^2 + \frac{1}{(2\pi)^2}\int_{\theta_1>\theta_2} d\theta_1 d\theta_2 \\
&\qquad e^{-\mathrm{i}t(E_{\theta_1}+E_{\theta_2})+\mathrm{i}x(p_{\theta_1}+p_{\theta_2})}\,|\langle \mathrm{vac}|\hat{\epsilon}(0,0)|\theta_1,\theta_2\rangle|^2 \\
&= |a|^2M^2 + \frac{M^2}{(2\pi)^4}\int_{\theta_1>\theta_2} d\theta_1 d\theta_2\, e^{-\mathrm{i}t(E_{\theta_1}+E_{\theta_2})+\mathrm{i}x(p_{\theta_1}+p_{\theta_2})}\sinh^2\left(\frac{\theta_1-\theta_2}{2}\right).
\end{aligned}$$

In order to get an expression that involves convergent integrals, expressed solely in terms of $s = \sqrt{x^2-t^2}$, we need to shift the θ_1 and θ_2 variables using relativistic invariance: the fact that the form factors only depend on the difference of the rapidities (which has to do with the fact that $\hat{\epsilon}$ is spinless). Note that it is possible to shift θ in $xp_\theta - tE_\theta$ in order to have only a p_θ part:

$$xp_{\theta+\beta} - tE_{\theta+\beta} = (x\cosh\beta - t\sinh\beta)p_\theta - (t\cosh\beta - x\sinh\beta)E_\theta$$

so we only need to choose β such that

$$\frac{t}{x} = \tanh\beta.$$

Clearly, then, $\sinh\beta = At$ and $\cosh\beta = Ax$ for some A, which can easily be evaluated through $\cosh\beta^2 - \sinh\beta^2 = 1 \Rightarrow A = 1/\sqrt{x^2-t^2}$. Hence, we have

$$xp_{\theta+\beta} - tE_{\theta+\beta} = \sqrt{x^2-t^2}p_\theta.$$

Doing this shift for both θ_1 and θ_2, we find

$$\langle \mathrm{vac}|\hat{\epsilon}(x,t)\hat{\epsilon}(0,0)|\mathrm{vac}\rangle = |a|^2M^2 + \frac{M^2}{(2\pi)^4}\int_{\theta_1>\theta_2} d\theta_1 d\theta_2\, e^{\mathrm{i}s(p_{\theta_1}+p_{\theta_2})}\sinh^2\left(\frac{\theta_1-\theta_2}{2}\right)$$

where $s = \sqrt{x^2 - t^2}$. In other words, a shift of θ is just a relativistic boost, which preserves the relativistic (space-like) distance $\sqrt{x^2 - t^2}$. Hence, by invariance under shift of rapidities, we can just replace the space-time point (x, t) by any other space-time point at the same relativistic distance from $(0, 0)$ (and in the same region, here space-like), for instance the point $(\sqrt{x^2 - t^2}, 0)$. Finally, in order to have a convergent integral, we shift the integration contour itself, $\theta \mapsto \theta + \mathrm{i}\pi/2$, using $p_{\theta+\mathrm{i}\pi/2} = \mathrm{i}E_\theta$, which gives

$$\langle \mathrm{vac}|\hat{\epsilon}(x,t)\hat{\epsilon}(0,0)|\mathrm{vac}\rangle = |a|^2M^2 + \frac{M^2}{(2\pi)^4}\int_{\theta_1>\theta_2} d\theta_1 d\theta_2\, e^{-s(E_{\theta_1}+E_{\theta_2})}\sinh^2\left(\frac{\theta_1-\theta_2}{2}\right).$$

This is nicely convergent, and can be evaluted in terms of modified Bessel functions.

For the field $\hat{\sigma}$, the same principle holds, but the form-factor series expansion is an infinite series. If we truncate it to the two-particle sector, we get something very similar to what we got for the $\hat{\epsilon}$ field:

$$\langle \mathrm{vac}|\hat{\sigma}(x,t)\hat{\sigma}(0,0)|\mathrm{vac}\rangle = |b|^2M^{\frac{1}{4}} + \frac{|b|^2M^{\frac{1}{4}}}{(2\pi)^2}\int_{\theta_1>\theta_2} d\theta_1 d\theta_2\, e^{-s(E_{\theta_1}+E_{\theta_2})}\tanh^2\left(\frac{\theta_2-\theta_1}{2}\right).$$

2.

The relativistic covariance equation tells us that under shift of θ's in the form factors, we get an overall factor that defines the spin. In both cases $\hat{\epsilon}$ and $\hat{\sigma}$, the form factors only depend on the difference of rapidities, hence are invariant under shift of all rapidities by a common constant, so both have zero spin.

If the model is integrable, then an exchange of two neighbouring rapidities gives a factor that is the scattering matrix (generalised Watson's theorem). For the two-particle form factor of $\hat{\epsilon}$, we see that exchanging θ_1 and θ_2 gives a minus sign, so we could say that the scattering matrix is $S(\theta) = -1$ (it is a constant, independent of θ). For the form factors of the $\hat{\sigma}$ field, observe that the product involves all rapidity differences $\theta_k - \theta_j$ with $k > j$, each of them appearing in exactly one factor. We could think of that as a product over all pairings of rapidities. Exchanging two rapidities will still give a product with all rapidity differences appearing exactly once, but some rapidity differences may be in a different order. Hence, all factors will be the same up to, possibly, a sign. A sign, in a given factor or pairing, occurs if and only if the order of the two rapidities in the resulting rapidity difference is $\theta_j - \theta_k$ for $j < k$ – the "wrong" order. But if we exchange only two neighbouring rapidities, all pairings will have the same correct order, except for the pairing that involves these two rapidities: only one factor acquires a minus sign, so there is an overall minus sign under exchange of rapidities. Hence, this is in agreement with $S(\theta) = -1$.

From the semi-locality form factor equation, if a field has bosonic statistics $f_{\hat{\mathcal{O}}} = 0$, then a shift by $2\pi\mathrm{i}$ of the last rapidity gives a form factor where this last rapidity is placed at the beginning, times a phase that directly defines the semi-locality index. For the $\hat{\epsilon}$ field, a shift by $2\pi\mathrm{i}$ of the two-particle form factor gives a minus sign times the same function, so that we have

$$\langle \mathrm{vac}|\hat{\epsilon}(0,0)|\theta_1,\theta_2+2\pi\mathrm{i}\rangle = -\langle \mathrm{vac}|\hat{\epsilon}(0,0)|\theta_1,\theta_2\rangle = \langle \mathrm{vac}|\hat{\epsilon}(0,0)|\theta_2,\theta_1\rangle$$

hence we find that $\omega(\hat{\epsilon}, \hat{\Psi}) = 0$ (or any integer). For the field $\hat{\sigma}$, we note that each factor containing θ_{2l} in the $2l$-particle form factor is invariant under a shift by $2\pi\mathrm{i}$:

$$\langle \mathrm{vac}|\hat{\sigma}(0,0)|\theta_1, \theta_2, \ldots, \theta_{2l} + 2\pi\mathrm{i}\rangle = \langle \mathrm{vac}|\hat{\sigma}(0,0)|\theta_1, \theta_2, \ldots, \theta_{2l}\rangle.$$

On the other hand, bringing the rapidity θ_{2l} back to the front involves $2l-1$ exchanges of neighbouring rapidities, and each exchange gives a factor -1. Since $2l-1$ is odd, we find an overall minus sign:

$$\langle \mathrm{vac}|\hat{\sigma}(0,0)|\theta_1, \theta_2, \ldots, \theta_{2l} + 2\pi\mathrm{i}\rangle = -\langle \mathrm{vac}|\hat{\sigma}(0,0)|\theta_{2l}, \theta_1, \theta_2, \ldots, \theta_{2l-1}\rangle.$$

Hence, $\omega(\hat{\sigma}, \hat{\Psi}) = 1/2$.

The remaining form factor equations are meromorphicity and the kinematic pole. Meromorphicity is clear for both fields. The two-particle form factor of the $\hat{\epsilon}$ field does not have any pole. But from the two-particle kinematic pole equation, the residue is proportional to

$$1 - (-1)^{f_{\hat{\mathcal{O}}} f_{\hat{\Psi}}} e^{2\pi\mathrm{i}\omega(\hat{\mathcal{O}}, \hat{\Psi})}.$$

This is exactly 0 for the field $\hat{\epsilon}$ (since it is bosonic and has semi-locality index 0). Hence this is fine. The four-particle form factor is zero, so its pole is also zero. But the two-particle form factor is non-zero, so we need to check that the kinematic pole equation also gives zero there. We get, for the residue in the case of $n = 4$ particles, the factor (using that $\hat{\epsilon}$ is bosonic and has semi-locality index 0)

$$1 - S(\theta_3 - \theta_2)S(\theta_3 - \theta_1).$$

This is indeed zero because the scattering matrix is just -1. For the field $\hat{\sigma}$, in any non-zero form factor, there is one factor (or pairing) that has a pole:

$$\tanh\left(\frac{\theta_{2l} - \theta_{2l-1}}{2}\right) \sim \frac{2}{\theta_{2l} - \theta_{2l-1} - \mathrm{i}\pi}.$$

The residue of that pole is evaluated by taking $\theta_{2l} = \theta_{2l-1} + \mathrm{i}\pi$ in all other factors – the pairings between θ_{2l} and $\theta_1, \theta_2, \ldots, \theta_{2l-2}$. These $+\mathrm{i}\pi$ shifts transform the tanh factors into coth factors, for all these pairings. But also, there are tanh factors corresponding to the pairings between θ_{2l-1} and $\theta_1, \theta_2, \ldots, \theta_{2l-2}$. Hence, for each coth factor, there is a corresponding tanh factor, so all these cancel out. This cancels all pairings involving either θ_{2l} or θ_{2l-1}. Hence, we find the pole

$$\frac{2}{\theta_{2l} - \theta_{2l-1} - \mathrm{i}\pi} b M^{\frac{1}{8}} \mathrm{i}^l \prod_{1 \le j < k \le 2l-2} \tanh\left(\frac{\theta_k - \theta_j}{2}\right)$$

which is just

$$\frac{2\mathrm{i}}{\theta_{2l} - \theta_{2l-1} - \mathrm{i}\pi} \langle \mathrm{vac}|\hat{\sigma}(0,0)|\theta_1, \theta_2, \ldots, \theta_{2l-2}\rangle.$$

From the kinematic pole equation, $\frac{2\mathrm{i}}{\theta_{2l} - \theta_{2l-1} - \mathrm{i}\pi}$ should be $\frac{\mathrm{i}}{\theta_{2l} - \theta_{2l-1} - \mathrm{i}\pi}$ times the factor

$$1 + S(\theta_{2l-1} - \theta_{2l-2})S(\theta_{2l-1} - \theta_{2l-3}) \cdots S(\theta_{2l-1} - \theta_1)$$

where we used the fact that $\hat{\sigma}$ is bosonic and has semi-locality index $1/2$. In the second term, there are exactly $2l-2$ scattering matrix factors; this is even and each scattering matrix is just -1, so this gives an overall $1 + 1 = 2$. Hence this is in agreement with our previous result.