Homework 1 – due 12 March 2009

A way of defining the free, real bosonic QFT model is by providing the various objects of the general QFT axioms. We start with the canonical algebra of the modes:

$$[\hat{A}(\theta), \hat{A}^\dagger(\theta')] = 4\pi\delta(\theta - \theta'), \quad [\hat{A}(\theta), \hat{A}(\theta')] = [\hat{A}^\dagger(\theta), \hat{A}^\dagger(\theta')] = 0 \tag{1}$$

where $\theta, \theta' \in \mathbb{R}$. Then:

- **Hilbert space.** We construct $\mathcal{H}$, the Hilbert space, as the Fock space over the algebra (1), with the Hermitian structure obtained by saying that † gives the Hermitian conjugate. That is, there is a vacuum state $|\text{vac}\rangle$ with

$$\hat{A}(\theta)|\text{vac}\rangle = 0 \quad \forall\theta$$

  and a basis of $\mathcal{H}$ is obtained by

$$|\theta_1, \ldots, \theta_n\rangle = \hat{A}^\dagger(\theta_1) \cdots \hat{A}^\dagger(\theta_n)|\text{vac}\rangle, \quad \theta_1 \geq \ldots \geq \theta_n.$$

- **Hamiltonian.** We construct the Hamiltonian operator as:

$$\hat{H} = \frac{1}{4\pi} \int d\theta \, m \cosh(\theta) \hat{A}^\dagger(\theta) \hat{A}(\theta) \tag{2}$$

  (integration symbols without limits are over $\mathbb{R}$). It is Hermitian and has the property that $\hat{H}|\text{vac}\rangle = 0$, and that the energies of all other staes are positive, as it should.

- **Relativistic invariance.** We construct the momentum operator and the generator of boosts as

$$\hat{P} = \frac{1}{4\pi} \int d\theta \, m \sinh(\theta) \hat{A}^\dagger(\theta) \hat{A}(\theta), \quad \hat{B} = \frac{1}{4\pi} \int d\theta \left(-i\frac{d}{d\theta} \hat{A}^\dagger(\theta)\right) \hat{A}(\theta). \tag{3}$$

  They are Hermitian and have the properties that $\hat{P}|\text{vac}\rangle = \hat{B}|\text{vac}\rangle = 0$ as they should again. The boost generator is slightly formal because of the derivative of the mode operator involved, but it can more precisely be defined through the corresponding group action on any state:

$$e^{i\alpha\hat{B}}|\theta_1, \ldots, \theta_n\rangle = |\theta_1 + \alpha, \ldots, \theta_n + \alpha\rangle. \tag{4}$$

  We check that they satisfy the correct Poincaré-algebra commutation relations:

$$[\hat{H}, \hat{P}] = 0, \quad [\hat{B}, \hat{P}] = i\hat{H}, \quad [\hat{B}, \hat{H}] = i\hat{P}. \tag{5}$$

- **Locality.** We write all these operators as integration of local Hermitian densities:

$$\hat{H} = \int dx\, \hat{h}(x), \quad \hat{P} = \int dx\, \hat{p}(x), \quad \hat{B} = \int dx\, x\hat{h}(x) \tag{6}$$

where

$$\begin{aligned}
\hat{h}(x) &= \int \frac{d\theta d\theta'}{(4\pi)^2}\Bigg[\left(\frac{-E_\theta E_{\theta'} - p_\theta p_{\theta'} + m^2}{2}\right)\left(e^{i(p_\theta+p_{\theta'})x}A(\theta')A(\theta) + h.c.\right) \\
&\quad + \left(E_\theta E_{\theta'} + p_\theta p_{\theta'} + m^2\right) e^{i(p_\theta - p_{\theta'})x} A^\dagger(\theta')A(\theta)\Bigg] \qquad (7)\\
\hat{p}(x) &= \int \frac{d\theta d\theta'}{(4\pi)^2}\Bigg[\left(\frac{E_\theta p_{\theta'} + E_{\theta'} p_\theta}{2}\right)\left(e^{i(p_\theta+p_{\theta'})x}A(\theta')A(\theta) + h.c.\right) \\
&\quad + \left(E_\theta p_{\theta'} + E_{\theta'} p_\theta\right) e^{i(p_\theta - p_{\theta'})x} A^\dagger(\theta')A(\theta)\Bigg]. \qquad (8)
\end{aligned}$$

We check that they satisfy the locality relations:

$$[\hat{h}(x), \hat{h}(x')] = [\hat{p}(x), \hat{p}(x')] = [\hat{h}(x), \hat{p}(x')] = 0 \quad \text{for} \quad x \neq x' \tag{9}$$

and the homogeneous density relations

$$[\hat{P}, \hat{h}(x)] = i\frac{d}{dx}\hat{h}(x), \quad [\hat{P}, \hat{p}(x)] = i\frac{d}{dx}\hat{p}(x). \tag{10}$$

——— **QUESTION 1**

Show explicitly that indeed, from the densities (7) and (8), we find the three equations of (6), that the relations (10) hold, and that we have

$$\langle \text{vac}|[\hat{h}(x), \hat{p}(x')]|\text{vac}\rangle = 0 \quad \text{for} \quad x \neq x'.$$

Answer. First, we use

$$\int dx\, e^{i(p_\theta - p_{\theta'})x} = 2\pi\delta(p_\theta - p_{\theta'}) = \frac{2\pi}{E_\theta}\delta(\theta-\theta'), \quad \int dx\, ix e^{i(p_\theta - p_{\theta'})x} = 2\pi\delta'(p_\theta - p_{\theta'}) = \frac{2\pi}{E_\theta}\frac{d}{d\theta}\left(\frac{\delta(\theta-\theta')}{E_\theta}\right),$$

so that we get

$$
\begin{aligned}
\int dx\, \hat{h}(x) &= \int \frac{d\theta}{8\pi E_\theta}\left[\left(\frac{-E_\theta^2+p_\theta^2+m^2}{2}\right)\left(A(\theta)^2+h.c.\right)+\left(E_\theta^2+p_\theta^2+m^2\right)A^\dagger(\theta)A(\theta)\right] \\
&= \int \frac{d\theta}{4\pi} E_\theta A^\dagger(\theta)A(\theta), \\
\int dx\, \hat{p}(x) &= \int \frac{d\theta}{8\pi E_\theta}\left[\left(\frac{-E_\theta p_\theta+E_\theta p_\theta}{2}\right)\left(A(\theta)^2+h.c.\right)+2E_\theta p_\theta A^\dagger(\theta)A(\theta)\right] \\
&= \int \frac{d\theta}{4\pi} p_\theta A^\dagger(\theta)A(\theta), \\
\int dx\, x\hat{h}(x) &= -i\int \frac{d\theta}{16\pi}\Bigg[-A(\theta)\frac{d}{d\theta}A(\theta)+\frac{p_\theta A(\theta)}{E_\theta}\frac{d}{d\theta}\frac{p_\theta A(\theta)}{E_\theta}+m^2\frac{A(\theta)}{E_\theta}\frac{d}{d\theta}\frac{A(\theta)}{E_\theta}+h.c. \\
&\qquad +2\left(\frac{d}{d\theta}A^\dagger(\theta)\right)A(\theta)+2\left(\frac{d}{d\theta}\frac{p_\theta A^\dagger(\theta)}{E_\theta}\right)\frac{p_\theta A(\theta)}{E_\theta}+2m^2\left(\frac{d}{d\theta}\frac{A^\dagger(\theta)}{E_\theta}\right)\frac{A(\theta)}{E_\theta}\Bigg] \\
&= -i\int \frac{d\theta}{16\pi}\Bigg[\frac{1}{2}\frac{d}{d\theta}\left(\left(-1+\frac{p_\theta^2}{E_\theta^2}+\frac{m^2}{E_\theta^2}\right)A(\theta)^2\right)+h.c. \\
&\qquad +2\left(1+\frac{p_\theta^2}{E_\theta^2}+\frac{m^2}{E_\theta^2}\right)\left(\frac{d}{d\theta}A^\dagger(\theta)\right)A(\theta)+\frac{d}{d\theta}\left(\frac{p_\theta^2}{E_\theta^2}+\frac{m^2}{E_\theta^2}\right)A^\dagger(\theta)A(\theta)\Bigg] \\
&= \frac{1}{4\pi}\int d\theta\left(-i\frac{d}{d\theta}\hat{A}^\dagger(\theta)\right)\hat{A}(\theta).
\end{aligned}
$$

As for the relations (10), we must use

$$
[\hat{P}, A(\theta)] = -p_\theta A(\theta), \quad [\hat{P}, A^\dagger(\theta)] = p_\theta A^\dagger(\theta)
$$

which immediately gives

$$
\begin{aligned}
&[\hat{P}, e^{i(p_\theta-p_{\theta'})x}A^\dagger(\theta')A(\theta)] = i\frac{d}{dx}\left(e^{i(p_\theta-p_{\theta'})x}A^\dagger(\theta')A(\theta)\right), \\
&[\hat{P}, e^{i(p_\theta+p_{\theta'})x}A(\theta')A(\theta)] = i\frac{d}{dx}\left(e^{i(p_\theta+p_{\theta'})x}A(\theta')A(\theta)\right).
\end{aligned}
$$

Finally, for evaluating the commutator $\langle \mathrm{vac}|[\hat{h}(x), \hat{p}(x')]|\mathrm{vac}\rangle$, we start by looking at $\langle \mathrm{vac}|\hat{h}(x)\hat{p}(x')|\mathrm{vac}\rangle$. Only the terms in $A^\dagger A^\dagger$ stay for the factor $\hat{p}(x')$, and only the terms in AA stay for the factor $\hat{h}(x)$. Hence, we have

$$
\begin{aligned}
\langle \mathrm{vac}|\hat{h}(x)\hat{p}(x')|\mathrm{vac}\rangle &= \langle \mathrm{vac}|\int \frac{d\theta d\theta'}{(4\pi)^2}\left(\frac{-E_\theta E_{\theta'}-p_\theta p_{\theta'}+m^2}{2}\right)e^{i(p_\theta+p_{\theta'})x}A(\theta')A(\theta)\times \\
&\quad \times \int \frac{d\beta d\beta'}{(4\pi)^2}\left(\frac{E_\beta p_{\beta'}+E_{\beta'}p_\beta}{2}\right)e^{-i(p_\beta+p_{\beta'})x'}A^\dagger(\beta')A^\dagger(\beta)|\mathrm{vac}\rangle
\end{aligned}
$$

There are two Wick contractions, but thanks to the symmetry of the factors in β, β', they lead to the same thing. Hence, we are left with

$$
\langle \mathrm{vac}|\hat{h}(x)\hat{p}(x')|\mathrm{vac}\rangle = \int \frac{d\theta d\theta'}{2(4\pi)^2}\left(-E_\theta E_{\theta'}-p_\theta p_{\theta'}+m^2\right)\left(E_\theta p_{\theta'}+E_{\theta'}p_\theta\right)e^{i(p_\theta+p_{\theta'})(x-x')}
$$

On the other hand, the term $\langle \text{vac}|\hat{p}(x')\hat{h}(x)|\text{vac}\rangle$ gives exactly the same answer, up to the change of sign $\theta \mapsto -\theta,\ \theta' \mapsto -\theta'$, which just changes the sign of the whole integral. Substracting, we get twice the same answer, and using the symmetry under $\theta \leftrightarrow \theta'$, we find

$$\langle \text{vac}|[\hat{h}(x), \hat{p}(x')]|\text{vac}\rangle \quad = \quad 2\int \frac{d\theta d\theta'}{(4\pi)^2}\left(-E_\theta E_{\theta'} - p_\theta p_{\theta'} + m^2\right) E_\theta p_{\theta'} e^{i(p_\theta+p_{\theta'})(x-x')}$$

In order to recognise that this is zero for $x \neq x'$, let us evaluate

$$\int d\theta\, E_\theta e^{ip_\theta(x-x')} = \pm i \int d\theta\, p_\theta e^{-E_\theta|x-x'|} = 0$$

where we shifted the contour by $\pm i\pi/2$ for $x - x' > 0$ or $x - x' < 0$. The last equation follows by anti-symmetry of p_θ. Note that the first integral is in fact not convergent, but conditionally convergent; it was transformed into a convergent integral by contour shift, made possible by $x \neq x'$. Similarly, we find

$$\int d\theta\, E_\theta p_\theta e^{ip_\theta(x-x')} = -\int d\theta\, p_\theta E_\theta e^{-E_\theta|x-x'|} = 0.$$

Now, if we look at all the terms we got for $\langle \text{vac}|[\hat{h}(x), \hat{p}(x')]|\text{vac}\rangle$ above, each of them factorises into an integal over θ and an integral over θ', and for each of the terms, there is always one of these two factors that is an integal of one of the types above, so that is zero (the other factor is not of the type above, so is non-zero and can be evaluated by similar methods). One could expect that for $x - x'$ general, what we have is in fact delta-functions, as suggested by the equations:

$$\int d\theta\, E_\theta e^{ip_\theta(x-x')} = \int dp\, e^{ip(x-x')} = 2\pi\delta(x - x')$$

and

$$\int d\theta\, E_\theta p_\theta e^{ip_\theta(x-x')} = \int dp\, p e^{ip(x-x')} = -2\pi i\delta'(x - x').$$

However, this does not fully decipher the form of $\langle \text{vac}|[\hat{h}(x), \hat{p}(x')]|\text{vac}\rangle$, because the delta-functions are multiplied by functions that are divergent at $x = x'$. In order to know what is happening, we need a more precise way of obtaining finite integrals over θ. A way of doing this, when evaluating the product of two local fields like $\hat{h}(x)\hat{p}(x')$ for instance, is to put these fields at different "imaginary time", $t = -i\tau$. So, in this example we should calculate instead $\hat{h}(x,\tau)\hat{p}(x',0)$ with $\tau > 0$. Then, all integrals are convergent, as the parameter τ implements a cut-off on each individual θ variables. This justifies the contour change on individual θ variables for evaluating the conditionally convergent intergals. For the commutator, one must then evaluate the limit $\tau \to 0$ of $\hat{h}(x,\tau), \hat{p}(x',0) - \hat{p}(x',\tau)\hat{h}(x,0)$, which is a generalised function of x. Our calculation essentially showed that this limit, in the vacuum, is zero for any fixed $x \neq x'$. A complete analysis of what happens "at $x = x'$", or under integration over x, is beyond the scope of the present homework...

A local field is an operator $\hat{\mathcal{O}}(x)$ with two properties:

$$[\hat{P},\hat{\mathcal{O}}(x)] = i\frac{d}{dx}\hat{\mathcal{O}}(x),\quad [\hat{h}(x),\hat{\mathcal{O}}(x')] = 0 \quad \text{for} \quad x\neq x'. \tag{11}$$

We can make more precise the second property. The commutator can be written

$$[\hat{h}(x),\hat{\mathcal{O}}(x')] = \sum_{n=0}^{\infty}\delta^{(n)}(x-x')\hat{\mathcal{O}}_n(x') \tag{12}$$

for some operators $\hat{\mathcal{O}}_n(x')$, and where $\delta^{(n)}(x-x')$ is the n^{th} derivative of $\delta(x-x')$. In general, this series in fact terminates, and we certainly have $\hat{\mathcal{O}}_0(x) = [\hat{H},\hat{\mathcal{O}}(x)] = -id\hat{\mathcal{O}}(x)/dt$.

——— **QUESTION 2**

By considering

$$\hat{Q} = \int_{x'-\epsilon}^{x'+\epsilon} dx\,(x-x')^n[\hat{h}(x),\hat{\mathcal{O}}(x')],$$

show that the operators $\hat{\mathcal{O}}_n(x')$ in (12) are also local fields. In particular, deduce that $[\hat{H},\hat{\mathcal{O}}(x)]$ is a local field.

Answer. First, let us verify the commutation relations with $\hat{P}$. We have

$$\begin{aligned}
[\hat{P},\hat{Q}] &= \int_{x'-\epsilon}^{x'+\epsilon} dx\,(x-x')^n([[\hat{P},\hat{h}(x)],\hat{\mathcal{O}}(x')] + [\hat{h}(x),[\hat{P},\hat{\mathcal{O}}(x')]]) \\
&= \int_{x'-\epsilon}^{x'+\epsilon} dx\,(x-x')^n i\left(\frac{d}{dx}+\frac{d}{dx'}\right)[\hat{h}(x),\hat{\mathcal{O}}(x')] \\
&= \int_{x'-\epsilon}^{x'+\epsilon} dx\, i\frac{d}{dx'}\Big((x-x')^n[\hat{h}(x),\hat{\mathcal{O}}(x')]\Big) + i\epsilon^n[\hat{h}(\epsilon),\hat{\mathcal{O}}(x')] - i(-\epsilon)^n[\hat{h}(-\epsilon),\hat{\mathcal{O}}(x')] \\
&= i\frac{d}{dx'}\left(\int_{x'-\epsilon}^{x'+\epsilon} dx\,(x-x')^n[\hat{h}(x),\hat{\mathcal{O}}(x')]\right)
\end{aligned}$$

where we did not need to use the fact that $\hat{\mathcal{O}}(x)$ is local, just its homogeneity property. Hence, $\hat{Q}$ is homogeneous as a function of x'. Now, for the locality we have that obviously $[\hat{h}(x''),\hat{Q}]=0$ for any x'' with $|x''-x'|>\epsilon$. But, using (12) we have

$$\hat{Q} = n!\hat{\mathcal{O}}_n(x')$$

for any $\epsilon>0$. Hence, we may take $\epsilon>0$ as small as we wish, without changing $\hat{Q}$, so that we deduce that $[\hat{h}(x''),\hat{\mathcal{O}}_n(x')]=0$ for any $x''\neq x'$. With $n=0$, this implies that $[\hat{H},\hat{\mathcal{O}}(x)]$ is a local field.

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We showed in class that if $\int dx\,\hat{u}(x)=0$ where $\hat{u}(x)$ is a local field, then it must be that $\hat{u}(x) = id\hat{v}(x)/dx$ for $\hat{v}(x)$ another local field. This, in particular, implies that if we have a conserved charge $\hat{Q}=\int dx\,\hat{q}(x)$ (that is, with $[\hat{H},\hat{Q}]=0$) where $\hat{q}(x)$ is a local field, then $\hat{u}(x)=[\hat{H},\hat{q}(x)]$ is a local field (by the theorem of question 2) with $\int dx\,\hat{u}(x)=0$ (by conservation of $\hat{Q}$), so that $\hat{u}(x) = id\hat{v}(x)/dx$ for $\hat{v}(x)$ another local field. This implies that there is a Noether current:

$$\frac{d}{dt}\hat{q}(x) + \frac{d}{dx}\hat{v}(x) = 0$$

The Hamiltonian $\hat{H}$ and momentum $\hat{P}$ operators naturally have dimension 1, hence their densities have dimension 2. The generator of boosts $\hat{B}$, then, has dimension 0. Looking at all possible commutators between the densities $\hat{h}(x)$ and $\hat{p}(x)$, we can easily evaluate the dimension of the generated local fields (the $\hat{\mathcal{O}}_n(x)$ in the series (12)). There is a general statement that **we can never generate local fields of dimension 1 from densities associated to space-time symmetries**. Local fields of dimension 1 only occur as densities associated to internal symmetries.

——— **QUESTION 3**

By considering $[\hat{B}, \hat{H}] = i\hat{P}$ and using the concepts of the discussion above, show that

$$\frac{\partial}{\partial t}\hat{h}(x) + \frac{\partial}{\partial x}\hat{p}(x) = 0.$$

Answer. First, from the fact that $\hat{h}(x)$ is the density of a local conserved charge and from the discussion above, it must be that

$$[\hat{H}, \hat{h}(x)] = i\frac{d}{dx}\hat{v}(x)$$

for some local field $\hat{v}(x)$. Then,

$$\begin{aligned}
0 = [\hat{B}, \hat{H}] - i\hat{P} &= \int dx\, \left(x[\hat{h}(x), \hat{H}] - i\hat{p}(x)\right) \\
&= \int dx\, \left(-ix\frac{d}{dx}\hat{v}(x) - i\hat{p}(x)\right) \\
&= \int dx\, \left(i\hat{v}(x) - i\hat{p}(x)\right).
\end{aligned}$$

Since the integrand is a local field, and since it integrates to zero, it must be a derivative of another local field:

$$i\hat{v}(x) - i\hat{p}(x) = i\frac{d}{dx}\hat{w}(x).$$

But by dimensional analysis, $\hat{w}(x)$ has dimension 1, and since there are no such fields in the field algebra associated to space-time symmetries, we must have $\hat{w}(x) = 0$. Another way of saying this is that we may well re-define $\hat{p}(x)$ by absorbing $d\hat{w}(x)/dx$ without changing $\hat{P}$. This shows that $\hat{v}(x) = \hat{p}(x)$, which indeed implies what we had to prove.

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——— **QUESTION 4**

Using the result of question 3, show that if $\hat{\mathcal{O}}(x)$ is a local field, then we have

$$[\hat{p}(x), \hat{\mathcal{O}}(x')] = 0 \quad \text{for} \quad x \neq x'.$$

Answer. We first evaluate, for $x \neq x'$,

$$[[\hat{H}, \hat{h}(x)], \hat{\mathcal{O}}(x')] = [\hat{H}, [\hat{h}(x), \hat{\mathcal{O}}(x')]] - [\hat{h}(x), [\hat{H}, \hat{\mathcal{O}}(x')]] = 0$$

where the last equality follows from the fact that $\hat{\mathcal{O}}(x')$ is a local field, and that $[\hat{H}, \hat{\mathcal{O}}(x')]$ also is a local field. Hence, from the result of question 3, we have that, still for $x \neq x'$,

$$0 = [\frac{d}{dx}\hat{p}(x), \hat{\mathcal{O}}(x')] = \frac{d}{dx}[\hat{p}(x), \hat{\mathcal{O}}(x')].$$

Applying $\int_{x''}^{\infty} dx$ for $x'' > x'$, we obtain $[\hat{p}(x''), \hat{\mathcal{O}}(x')] = 0$ for $x'' > x'$. Likewise, applying $\int_{-\infty}^{x''} dx$ for $x'' < x'$, we obtain $[\hat{p}(x''), \hat{\mathcal{O}}(x')] = 0$ for $x'' < x'$.
