Two-dimensional quantum field theory

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1 Introduction

1.1 Quantising a quadratic field

There are many ways of understanding QFT, and they are all quite useful. The first one, from which QFT takes its name, is as a quantum version of a classical field theory. Imagine, for instance, a classical string, along the horizontal, with oscillations only in the vertical direction. We usually think of it as many particles that can move in the vertical direction, connected to each other if they are neighbours with a spring. In fact, it will be quite useful for us to have a slightly generalised situation, with a string where each particle is also connected to a fixed spring that wants to keep the particle at some height 0. When we take the distance between the particles very small, we get an equation for a continuous height function $\phi(x)$. This is a classical field: it has a value (a degree of freedom) at every point x in space (one-dimensional). Under the dynamics, we get a time dependent function on $\mathbb{R}$. Naturally, there is a momentum density $\pi(x)$ associated to it, with canonical Poisson bracket relations $\{\phi(x), \pi(x')\} = \delta(x - x')$, and a Hamiltonian, which for our string is

$$H = \frac{1}{2} \int dx \left[\pi^2(x) + \left(\frac{\partial \phi(x)}{\partial x} \right)^2 + m^2 \phi^2(x) \right] . \tag{1.1}$$

The parameter m is what sets the force of the fixed springs. The term in π^2 is the kinetic energy, and the other terms are the potential energies due to inter-particle springs and fixed springs. This Hamiltonian gives the usual $\pi(x) = \frac{\partial \phi(x)}{\partial t}$ (here I use partial derivatives because of the x dependence), as well as the string equation with unit wave velocity, $\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2} - m^2 \phi$ (note that with velocity set to 1, x and t have the same units, and the parameter m has units of inverse distance).

We would like to perform the canonical quantisation of this classical field. The elements of canonical quantisations are:

- A map from classical variables to quantum observables, $A \mapsto \hat{A}$;
- A Lie algebra on the quantum observables containing the Hamiltonain and defined by the Lie bracket
$$[\hat{A}, \hat{B}] = i\widehat{\{A, B\}} \ ;$$
- A Hilbert space forming a representation of the algebra of quantum observables, where the Hamiltonian has eigenvalues that are finite and bounded from below.

Let us try to quantise the string in this way. The first two points are rather straightforward; in particular, the quantum version of the Hamiltonian does not have ordering problems, so we directly have,

for instance, $\widehat{\pi^2}(x) = \hat{\pi}^2(x)$. But the last point is what makes the problem non-trivial. In order consider the last point, we would like to construct quantum observables that allow us to form a Hilbert space by directly constructing a basis where the Hamiltonian is diagonalised; then we may assess its eigenvalues.

We know how to solve the string problem in classical mechanics: by Fourier transform. The linearity of the Fourier transform goes quite well with the linearity of the algebra of operators in quantum mechanics, and this saves us. Usually one introduces Fourier coefficients $A(p)$, with the variable p dual to x under Fourier transform. I will rather use a different variable: θ, defined by $p_\theta = m \sinh\theta$. This is the *rapidity* in relativity, where m is the mass of the relativistic particle and $E_\theta = m\cosh\theta$ is its energy, and gives a first flavor of relativity to our non-relativistic quantum string. But for now, it's just a choice of variable. Hence we define quantum observables $\hat{A}(\theta)$, with their "hermitian conjugate" $\hat{A}^\dagger(\theta)$, by saying that they are Fourier coefficients of the operator-fields $\hat{\phi}(x)$, $\hat{\pi}(x)$:

$$\hat{\phi}(x) = \int \frac{d\theta}{4\pi}\left(e^{ip_\theta x}\hat{A}(\theta) + e^{-ip_\theta x}\hat{A}^\dagger(\theta)\right) \ , \quad \hat{\pi}(x) = -i\int \frac{d\theta}{4\pi} E_\theta \left(e^{ip_\theta x}\hat{A}(\theta) - e^{-ip_\theta x}\hat{A}^\dagger(\theta)\right) \ . \tag{1.2}$$

The inversion of this is simply

$$\hat{A}(\theta) = \int dx\, e^{-ip_\theta x}(E_\theta\hat{\phi}(x) + i\hat{\pi}(x)) \ . \tag{1.3}$$

It is important to realise that this is nothing else than a change of variable; for any two real functions π and ϕ, there is a pair of complex conjugate functions A and $A^\dagger$ such that relations like those above hold. Although the variables θ and quantities that have to do with relativity appear, it is just for later convenience. The canonical commutation relations give

$$[\hat{A}(\theta), \hat{A}^\dagger(\theta')] = 4\pi\delta(\theta-\theta') \ , \quad [\hat{A}(\theta), \hat{A}(\theta')] = [\hat{A}^\dagger(\theta), \hat{A}^\dagger(\theta')] = 0 \tag{1.4}$$

and the Hamiltonian is

$$\hat{H} = \frac{1}{8\pi}\int d\theta E_\theta \left(\hat{A}^\dagger(\theta)\hat{A}(\theta) + \hat{A}(\theta)\hat{A}^\dagger(\theta)\right) \ . \tag{1.5}$$

Note that the *modes* $\hat{A}(\theta)$ are very much like those of a harmonic oscillator for each individual θ, and the Hamiltonian is a sum of that of such harmonic oscillators, with frequencies E_θ. Like in classical mechanics, this gives a simple expression for the time-evolved fields (in the Heisenberg picture), for instance:

$$e^{i\hat{H}t}\hat{\phi}(x)e^{-i\hat{H}t} = \hat{\phi}(x,t) = \int \frac{d\theta}{4\pi}\left(e^{ip_\theta x - iE_\theta}\hat{A}(\theta) + e^{-ip_\theta x + iE_\theta}\hat{A}^\dagger(\theta)\right) \ . \tag{1.6}$$

It is easy to check that this Heisenberg-picture field satisfies exactly the equation of motion of the classical field: this is a consequence of Ehrenfest theorem, and the fact that the potential energy is quadratic in the field.

Now all this is useful because the Fock space for the algebra (1.4), which is a module for canonical commutation relations like that, gives a construction of eigenstates of the Hamiltonian; this forms a Hilbert space which respects the Hermitian structure already implied by the notation $\hat{A}^\dagger$. It goes like this: first define a state $|\text{vac}\rangle$ by the fact that

$$\hat{A}(\theta)|\text{vac}\rangle = 0 \tag{1.7}$$

for all θ. Then construct a basis of states $|\theta_1, \ldots, \theta_n\rangle$ for $\theta_1 > \ldots > \theta_n$ by

$$|\theta_1, \ldots, \theta_n\rangle = \hat{A}^\dagger(\theta_1)\cdots\hat{A}^\dagger(\theta_n)|\text{vac}\rangle \tag{1.8}$$

(we need an ordering of the rapidities to have a basis, with independent states, because the $\hat{A}^\dagger(\theta)$ commute with each other). Using the algebra relations (1.4), the action of any operator $\hat{A}(\theta)$ or $\hat{A}^\dagger(\theta)$ can be evaluated on any of these states. Hence this is a module. The Hilbert space structure is given by the Hermitian structure, along with $\langle \text{vac}|\text{vac}\rangle = 1$. Using the mode algebra (1.4), this gives

$$\langle \theta'_1, \dots, \theta'_{n'}|\theta_1, \dots, \theta_n\rangle = (4\pi)^n \delta_{n,n'} \prod_{j=1}^{n} \delta(\theta_j - \theta'_j) \tag{1.9}$$

for $\theta_1 > \ldots > \theta_n$ and $\theta'_1 > \ldots > \theta'_{n'}$. This basis looks like a basis of eigenstates of the Hamiltonian. Indeed, we have

$$\hat{H}|\theta_1, \dots, \theta_n\rangle = \left(\frac{1}{2}\int d\theta E_\theta \delta(0) + \sum_{j=1}^{n} E_{\theta_j}\right)|\theta_1, \dots, \theta_n\rangle \ . \tag{1.10}$$

But there is a problem: the energies are infinite (and very badly: both because of the integral over θ, and because of the $\delta(0)$)! This means that we did not manage to take care of the third point of canonical quantisation. In fact, it is *impossible* to perform the canonical quantisation of the classical field: we will always run into problems.

What we could do is simply *define* the Hamiltonian as the operator

$$\hat{H}_R = \frac{1}{4\pi}\int d\theta E_\theta \hat{A}^\dagger(\theta)\hat{A}(\theta) \ . \tag{1.11}$$

This is well defined on the Fock space that we constructed, and has minimal eigenvalue 0. With this Hamiltonian, we have what we call the quantum version of our classical field theory. What we have done is essentially defining a renormalised Hamiltonian, by substracting an infinite constant. Such a substraction does not affect the dynamics, so not much has change in our quantum theory; but it is a necessary adjustment, in order to have a properly quantum theory. Of course, substracting an infinite constant is not a well-defined mathematical operation; the proper understanding is in first quantising a "regularised" version of the classical system (discretised, or otherwise regularised) such its quantisation is well defined, then taking the limit where the regularisation parameter disappears. We would need to re-define the Hamiltonian operator by substracting a constant for every finite regularisation parameter, and in our case, as this limit is taken, that constant would go to infinity.

This is, in fact, a fundamental problem occurring when quantising a classical field theory: the usual continuum limit of a classical field is usually not a well-defined quantum mechanical system, we need to take this limit in a very particular way, substracting infinities. In the case of the string, it is quite simple, but if we had consider *corrections* to the spring-like interactions, like a term $\phi^4(x)$, the quantisaton process would have been much more complicated: this needs a full-fledged *renormalisation*, where not only the Hamiltonian is shifted, but other parameters in the model have to be adjusted also, with, in the limit, infinite shifts.

This problem can be understood in Feynman's formulation of quantum mechanics using path integrals. In this formulation, the probability amplitude for a particle to go from a point x' at time t' up to a point x'' at time t'', $\langle x''|e^{-i\hat{H}(t''-t')}|x'\rangle$ in the usual notation, is given by

$$\int_{x(t')=x',\ x(t'')=x''} [dx(t)] e^{-iS(x,dx/dt)} \tag{1.12}$$

where $S(x, dx/dt)$ is the classical action, and where we integrate over all possible paths $x(t)$ with appropriate initial and final conditions. Hence it looks like the particles can take almost any path. However,

the classical one is the most favorable because the phase $e^{-iS(x,dx/dt)}$ oscillates wildly, phases of nearby trajectories mostly cancelling each other, except at the minimum of the classical action, where it oscillates least. Yet, trajectories with wild turns will occur, in agreement with uncertainty principles, a bit like a Brownian motion. But if all particles of a string are subject to such a behaviour, there will be strong distortions, and in the continuum, infinite energies will occur. In fact, if we add too many corrections to our spring-like approximation, these wild distortions will impose that all possible corrections have to be considered; this would be a non-renormalisable theory.

Finally, it is worth investigating the relation between the Fock space that we constructed, and the naïve Hilbert space spanned by the basis of all possible classical field configurations. In quantum mechanics, the initial space that we are usually considering, before diagonalising the Hamiltonian, is that spanned by a basis parametrised by all possible classical configurations of the system; for instance, the basis $\{|x\rangle,\ x\in\mathbb{R}\}$ for a particle on the line. This is endowed with an inner product, with delta-function normalisation. This basis allows us to consider wave functions $\Psi:\ x\mapsto\langle x|\psi\rangle$ associated to any state $|\psi\rangle$, and to consider operators, like the Hamiltonian or the momentum, as acting on a space of functions. Diagonalising the Hamiltonian usually reduces this space of functions, and this is how the Hilbert space is reduced from the space spanned by $\{|x\rangle,\ x\in\mathbb{R}\}$. In the present case, the classical configurations are the field configurations $\{\phi:\ \mathbb{R}\to\mathbb{R}\}$. Hence, the naïve space is that spanned by the basis $\{|\phi\rangle,\ \phi:\ \mathbb{R}\to\mathbb{R}\}$, with inner product on basis elements given by a delta-function on the space of functions. Then, we may form wave functions $\Psi:\ \phi\mapsto\langle\phi|\psi\rangle$, and in our case, on these we have the following operators:

$$(\hat{\phi}(x)\Psi)(\phi)=\phi(x)\Psi(\phi),\quad (\hat{\pi}(x)\Psi)(\phi)=-i\frac{\delta}{\delta\phi(x)}\Psi(\phi). \tag{1.13}$$

Then, we may construct the wave function $\Psi_{\rm vac}$ associated to the vacuum state by solving the functional equation

$$\int dx\, e^{-ip_\theta x}\left(E_\theta\phi(x)+\frac{\delta}{\delta\phi(x)}\right)\Psi_{\rm vac}(\phi)=0 \tag{1.14}$$

corresponding to the equation $\hat{A}(\theta)|{\rm vac}\rangle=0$. In order to solve this equation, we isolate the functional derivative term by applying $\int d\theta\, e^{ip_\theta x'}E_\theta$ using $dp_\theta=d\theta E_\theta$, so that

$$\frac{\delta}{\delta\phi(x')}\Psi_{\rm vac}(\phi)=\frac{1}{2\pi}\int dx\int d\theta\, e^{-E_\theta|x-x'|}p_\theta^2\phi(x)\Psi_{\rm vac}(\phi). \tag{1.15}$$

Here, we have made a shift $\theta\mapsto\theta\pm i\pi/2$ in order to get a real decaying exponential. Solving this equation is simple:

$$\Psi_{\rm vac}(\phi)=\exp\left[\frac{1}{4\pi}\int d\theta\int dxdx'\, p_\theta^2e^{-E_\theta|x-x'|}\phi(x)\phi(x')\right]. \tag{1.16}$$

Note that we can re-write that in a more instructive form. Indeed, we have

$$\langle{\rm vac}|\phi(x)\phi(x')|{\rm vac}\rangle=\int d\theta\, e^{-E_\theta|x-x'|}, \tag{1.17}$$

so that, using the equations of motion,

$$\Psi_{\rm vac}(\phi)=\exp\left[\frac{1}{4\pi}\int dxdx'\,\langle{\rm vac}|\hat{\phi}(x)\frac{\partial^2}{\partial t^2}\hat{\phi}(x')|{\rm vac}\rangle\,\phi(x)\phi(x')\right]. \tag{1.18}$$

In terms of this, wave functions of excited states can be obtained, for instance:

$$\Psi_\theta(\phi)=2\int dx\, e^{ip_\theta x}E_\theta\phi(x)\Psi_{\rm vac}(\phi). \tag{1.19}$$

Note that the wave function for the vacuum is not a delta-function at the field configuration $\phi = 0$. This is in similar spirit to the usual harmonic oscillator in quantum mechanics, where the ground state is not the state $|x = 0\rangle$, because the uncertainty principle would imply a large momentum, or large energy.

1.2 Scaling limits and correlation functions

With interactions, as we said above, the renormalisation procedure is more complicated, and the effect is not just a shift of the Hamiltonian. The deepest understanding of the renormalisation procedure and of quantum field theory in general is through the scaling limit of quantum systems with many degrees of freedom. That is, we first quantise the particles of the string, or other degrees of freedom, then we think about how to take the continuum limit, instead of first taking the continuum limit then thinking about how to quantise. Let me take now a different example than the string, to make things clearer. Suppose we have a chain of quantum degrees of freedom. For instance, at each site of the chain, we have a two-dimensional quantum space (as it happens if this is a spin-1/2 degree of freedom, like the spin of the electron), and the full quantum space is the tensor product of all such two-dimensional spaces, for all sites of the chain. This describes all the possible states of the system. In the so-called "XXZ" Heisenberg spin chain, the Hamiltonian describes magnetic interactions between sites that are nearest neighbours:

$$H = J \sum_j \left(\sigma_j^{\mathrm{x}} \sigma_{j+1}^{\mathrm{x}} + \sigma_j^{\mathrm{y}} \sigma_{j+1}^{\mathrm{y}} + \Delta \sigma_j^{\mathrm{z}} \sigma_{j+1}^{\mathrm{z}} \right)$$

where J is the magnetic coupling. Here $\sigma_j^{\mathrm{x,y,z}}$ are Pauli matrices associated to independent sites $j \in \mathbb{Z}$:

$$\sigma^{\mathrm{x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma^{\mathrm{y}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma^{\mathrm{z}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{1.20}$$

The paremeter Δ is an anisotropy: the z direction of the magnetic interaction is different.

Many of the quantities of physical interest are obtained from *correlation functions*, quantum averages of products of observables of local degrees of freedom at different sites of the chain, for instance

$${}_\Delta\langle \mathrm{gs} | \sigma_j^{\mathrm{y}} \sigma_k^{\mathrm{y}} \sigma_l^{\mathrm{y}} \sigma_m^{\mathrm{y}} \cdots | \mathrm{gs} \rangle_\Delta .$$

Here, $|\mathrm{gs}\rangle_\Delta$ is the ground state of the chain, that depends on the parameter Δ in the Hamiltonian. For $\Delta > 1$, complicated calculations show that correlation functions for two separated sites decrease exponentially at large separations, for instance:

$${}_\Delta\langle \mathrm{gs} | \sigma_j^{\mathrm{y}} \sigma_k^{\mathrm{y}} | \mathrm{gs} \rangle_\Delta \propto e^{-\frac{|j-k|}{\xi}} \quad \text{as} \quad |j-k| \to \infty$$

for some *correlation length* ξ, that is a function of Δ (more precisely, this asymptotics holds when $\langle \mathrm{gs} | \sigma_j^{\mathrm{y}} | \mathrm{gs} \rangle = 0$ for all j, which we will assume for simplicity; otherwise, we would have to substract the product of ground-state expectation values). This correlation length is a function of the parameter Δ. Hence if we look at very large distances, we see no quantum correlations at all; this seems natural, since the interaction is just between nearest neighbours.

But even more complicated calculations show that the correlation length tends towards infinity as $\Delta \to 1^+$. Since correlation functions are related to *response functions*, this will imply singularities in the magnetic susceptibility, for instance, so that this is a second order (quantum) phase transition. Exactly at $\Delta = 1$, the correlation length is infinite, and the correlation function has asymptotics:

$${}_{\Delta=1}\langle \mathrm{gs} | \sigma_j^{\mathrm{y}} \sigma_k^{\mathrm{y}} | \mathrm{gs} \rangle_{\Delta=1} \propto |j-k|^{-2d} \quad \text{as} \quad |j-k| \to \infty$$

for some $d > 0$. In this asymptotic, we see that there is no particular scale that determines how the correlation function vanishes. This is the characteristics of a critical quantum system.

Now, if we "look" at very large distances while at the very same time we take the limit $\Delta \to 1^+$, we still see a power-law decay, but the proportionality constant in this power-law decay depends on the ratio of how far we look and how large $\xi(\Delta)$ is. But looking at large distances is like considering the inter-site spacing very small: it is a continuum limit. The proportionality constant is quite independent of the details of the local interactions, so it is universal, yet it is non-trivial. It is described by QFT, and the process of getting near to a (quantum) phase transition while looking at larger and larger distances is the scaling limit. Before we take the scaling limit, we can think of the chain as a *regularisation* of the quantum field theory, and the process of taking the scaling limit gives rise to the renormalisation process. Like in the chain we had to shift the Hamiltonian by a constant that was becoming infinity as we were going to the continuum, here we have to take $\Delta \to 1^+$ as we look at larger distances. To be more precise: there exists a quantum field $\hat{\mathcal{O}}(x)$ (like $\hat{\phi}(x)$ in the string case) in a quantum field theory, with the property that

$$\langle \mathrm{vac}|\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(x')|\mathrm{vac}\rangle \propto e^{-m|x-x'|} \quad \text{as} \quad m|x-x'| \to \infty$$

for some m (that takes care of the dimensions – the mass associated to $\hat{\mathcal{O}}(x)$), such that

$$\lim_{\Delta\to 1^+} \xi^{2d}(\Delta)\ {}_\Delta\langle \mathrm{gs}|\sigma^{\mathrm{y}}_{[s\xi(\Delta)]}\sigma^{\mathrm{y}}_{[s'\xi(\Delta)]}|\mathrm{gs}\rangle_\Delta = \langle \mathrm{vac}|\hat{\mathcal{O}}(sm^{-1})\hat{\mathcal{O}}(s'm^{-1})|\mathrm{vac}\rangle$$

where the limit is taken keeping the real parameters s and s' fixed (and $[\cdot]$ means that we must take the integer part). On the right-hand side, we have a correlation function in a QFT with a mass m associated to $\hat{\mathcal{O}}(x)$, and the operator $\hat{\mathcal{O}}(x)$ is an appropriate local field "representing" σ^y in scaling limit (below we will make precise the concepts of mass and of local field). The unique (positive) number d making the limit finite is the same as that involved in the asymptotics of the correlation functions of the spins at $\Delta = 1$ exactly. It is called the dimension of the field $\hat{\mathcal{O}}$. The QFT correlation function is a "scaling function", and is universal, in that it depends on the details of the initial quantum chain just through a finite number of parameters (models with very different interactions and basic constituents could give the same QFT). In this case, the QFT is integrable (it is related to the quantum sine-Gordon model). Similar scaling limits can be taken for other spin-chain correlation functions, in order to obtain other QFT correlation functions.

The true meaning of these proportionality constants, the QFT correlation functions, is through collective behaviours. Near to a critical point, there are large structures that start forming and being more stable to quantum fluctuations; they represent collective behaviours of the local degrees of freedom. The presence of such large structures has the consequence that correlation lengths diverge, as is observed at critical points. These large structures fluctuate also, but the fluctuations do not destroy them. The law for their fluctuations is controlled by the initial Hamiltonian only in a very indirect way. Since they are large structures, we need to consider large chains of local interactions in order to describe their fluctuations. This means that these fluctuations are not so sensitive to the exact form of the local interactions, so they are universal. Correlation functions of QFT describe exactly fluctuations of such large, stable structures. Thanks to modern advances in conformal field theory, in particular through conformal loop ensembles, it is possible to describe with mathematical precision such large structures. This opens the way to a better understanding of how universality comes about, and how the mathematical structure of quantum field theory arises.

1.3 Relativistic quantum particles and their relation to quantum fields

Let us come back to the quantisation of the non-relativistic string. Certainly, there seemed to be some structures of special relativity coming out. How far can we make the correspondence?

Let us now consider the quantisation of a *relativistic* particle. What do we expect from our knowledge of non-relativistic quantum mechanics? First, in relativity, energy and mass can be exchanged, so that there can be creation of particles. Hence, we should not consider just one particle, but any number of particles, so that the Hilbert space should be a direct sum of sectors with various particle numbers. Let us consider free particles (really, then there is no particle creation, but let's keep all particle numbers). Generalising what happens in non-relativistic quantum mechanics, we expect that for n identical particles, the state will be described by the n energies, or momenta, or better rapidities, of the particles. But since particles are identical, exchanging the rapidities of two particles does not change the state. Hence we expect the Hilbert space to have a basis of states like

$$|\theta_1, \dots, \theta_n\rangle \ , \quad \theta_1 > \dots > \theta_n \ . \tag{1.21}$$

These states should have the inner product (1.9), which is relativistic invariant, that is, invariant under $\theta \mapsto \theta + \beta$ for all rapidities involved. On these states, the energy is $\sum_{j=1}^n m \cosh \theta_j$, and the momentum is $\sum_{j=1}^n m \sinh \theta_j$. This is the same Hilbert space as for the quantum non-relativistic string, with the same energies as the eigenvalues of the *renormalised* Hamiltonian. Hence, we have a nice interpretation of the renormalised Hamiltonian: it gives the energies of the relativistic particles.

But we can go further with the identification. For relativistic particles, we certainly have a momentum operator $\hat{P}$ with eigenvalues $\sum_{j=1}^n m \sinh \theta_j$ on the states (1.21). Hence, this is also a well-defined operator on the Hilbert space of the quantum string, since it is the same Hilbert space. It can be written

$$\hat{P} = \frac{1}{4\pi} \int d\theta \, p_\theta A^\dagger(\theta) A(\theta). \tag{1.22}$$

But does it have a deeper meaning on the string? The answer is yes: it is the Noether charge associated to the symmetry under translation of the string. In the context of the classical string, it can be written as an integral of a local density,

$$P = -\int dx \, \pi(x) \frac{d}{dx} \phi(x) \tag{1.23}$$

but the quantum observable associated to this is again plagued by singularities. The correct definition is (1.22), and the fact that it is associated to a symmetry that changes the observables $\hat{\phi}$ and $\hat{\pi}$ only locally, can be recovered:

$$[\hat{H}, \hat{P}] = 0, \quad [\hat{P}, \hat{\phi}(x)] = i \frac{d}{dx} \hat{\phi}(x), \quad [\hat{P}, \hat{\pi}(x)] = i \frac{d}{dx} \hat{\pi}(x). \tag{1.24}$$

We will discuss more at length the notions of locality in the next sections.

Likewise, there is certainly a hermitian boost generator acting on the Hilbert space (since boost transformations are unitary transformaions), with $\hat{B}|\theta_1, \dots, \theta_n\rangle = -i(d/d\theta_1 + \dots + d/d\theta_n)|\theta_1, \dots, \theta_n\rangle$, which we can represent by

$$\hat{B} = \frac{1}{4\pi} \int d\theta \, A^\dagger(\theta) i \frac{d}{d\theta} A(\theta). \tag{1.25}$$

Again, it corresponds to a symmetry, albeit here a space-time symmetry, and it acts locally:

$$[\hat{B}, \hat{\phi}(x)] = ix\hat{\pi}(x), \quad [\hat{B}, \hat{\pi}(x)] = ix \left(\frac{d^2}{dx^2} - m^2 \right) \hat{\pi}(x). \tag{1.26}$$

The important thing to notice is the following algebra of space-time symmetries:

$$[\hat{H}, \hat{P}] = 0, \quad [\hat{B}, \hat{P}] = i\hat{H}, \quad [\hat{B}, \hat{H}] = i\hat{P}. \tag{1.27}$$

Let us now consider the (supposed) state $|x\rangle$ that represents one relativistic particle at position x (we will see that a relativistic particle cannot be exactly localised). Since we have a Hilbert space with all numbers of particles, we can construct a Hermitian operator, say $\hat{\phi}(x)$, that creates a particle at position x out of the vacuum: $|x\rangle = \hat{\phi}(x)|\text{vac}\rangle$. Let us look at the wave function of the one-particle state $|\theta\rangle$, that is, $\psi(x) = \langle x|\theta\rangle = \langle \text{vac}|\hat{\phi}(x)|\theta\rangle$. Remember that in non-relativistic quantum mechanics, the wave function of a free particle is an eigenfunction of the momentum operator, $\hat{P} = -id/dx$ (acting on wave functions). Here also momenum is conserved, so we should have $\hat{P}\psi(x) = p_\theta \psi(x)$. This is in agreement with the matrix element $\langle \text{vac}|\hat{\phi}(x)|\theta\rangle$ of the quantum height field $\hat{\phi}(x)$:

$$\langle \text{vac}|\hat{\phi}(x)|\theta\rangle = e^{ip_\theta x}. \tag{1.28}$$

We can also look at the wave function of the time-evolved state, $\psi(x,t) = \langle x|e^{-iHt}|\theta\rangle = \langle \text{vac}|\hat{\phi}(x,t)|\theta\rangle$ where we used the Heisenberg picture. This has again the correct expected form,

$$\langle \text{vac}|\hat{\phi}(x,t)|\theta\rangle = e^{ip_\theta x - iE_\theta t}. \tag{1.29}$$

Let us consider some arbitrary time-evolved wave function $\psi(x,t)$ for one particle. In non-relativistic quantum mechanics, the wave function obeys the Schroedinger equation, which is just $i\partial/\partial t = \hat{P}^2/(2m)$. Similarly, we could impose the relativistic dispersion relation, understanding $i\partial/\partial t$ as the energy: $(i\partial/\partial t)^2 - \hat{P}^2 = m^2$. This leads to the Klein-Gordon equation:

$$\frac{\partial^2}{\partial t^2}\psi(x,t) = \frac{\partial^2}{\partial x^2}\psi(x,t) - m^2\psi(x,t) \ . \tag{1.30}$$

This is exactly the wave equation for the string above, indeed satisfied by the matrix element $\langle \text{vac}|\hat{\phi}(x,t)|\psi\rangle$ of the time-evolved height field thanks to Ehrenfest's theorem, for any state $|\psi\rangle$ in the Hilbert space.

In order to fully understand the meaning of $\hat{\phi}(x,t)$ in the relativistic particle picture, consider the state $\hat{\phi}(x,t)|\text{vac}\rangle$ and evolve it for a time t: $e^{-iHt}\hat{\phi}(x,t)|\text{vac}\rangle = \hat{\phi}(x)|\text{vac}\rangle = |x\rangle$; this is a state with a particle at x. Hence, the state $\hat{\phi}(x,t)|\text{vac}\rangle$ is a state *at time zero* such that when we reach time t starting from it, we get a particle at position x. That is, we interpret $\hat{\phi}(x,t)$ as creating a particle at position x and time t out of a vacuum at time 0.

Note also that $\hat{\phi}(x,t)|\text{vac}\rangle$ does not overlap with any state $|\theta_1, \ldots, \theta_n\rangle$ with $n \neq 1$. That is, it is truely a one-particle state only.

Finally, this wave function transforms in the right way under boosts. In terms of the rapidity θ, a relativistic boost is just a shift $\theta \mapsto \theta + \beta$. This should correspond to $(x,t) \mapsto (x\cosh\beta - t\sinh\beta, t\cosh\beta - x\sinh\beta)$. Since time-evoled the wave function is

$$\psi_\theta(x,t) = \langle \text{vac}|\hat{\phi}(x,t)|\theta\rangle = e^{ip_\theta x - iE_\theta t} \ , \tag{1.31}$$

we can see that indeed $\psi_{\theta+\beta}(x,t) = \psi_\theta(x\cosh\beta - t\sinh\beta, t\cosh\beta - x\sinh\beta)$.

Hence we have found a direct relation between *quantum relativistic particles* and the *quantum non-relativistic string*:

- They have the same Hilbert space, with a quantum string mode of frequency $E_\theta = m\cosh\theta$ and wave number $p_\theta = m\sinh\theta$ corresponding to a particle of rapidity θ;
- The energy of the relativistic particles is the renormalised energy of the quantum string, the momentum and boost operators in the context of the relativistic particle are Noether charges associated to symmetries of the string;
- The Heisenberg-picture time-evolved height operator $\hat{\phi}(x,t)$ creates a particle at position x and time t out of the vacuum at time 0.

Consider now the quantity $\langle \text{vac}|\hat{\phi}(x')\hat{\phi}(x)|\text{vac}\rangle$. It has very different interpretations: in the string picture, it is the average of a product of the height observable at different points (it is a correlation function), and in the relativistic picture, it is the probability amplitude for a particle at x' to be present in a state with a particle at x. Interestingly, this probability amplitude is not zero for $x \neq x'$! That is, due to relativistic invariance, in a state with "a particle at x" there is in fact a probability of finding the particle at $x' \neq x$. However, this probability decays exponentially fast as $|x'-x|$ becomes large, like $e^{-m|x-x'|}$; we just cannot completely "localise" a particle.

But we do have some concept of locality: we know that $[\hat{\phi}(x), \hat{\phi}(x')] = 0$ for $x \neq x'$. This is just saying that the height observables at different positions are independent observables, as it should be, and it holds also for (x,t) and (x',t') at *space-like* distances, $(x-x')^2 > (t-t')^2$, thanks to Lorentz invariance. In the relativistic particle picture, it is saying that the probability amplitude for the propagation of a particle from a point of space-time to another at a space-like distance, is exactly cancelled by the probability amplitude for the propagation of the associated *anti-particle*. This does not hold if the distance is *time-like*. This is the concept of quantum locality in relativistic quantum mechanics. This concept of locality is satisfying in the relativistic particle viewpoint, because any physical process that will create a particle out of the vacuum must also create an anti-particle. The picture here is that of the *Dirac sea*: the vacuum is a sea filled up to some level, and creating a particle means taking an "element" of the sea above the sea level. This leaves a hole, that will propagate like a relativistic particle as well: the anti-particle. Then, the quantum correlation between such physical processes at space-like points will be zero, because the particle and anti-particle have cancelling amplitudes.

In relation to that, we note that the field $\hat{\phi}(x)$ not only create a particle, but also destroys a particle (which doesn't do anything to vacuum, but does something to states with more particles). In fact, the more precise statement is that it destroys the associated anti-particle, but in our simple example, the particle is its own anti-particle (there is no electric charge to invert, for instance, as there would be in the case of the electron and positron). This is also general: fields that are quantum mechanically local must both create a particle and destroy an anti-particle.

It is worth noting that the relation we have described is *not* exactly what is usually called "second quantisation". Indeed, in second quantisation, one considers the wave function as an operator, but in general the wave function is complex whereas our field $\hat{\phi}$ is hermitian. The usual second quantisation description is not entirely adequate, as it imposes the presence of an internal charge (like an electric charge). A more precise relation between quantum fields and relativistic particles is through the arguments presented here.

Finally, this relation between the quantum string and quantum relativistic particles wouldn't be so useful if we were only to look at free particles. But for *interacting* relativistic particles, the only clear

quantum mechanics theory is through this relation: the interaction is implemented by adding corrections to the quadratic potential in the string (or in other "free" quantum field theory models), and by defining the quantum theory through a renormalisation procedure.

2 Local relativistic quantum field theory

Let us now put together the main ingredients of the previous analysis of the free bosinic field alias the free relativistic particle, and express more formally what a quantum field theory is. In fact, there are two complementary ways of defining any given model of QFT: a local-density formulation, which has to do with the idea of the quantisation of a classical field theory, and a scattering-state formulation, which has to do with the idea of quantum relativistic particles. Both ways are expected to be equivalent: the same physical information (mainly, the correlation functions of local fields) can be obtained from both formulation. This is extremely non-trivial in general, but in integrable QFT, one goes a long way towards understanding this equivalence.

Both ways are based on basic quantum mechanics ingredients: a Hilbert space and observables acting on it. There are more ways of defining a QFT model, which are not based on that. For instance, one could think about an action formalism and a path-integral formulation. However, one must remember that, contrary to the impression one gets from standard QFT courses, an action by itself does not define a QFT model. Indeed, this does not specify the renormalisation scheme, because the action only fixes the classical field theory, which, as we saw, has divergencies upon naïve quantisation, so needs a regularisation and a way of taking this regulatisation away (a renormalisation scheme). It is an action and a proper definition of a path integral that would fix a QFT model. In the same spirit, one could provide a well-defined mathematical measure for random objects (fields or other objects). But this is beyond the scope of the present notes.

In fact, additionally to the fundamental quantum mechanics ingredients, we want a theory with relativistic invariance. Hence in both formulations presented below, we will start with the following data:

- **Hilbert space:** a Hilbert space $\mathcal{H}$.
- **Hamiltonian:** an operator $\hat{H}$ acting on $\mathcal{H}$, called the Hamiltonian, which is diagonalizable and whose eigenvalues are bounded from below; the vector associated with the lowest eigenvalue is the vacuum, $|\text{vac}\rangle \in \mathcal{H}$, and the lowest eigenvalue is set to 0.
- **Relativistic invariance:** operators $\hat{P}$ (the momentum operator) and $\hat{B}$ (the boost operator) which satisfy
$$[\hat{H}, \hat{P}] = 0 \ , \quad [\hat{B}, \hat{P}] = i\hat{H} \ , \quad [\hat{B}, \hat{H}] = i\hat{P} \tag{2.1}$$
as well as
$$\hat{H}|\text{vac}\rangle = \hat{P}|\text{vac}\rangle = \hat{B}|\text{vac}\rangle = 0. \tag{2.2}$$

It is important to note that these data are not enough to fully fix a model of quantum field theory, although it could appear so from general principles of quantum mechanics, since we have both the Hilbert space and the Hamiltonian. In fact, in quantum mechanics, we also usually want to have a set of observables (including the Hamiltonian) forming a closed algebra; or, an embedding of the Hilbert space

into a space of functions (the wave functions). Something similar happens in quantum field theory, but it is slightly more complicated: from one viewpoint, this is because QFT has to do with infinitely many degrees of freedom, which need to be "organised" in a definite way (local-density formulation); from another viewpoint, this is because QFT has to do with relativistic particles, whose interactions need to be specified (scattering-state viewpoint).

2.1 The local-density formulation

This formulation concentrates not on the Hilbert space, but on the operators $\hat{H}$, $\hat{P}$ and $\hat{B}$, the algebra they satisfy, and a way of expressing them as integration of local densities over space.

The first problem to solve is that we need to define what "local" means, and what space is. We must start with some operator that will partly encode these things. Consider an operator $\hat{h}(x)$ that depends on a variable $x \in \mathbb{R}$; we will refer to the map from $\mathbb{R}$ to operators given by $x \mapsto \hat{h}(x)$ simply as $\hat{h}$. We will say that an operator $\hat{\mathcal{O}}_{x_0}$ is a **local field at** x_0 with respect to $\hat{h}$ if

$$[\hat{\mathcal{O}}_{x_0}, \hat{h}(x)] = 0 \quad \text{for} \quad x \neq x_0. \tag{2.3}$$

This is to be understood as an operator statement, valid when we evaluate any matrix element, and as a generalised function statement in terms of the variable x. Generalised functions must be integrated with appropriate test functions in order to have ordinary function statements. The statement above would say that, for $f(x)$ infinitely differentiable on the whole $x \in \mathbb{R}$,

$$\int dx\, f(x)(x-x_0)^n [\hat{\mathcal{O}}_{x_0}, \hat{h}(x)] = 0 \quad \text{for } n \text{ large enough.} \tag{2.4}$$

Of course, generalised functions are obtained by taking some parameter to 0 after the integration. The parameter that can be taken to zero here is the *imaginary time* τ, related to the real time t by $t = -i\tau$. That is, we say that

$$\lim_{\tau \to 0^+} \int dx\, f(x)(x-x_0)^n \left(e^{\hat{H}\tau} \hat{\mathcal{O}}_{x_0} e^{-\hat{H}\tau} \hat{h}(x) - e^{\hat{H}\tau} \hat{h}(x) e^{-\hat{H}\tau} \hat{\mathcal{O}}_{x_0} \right) = 0 \quad \text{for } n \text{ large enough.} \tag{2.5}$$

Here, for any finite $\tau > 0$, the result is a well-defined operator even before integration, for any fixed x. The effect of $e^{-\hat{H}\tau}$ between the operators is that inserting a decomposition of the identity in terms of energy eigenstates, we immediately get integrals that are convergent for $\tau > 0$.

An additional comment is in order: when we say that $\hat{h}(x)$ is an operator, we mean it in a certain "extended" way. More precisely, we mean that all its matrix elements can be calculated unambiguously. However, we do not mean that two such $\hat{h}(x)$ can be multiplied with each other: the insertion of the decomposition of the identity into energy eigenstates may give sums/integrals that are not convergent. In particular, we definitely expect that $\hat{h}(x)^2$ is not a well-defined operator: its matrix elements are not well-defined. These problems occur because we have an infinite-dimensional Hilbert space. Hence, when we say that $\hat{h}(x)$ is an operator, we are not exactly talking about a linear map on the Hilbert space, but rather about a collection of matrix elements. Then, it is in this sense again that $e^{\hat{H}\tau} \hat{\mathcal{O}}_{x_0} e^{-\hat{H}\tau} \hat{h}(x)$, for instance, is a well-defined operator: it is a well-defined collection of matrix elements. For local fields, this is always the way we will undertstand the term "operator".

We will say that an x-dependent operator $\hat{\mathcal{O}}(x)$, or more precisely, a map $\hat{\mathcal{O}} : x \mapsto \hat{\mathcal{O}}(x)$ from $\mathbb{R}$ to the space of operators on $\mathcal{H}$, is a **local field** with respect to $\hat{h}$ if $\hat{\mathcal{O}}(x)$ is a local field at x for any fixed

$x \in \mathbb{R}$. Finally, given another operator $\hat{P}$, we will say that $\hat{\mathcal{O}}$ is a **homogeneous local field** with respect to $\hat{h}$ and $\hat{P}$ if it is a local field with respect to $\hat{h}$, and if, additionally,

$$[\hat{P}, \hat{\mathcal{O}}(x)] = i\frac{d}{dx}\hat{\mathcal{O}}(x) \tag{2.6}$$

for any $x \in \mathbb{R}$.

Armed with these definitions, we now provide the addional data for the local-density formulation of QFT:

- **Locality:** we have

$$\hat{H} = \int dx\, \hat{h}(x) \ , \quad \hat{P} = \int dx\, \hat{p}(x) \ , \quad \hat{B} = \int dx\, \hat{b}(x)$$

  where $\hat{b}$ is a local field, and $\hat{h}$ and $\hat{p}$ are homogeneous local fields, with respect to $\hat{h}$ and $\hat{P}$.

In the following, we will omit to write "with respect to $\hat{h}$ and $\hat{P}$," as this will be implicitly assumed, unless it is stated otherwise.

The goal, from the viewpoint of constructing QFT, is first to find a Hilbert space $\mathcal{H}$ where the local-density formulation holds, and then to find all homogeneous local fields on this Hilbert space and their correlation functions. These are the objects that have a physical content: they correspond to the scaling limit of lattice correlation functions, for instance. Of course, identifying the resulting QFT model with an underlying lattice model is non-trivial. Usually this is done based on the expected symmetry algebra and some explicit checks through different techniques.

Although this local-density formulation does not seem to provide a lot of direct information about the QFT model, the idea is that it still completely fixes it. In fact, many nice general conclusions can be extracted from this formulation.

2.1.1 Symmetry generators and local densities

The first thing we wish to clarify, though, is why it is called a local-density formulation. A **symmetry generator** is an operator $\hat{Q}$ with the properties that $\hat{Q}|\text{vac}\rangle = 0$, and that for any local fields $\hat{\mathcal{O}}$, $[\hat{Q}, \hat{\mathcal{O}}(x)]$ is also, as a function of x, a local field. The meaning is that $\hat{Q}$ generates a transormation in the space of local fields that preserves the correlation functions $\langle\text{vac}|\hat{\mathcal{O}}_1(x_1)\cdots\hat{\mathcal{O}}_n(x_n)|\text{vac}\rangle$. More precisely,

$$\langle\text{vac}|[\hat{Q}, \hat{\mathcal{O}}_1(x_1)]\cdots\hat{\mathcal{O}}_n(x_n)|\text{vac}\rangle + \ldots + \langle\text{vac}|\hat{\mathcal{O}}_1(x_1)\cdots[\hat{Q}, \hat{\mathcal{O}}_n(x_n)]|\text{vac}\rangle = \langle\text{vac}|[\hat{Q}, \hat{\mathcal{O}}_1(x_1)\cdots\hat{\mathcal{O}}_n(x_n)]|\text{vac}\rangle = 0. \tag{2.7}$$

These are the so-called **Ward identities** associated to a symmetry transformation.

A **local density** is a local field such that upon integration over its argument x, one gets a symmetry generator. By definition, $\hat{H}$, $\hat{P}$ and $\hat{B}$ indeed do annihilate the vacuum, but in order to conclude that $\hat{h}(x)$, $\hat{p}(x)$ and $\hat{b}(x)$ are local densities, we need to check the second condition, that $\hat{H}$, $\hat{P}$ and $\hat{B}$ generate a transformation on the space of local fields. This is, of course, fully expected. Since the derivative of a local field is also a local field, and since $[\hat{P}, \hat{\mathcal{O}}(x)] = id\hat{\mathcal{O}}(x)/dx$, then clearly $\hat{p}(x)$ is indeed a local density. For $\hat{h}(x)$ and $\hat{b}(x)$, we need to derive some more general properties from the general definition above, but it will turn out that they are also local densities. By the relation to symmetry transformations,

local densities often have "protected" dimensions: the renormalisation of the QFT does not change the dimension of local densities from that of their classical counterpart (their naïve dimension).

2.1.2 Some general consequences

We can show that it is sufficient to have the linear space (over $\mathbb{C}$) of homogeneous local fields, in order to build the linear space (over functions of x) of local fields. Indeed, suppose $\hat{\mathcal{O}}_{x_0}$ is a local field at x_0. Then, since $\hat{h}$ is itself homogeneous, $e^{-ia\hat{P}}\hat{\mathcal{O}}_{x_0}e^{ia\hat{P}}$ is a local field at x_0+a. Hence, from the linear space of local fields at x_0, we may construct the linear space of local field at any other point. But also, $e^{-ix\hat{P}}\hat{\mathcal{O}}_0 e^{ix\hat{P}}$ is, as a function of x, a homogeneous local field, and any homogeneous local field can be put at some point $x=0$ to produce a local field at 0. Hence, the linear space of local fields at any fixed point, say 0, is isomorphic to the linear space of homogeneous local fields. For instance, if $\hat{\mathcal{O}}^{(i)},\ i\in I$ span the space of homogeneous local fields (for some set of indices I), then any local field at x_0 can be written $\hat{\mathcal{O}}_{x_0}=\sum_{i\in I} f_i\hat{\mathcal{O}}^{(i)}(x_0)$, so that any local field $\hat{\mathcal{O}}$ in general can be written $\hat{\mathcal{O}}(x)=\sum_{i\in I} f_i(x)\hat{\mathcal{O}}^{(i)}(x)$.

The derivative with respect to x of a homogeneous local field also is a homogeneous local field. Taking the space of homogeneous local fields and substracting from it the space of its first derivative, we obtain a space which may be said of **degree** 0. From it, by differentiating, we can re-construct the full space, which may be divided into subspaces of fixed degree equal to the number of derivatives. Any local field has in fact a finite maximal degree: its overlap with the subspace of degree n is zero for n large enough. Intuitively, this is because infinite derivatives give rise to finite displacements, spoiling the locality property.

We now derive two important results. First, if $\int dx\,\hat{u}(x)=0$ for some local field $\hat{u}(x)$, then it must be that $\hat{u}(x)=id\hat{v}(x)/dx$ for some other local field $\hat{v}(x)$ (the factor i is for convenience). Indeed, let us consider the operator

$$\hat{v}(x)=-i\int_{-\infty}^{x}\hat{u}(x). \tag{2.8}$$

We have that $[\hat{v}(x),\hat{h}(x')]=0$ for $x<x'$ by locality of $\hat{u}$, and that $[\hat{v}(x),\hat{h}(x')]$ is independent of x for any $x>x'$ because $id/dx\,[\hat{v}(x),\hat{h}(x')]=[\hat{u}(x),\hat{h}(x')]=0$ by locality of $\hat{u}$ again. But $\lim_{x\to\infty}\hat{v}(x)=\int dx\,\hat{u}(x)=0$, hence $[\hat{v}(x),\hat{h}(x')]=0$ for $x>x'$ as well. Hence $\hat{v}(x)$ is a local field, and $id\hat{v}(x)/dx=\hat{u}(x)$.

Second, let us consider a local field $\hat{\mathcal{O}}$ and the commutation relations

$$[\hat{h}(x),\hat{\mathcal{O}}(x')]=\sum_{n=0}^{\infty}\delta^{(n)}(x-x')\hat{\mathcal{O}}_n(x') \tag{2.9}$$

for some operators $\hat{\mathcal{O}}_n(x')$, and where $\delta^{(n)}(x-x')$ is the n^{th} derivative of $\delta(x-x')$. This is a general consequence of locality of $\hat{\mathcal{O}}$, and of the fact that the result of the commutator is to be understood in terms of generalised functions. In general, this series in fact terminates, and we certainly have $\hat{\mathcal{O}}_0(x)=[\hat{H},\hat{\mathcal{O}}(x)]=-id\hat{\mathcal{O}}(x)/dt$. By considering

$$\hat{A}(x',n,\epsilon)=\int_{x'-\epsilon}^{x'+\epsilon}dx\,(x-x')^n[\hat{h}(x),\hat{\mathcal{O}}(x')],$$

we can show that $\hat{\mathcal{O}}_n$ in (2.9) are also local fields, and that if $\hat{\mathcal{O}}$ is a homogeneous local field, then also $\hat{\mathcal{O}}_n$ are. First, for the locality, we have that obviously $[\hat{h}(x''),\hat{A}(x',n,\epsilon)]=0$ for any x'' with $|x''-x'|>\epsilon$,

since

$$[\hat{h}(x''), \hat{A}(x', n, \epsilon)] = \int_{x'-\epsilon}^{x'+\epsilon} dx\, (x-x')^n \left([[\hat{h}(x''), \hat{h}(x)], \hat{\mathcal{O}}(x')] + [\hat{h}(x), [\hat{h}(x''), \hat{\mathcal{O}}(x')]] \right).$$

But, using (2.9) we have

$$\hat{A}(x', n, \epsilon) = (-1)^n n! \hat{\mathcal{O}}_n(x')$$

for any $\epsilon > 0$. Hence, we may take $\epsilon > 0$ as small as we wish, without changing $\hat{A}(x', n, \epsilon)$, so that we deduce that $[\hat{h}(x''), \hat{\mathcal{O}}_n(x')] = 0$ for any $x'' \neq x'$. In particular, $[\hat{H}, \hat{\mathcal{O}}(x)]$ is a local field (as a function of x), so that $\hat{h}$ is indeed a local density.

Concerning homogeneity, let us verify the commutation relations with $\hat{P}$, under the assumption that $\hat{\mathcal{O}}$ is homogeneous. We have

$$\begin{aligned}
[\hat{P}, \hat{A}(x', n, \epsilon)] &= \int_{x'-\epsilon}^{x'+\epsilon} dx\, (x-x')^n ([[\hat{P}, \hat{h}(x)], \hat{\mathcal{O}}(x')] + [\hat{h}(x), [\hat{P}, \hat{\mathcal{O}}(x')]]) \\
&= \int_{x'-\epsilon}^{x'+\epsilon} dx\, (x-x')^n i \left(\frac{d}{dx} + \frac{d}{dx'} \right) [\hat{h}(x), \hat{\mathcal{O}}(x')] \\
&= \int_{x'-\epsilon}^{x'+\epsilon} dx\, i \frac{d}{dx'} \left((x-x')^n [\hat{h}(x), \hat{\mathcal{O}}(x')] \right) + i\epsilon^n [\hat{h}(\epsilon), \hat{\mathcal{O}}(x')] - i(-\epsilon)^n [\hat{h}(-\epsilon), \hat{\mathcal{O}}(x')] \\
&= i \frac{d}{dx'} \left(\int_{x'-\epsilon}^{x'+\epsilon} dx\, (x-x')^n [\hat{h}(x), \hat{\mathcal{O}}(x')] \right)
\end{aligned}$$

where we did not need to use the fact that $\hat{\mathcal{O}}$ is local, just its homogeneity property. Hence, $\hat{A}(x', n, \epsilon)$ is homogeneous as a function of x'.

2.1.3 Local charges and Noether's theorem

A **local charge** (or local conserved charge) is an operator $\hat{Q}$ that commutes with the Hamiltonian $[\hat{Q}, \hat{H}] = 0$, and that can be written as the integral of a local field

$$\hat{Q} = \int dx\, \hat{q}(x), \quad \hat{q} \text{ local field.} \tag{2.10}$$

A simple consequence of these properties is that $[\hat{H}, \hat{q}(x)]$ is in fact the derivative of a local field. Indeed, from the results of 2.1.2, we know that $[\hat{H}, \hat{q}(x)]$ is, as function of x, a local field, and from the property of $\hat{Q}$ we know that $\int dx\, [\hat{H}, \hat{q}(x)] = 0$. Hence from the results of 2.1.2 again, we have $[\hat{H}, \hat{q}(x)] = id\hat{v}(x)/dx$. This leads to the local conservation equation

$$\frac{\partial}{\partial t} \hat{q}(x,t) + \frac{\partial}{\partial x} \hat{v}(x,t) = 0 \tag{2.11}$$

where the time evolution is obtained through, for instance, $\hat{q}(x,t) = e^{i\hat{H}t} \hat{q}(x) e^{-i\hat{H}t}$, as usual in the Heisenberg picture. (In fact, we obtained this equation for $t = 0$, but it can be generalised to any t by simply evolving in the Heisenberg picture, and using the fact that this evolution commutes with the $d/dt\cdot \equiv i[\hat{H}, \cdot]$ and $d/dx\cdot \equiv -i[\hat{P}, \cdot]$ operations). Noether's theorem then is the statement that certain symmetry generators that are compatible with the time evolution are local charges (that is, have local densities). Symmetry generators that are compatible with the time evolution indeed commute with the Hamiltonian, and the fact that symmetry generators transform local fields into other local fields seem

to be in agreement with an expression as an integral over a local field. However, there are details that need to be clarified to establish a more precise correspondence, which we will not go into here. Also, compatibility with the time evolution is not a necessity, but in the cases where there is no compatibility, one must extract an "explicit" time dependence and an "internal" time dependence. Again, we will not go into these details.

2.1.4 Relativistic invariance and the boost generator

It is possible to derive the exact form of the local field $\hat{b}$ involved in the definition of the boost operator $\hat{B}$, and from this, the current conservation equation associated to the Hamiltonian density.

First, let us write

$$\hat{b}(x) = x\hat{h}(x) + \sum_{n=0}^{N} f_n(x) \cdot \hat{\mathcal{O}}^{(n)}(x) \tag{2.12}$$

where $\hat{\mathcal{O}}^{(n)}$ is a vector in the sub-space of homogeneous local fields of degree n, and f_n is a vector-valued function on $\mathbb{R}$. We wrote the term $x\hat{h}(x)$ explicitly out of the sum, which can be done without loss of generality. The number N is the maximal degree associated to $\hat{b}$. For definiteness, we will consider $d\hat{\mathcal{O}}^{(n)}(x)/dx = \hat{\mathcal{O}}^{(n+1)}(x)$, which is just a choice of bases of homogeneous local fields of fixed degrees. We consider the equation

$$[\hat{B}, \hat{P}] = i\hat{H},$$

which leads to

$$\int dx \left(x\left(-i\frac{d}{dx}\hat{h}(x)\right) + \sum_{n=0}^{N} f_n(x) \cdot \left(-i\frac{d}{dx}\hat{\mathcal{O}}^{(n)}(x)\right)\right) = i\hat{H}. \tag{2.13}$$

Integrating by part (using the fact that the boundary terms must vanish because the integral defining $\hat{B}$ is well defined), we find that

$$\sum_{n=0}^{N} \left(\frac{d}{dx} f_n(x)\right) \cdot \hat{\mathcal{O}}^{(n)}(x) = \hat{u}(x) \tag{2.14}$$

where $\hat{u}$ is a local field such that its integral $\int dx\, \hat{u}(x)$ gives 0. By the results of 2.1.2, it must be a derivative of a local field, so that we can write

$$\sum_{n=0}^{N} \left(\frac{d}{dx} f_n(x)\right) \cdot \hat{\mathcal{O}}^{(n)}(x) = \frac{d}{dx}\left(\sum_{n=0}^{N-1} g_n(x) \cdot \hat{\mathcal{O}}^{(n)}(x)\right). \tag{2.15}$$

Looking at the degree N, we find the equation

$$g_{N-1}(x) = \frac{d}{dx} f_N(x). \tag{2.16}$$

Lower degrees give the recurrence relations

$$\frac{d}{dx} g_n(x) + g_{n-1}(x) = \frac{d}{dx} f_n(x), \quad n = 1, 2, \ldots, N-1 \tag{2.17}$$

and for degree 0, we have

$$g_0(x) + C = f_0(x) \tag{2.18}$$

for some constant vector C. These equations impose only one condition on the $f_n(x)$. Indeed, solving for g_n recursively from $n = N$, we find

$$g_{N-2}(x) = \frac{d}{dx} f_{N-1}(x) - \frac{d^2}{dx^2} f_N(x), \quad g_{N-3}(x) = \frac{d}{dx} f_{N-2}(x) - \frac{d^2}{dx^2} f_{N-1}(x) + \frac{d^3}{dx^3} f_N(x), \quad \ldots \tag{2.19}$$

so that

$$g_0(x) = \sum_{n=1}^{N} (-1)^{n-1} \frac{d^n}{dx^n} f_n(x). \tag{2.20}$$

Hence, we must have

$$\sum_{n=0}^{N} (-1)^n \frac{d^n}{dx^n} f_n(x) = C. \tag{2.21}$$

But if we write $f_0 = C + dF_0(x)/dx$ and $f_n(x) = dF_n(x)/dx$ for $n = 1, 2, \ldots, N$, which can be done without loss of generality, we can integrate the equation above and we see that we have

$$\sum_{n=0}^{N} (-1)^n \frac{d^n}{dx^n} F_n(x) = D \tag{2.22}$$

for some constant vector D. But as we just derived, this is the unique condition that we need in order to be able to write

$$\sum_{n=0}^{N} \left(\frac{d}{dx} F_n(x) \right) \cdot \hat{\mathcal{O}}^{(n)}(x) = \frac{d}{dx} \hat{v}(x)$$

for some local field $\hat{v}$. Hence, we have

$$\hat{b}(x) = x\hat{h}(x) + C \cdot \hat{\mathcal{O}}^{(0)}(x) + \frac{d}{dx}\hat{v}(x) = x \left(\hat{h}(x) - \frac{d}{dx} C \cdot \hat{\mathcal{O}}^{(0)}(x) \right) + \frac{d}{dx} \left(\hat{v}(x) + xC \cdot \hat{\mathcal{O}}^{(0)}(x) \right).$$

The total-derivative term can be taken out because it does not change the integral $\hat{B} = \int dx\, \hat{b}(x)$ (here, assuming vanishing asymptotic values), and for the first term, we may absorbe $\frac{d}{dx} C \cdot \hat{\mathcal{O}}^{(0)}(x)$ into $\hat{h}(x)$ because it is a total derivative term that does not change the result of $\hat{H} = \int dx\, \hat{h}(x)$ (again with the same assumption – we will keep such assumptions implicit from now on)[1]. Hence, we have shown that an appropriate choice of $\hat{h}(x)$ gives

$$\hat{b}(x) = x\hat{h}(x) \tag{2.23}$$

so that

$$\hat{B} = \int dx\, x\hat{h}(x). \tag{2.24}$$

Using the algebra relations, it is easy to get other forms of $\hat{B}$ that are in agreement with the algebra, corresponding to shifts in space and time:

$$e^{i\hat{H}t - i\hat{P}x} \hat{B} e^{-i\hat{H}t + i\hat{P}x} = \hat{B} + t\hat{P} - x\hat{H} = \int dx' \left((x' - x)\hat{h}(x') + t\hat{p}(x') \right). \tag{2.25}$$

Using the relation (2.9) and the results of 2.1.2, we see that the commutator of $\hat{B}$ with a local field, $[\hat{B}, \hat{\mathcal{O}}(x)]$, is also a local field, so that $\hat{b}$ is indeed a local density, as we claimed.

2.1.5 Relativistic invariance and conservation of the energy current

Another consequence of relativistic invariance is as follows. Considering

$$[\hat{B}, \hat{H}] = i\hat{P}$$

[1]There is one subtlety in changing the Hamitonian density: we may change the actual local structure of the theory! That is, we must make sure that $\hat{\mathcal{O}}^{(0)}(x)$ has the right properties so that the set of local fields does not change by the shift of $\hat{h}(x)$.

we can obtain the conservation equation associated to the Hamiltonian density $\hat{h}$. First, from the fact that $\hat{H}$ is local charge and the results of 2.1.3, we have

$$[\hat{H}, \hat{h}(x)] = i\frac{d}{dx}\hat{v}(x)$$

for some homogeneous local field $\hat{v}$ (the field $\hat{v}$ must be homogeneous because $[\hat{H}, \hat{h}(x)]$ is, as a function of x). Then,

$$\begin{aligned} 0 = [\hat{B}, \hat{H}] - i\hat{P} &= \int dx \, \left(x[\hat{h}(x), \hat{H}] - i\hat{p}(x) \right) \\ &= \int dx \, \left(-ix\frac{d}{dx}\hat{v}(x) - i\hat{p}(x) \right) \\ &= \int dx \, \left(i\hat{v}(x) - i\hat{p}(x) \right). \end{aligned}$$

Since the integrand is a local field, and since it integrates to zero, it must be a derivative of another local field:

$$i\hat{v}(x) - i\hat{p}(x) = i\frac{d}{dx}\hat{w}(x).$$

But we may re-define $\hat{p}(x)$ by absorbing $d\hat{w}(x)/dx$ without changing $\hat{P}$. Hence, with an appropriate choice of $\hat{p}(x)$, we have $\hat{v}(x) = \hat{p}(x)$, which implies

$$\frac{\partial}{\partial t}\hat{h}(x,t) + \frac{\partial}{\partial x}\hat{p}(x,t) = 0. \tag{2.26}$$

This is the conservation equation for one component of the stress-energy tensor. It implies, in particular, that we may make the stress-energy tensor symmetric[2].

Note that these discussions show that any field of dimension 1 could be absorbed away into definitions of $\hat{h}$ and $\hat{p}$. This is a general principle: **we can never generate local fields of dimension 1 from densities associated to space-time symmetries** (when the stress-energy tensor is symmetric). Local fields of dimension 1 only occur as densities associated to internal symmetries.

The relation (2.26) tells us that local fields are also local with respect to $\hat{p}(x)$:

$$[\hat{p}(x), \hat{\mathcal{O}}(x')] = 0 \quad \text{for} \quad x \neq x'.$$

We first evaluate, for $x \neq x'$,

$$[[\hat{H}, \hat{h}(x)], \hat{\mathcal{O}}(x')] = [\hat{H}, [\hat{h}(x), \hat{\mathcal{O}}(x')]] - [\hat{h}(x), [\hat{H}, \hat{\mathcal{O}}(x')]] = 0$$

where the last equality follows from the fact that $\hat{\mathcal{O}}(x')$ is a local field, and that $[\hat{H}, \hat{\mathcal{O}}(x')]$ also is a local field. Hence, from (2.26), we have that, still for $x \neq x'$,

$$0 = [\frac{d}{dx}\hat{p}(x), \hat{\mathcal{O}}(x')] = \frac{d}{dx}[\hat{p}(x), \hat{\mathcal{O}}(x')].$$

Applying $\int_{x''}^{\infty} dx$ for $x'' > x'$, we obtain $[\hat{p}(x''), \hat{\mathcal{O}}(x')] = 0$ for $x'' > x'$. Likewise, applying $\int_{-\infty}^{x''} dx$ for $x'' < x'$, we obtain $[\hat{p}(x''), \hat{\mathcal{O}}(x')] = 0$ for $x'' < x'$.

2.1.6 The algebra of local densities associated to Poincaré invariance

[2]This is true up to the subtlety alluded to in the construction of the boost density $\hat{b}(x)$ above.